Summing over Geometries in String Theory

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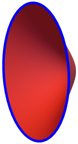
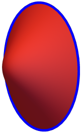
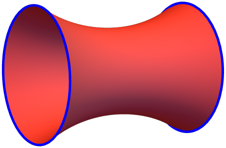
Based on [2008.07533] and [2102.12355]

Motivation

- Gravity should emerge from string theory at large distances.

- The gravitational path integral includes a sum over all distinct topologies with given boundary conditions, even those that do not contain saddles.
  e.g. JT-gravity:

$Z(\mathsf{S}^1 \sqcup \mathsf{S}^1) =$   $+$  $+ \ldots$

[Saad, Shenker, Stanford '19; . . .]

String theory

- We usually formulate perturbative string theory on a fixed background and study perturbative string corrections around a given background.
- This procedure is only consistent if the background is a saddle of the effective spacetime theory $\Longleftrightarrow$ the worldsheet is a CFT.
- So how does the semiclassical sum over geometries emerges from the string path integral?

AdS/CFT correspondence

- In the AdS/CFT correspondence, we can be more precise because we know the answer for the string partition function: the dual CFT partition function.
- There are exact (no ensemble average!) AdS/CFT dualities:

$$\text{AdS}_5 \times \text{S}^5 \Longleftrightarrow \mathcal{N} = 4 \text{ SYM} \ ,$$
$$\text{AdS}_3 \times \text{S}^3 \times \mathbb{T}^4 \Longleftrightarrow \text{Sym}^N(\mathbb{T}^4) \ .$$

- How do we reproduce the exact boundary partition function from string theory?

A naive prescription

- Inspired by semiclassical gravity, one might think

$$Z = \sum_{\text{bulk saddles } \mathcal{M}} \exp\left(\sum_{g=0}^{\infty} g_\text{s}^{2g-2} \int \mathscr{D}[\text{fields}]\ \mathrm{e}^{-S_\mathcal{M}[\text{fields}]}\right) + \text{D-instantons} + \textbf{???}\ ,$$

  where "fields" stands for all the worldsheet fields.
- This is wrong in general! The sum over saddles overcounts configurations, since they can be represented by highly excited string states (gas of gravitons) on other geometries.

Tensionless $\mathrm{AdS}_3/\mathrm{CFT}_2$

- There is a proposal for an exact AdS/CFT dual pair (in the sense that both the string side and the CFT side are under very good computational control): [LE, Gaberdiel, Gopakumar '18,. . .]

[see Matthias' talk]

$$\text{Strings on } \mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4 \text{ with one unit of NS-NS flux}$$
$$\Longleftrightarrow \mathrm{Sym}^N(\mathbb{T}^4)\ .$$

- We can study this theory on different backgrounds that are asymptotically $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4$ and address this question directly.

Result

- There is no sum over geometry!
- We simply have (when interpreted correctly)

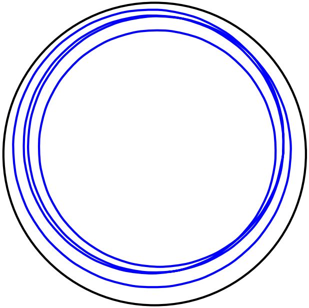
$$Z_{\mathsf{CFT}} = \exp\left(\sum_{g=0}^{\infty} g_{\mathsf{s}}^{2g-2} \int \mathscr{D}[\text{fields}]\ \mathrm{e}^{-S_{\mathcal{M}}[\text{fields}]}\right) \ .$$

  for any saddle geometry $\mathcal{M}$.
- All other semiclassical background geometries arise as highly excited string states.
- Perturbative string theory with asymptotic boundary conditions $\mathsf{AdS}_3 \times \mathsf{S}^3 \times \mathbb{T}^4$ with unit of NS-NS flux is background independent.

Geometry in the tensionless limit

- In the tensionless limit, strings can become very large.
- On AdS_3, string states wind around the asymptotic boundary of AdS_3.
- Even the graviton is such a non-local string state
  $\Rightarrow$ No local notion of spacetime geometry?
- Background geometry is still well-defined in the form of the worldheet theory.
- So 'summing over worldsheet theories' will be our proxy for 'summing over geometries'

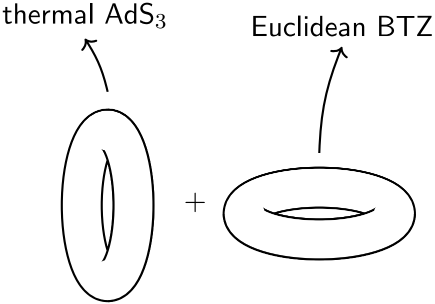
$t = 0$ slice of AdS_3



Thermal partition function

- Let's consider the example of a single torus boundary.
- In 3d-gravity, we sum over all modular images of thermal AdS_3 ($SL(2,\mathbb{Z})$ family of Euclidean black holes) [Maloney, Witten '07]

$Z =$



$+ \cdots$

Do this in string theory!

$+$ conical defects? [Benjamin, Collier, Maloney '20]

$+$ non-saddle geometries? [Maxfield, Turiaci '20] [Cotler, Jensen '20]

The string partition function on thermal AdS$_3$

- In the following, we explain how to compute the string partition function on thermal AdS$_3$.
- Other geometries will be similar.
- One can show that the perturbative string partition function is one-loop exact in g_{s}. [LE '21]
- So the partition function receives a sphere contribution and a torus contribution.
- sphere $\propto \frac{1}{g_{\mathsf{s}}^2} \propto \frac{1}{G_{\mathsf{N}}} \propto$ on-shell gravity action $= \frac{c\pi}{6}\,\mathrm{Im}\,\tau_{\mathsf{bdry}}$
- It is not known how to compute this directly in string theory and we will borrow the gravity result.

Grand canonical ensemble

- In the correspondence with the symmetric orbifold $\mathsf{Sym}^N(\mathbb{T}^4)$, N is the number of strings in the background.
- Since perturbative strings wind AdS_3 asymptotically, they contribute to this number and hence perturbative string theory computes the grand canonical partition function

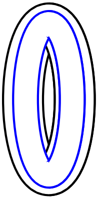
[Kim, Porrati '15]

$$\mathfrak{Z}_{\mathsf{Sym}(\mathbb{T}^4)} = \sum_{N=0}^{\infty} p^N Z_{\mathsf{Sym}^N(\mathbb{T}^4)}$$
$$= \exp\left(\sum_{a,d=1}^{\infty} \sum_{b=1}^{d} \frac{p^{ad}}{ad} Z_{\mathbb{T}^4}\left(\frac{a\tau_{\mathsf{bdry}}+b}{d}\right)\right) .$$

- This expresses the partition function as a sum over holomorphic covering surfaces.

Torus partition function

- Thus, we need to compute the string torus partition function.
- Thermal AdS_3 is obtained via a $\mathbb{Z}$-orbifold of global AdS_3.
  $\Rightarrow$ The worldsheet theory is the $\mathbb{Z}$-orbifold of the worldsheet theory of global AdS_3.



- We use the hybrid formalism to describe the worldsheet theory of global AdS_3. [Berkovits, Vafa, Witten '99]

$$\text{PSU}(1,1|2)_1/\mathbb{Z} \times \text{top. twisted } \mathbb{T}^4 \times \text{ghosts}$$

[see Matthias' lectures at pre-strings]

Torus partition function

- The worldsheet sphere + torus partition function takes the form

$$\frac{\text{Im}\,\tau_{\text{bdry}}}{2}\sum_{a,b,c,d\in\mathbb{Z}} p^{\det\begin{pmatrix} a & b \\ c & d\end{pmatrix}}\delta^2\big(\tau_{\text{bdry}}(c\tau+d)-a\tau-b\big)Z^{\mathbb{T}^4}\begin{bmatrix}\frac{b}{2}\\ \frac{a}{2}\end{bmatrix}(\tau)\,.$$

- $\sum_{a,b,c,d}$ is the sum over worldsheet instanton sectors.
- The partition function localizes in the moduli space of Riemann surface to tori that cover the boundary torus holomorphically. [Matthias' talk, Rajesh @ Strings 2020]

The string partition function

- For the string partition function, one integrates over the moduli space of tori.
- This is straightforward because of the presence of the δ-function:

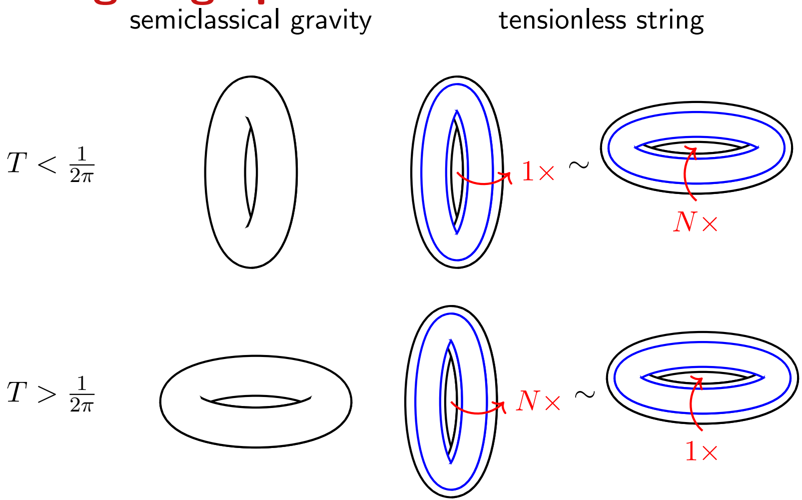
$$\mathfrak{Z}_{\text{thermal AdS}_3} = \exp\left(\sum_{a,d=1}^{\infty}\sum_{b=1}^{d}\frac{p^{ad}}{ad}Z_{\mathbb{T}^4}\left(\frac{a\tau_{\text{bdry}}+b}{d}\right)\right) \stackrel{!}{=} \mathfrak{Z}_{\text{Sym}(\mathbb{T}^4)} \ .$$

- One can repeat the same exercise for other backgrounds [LE '20]

$$\mathfrak{Z}_{\text{thermal AdS}_3} = \mathfrak{Z}_{\text{Euclidean BTZ}} = \mathfrak{Z}_{\text{conical defect}} = \mathfrak{Z}_{\text{Sym}(\mathbb{T}^4)} \ .$$

- These equalities hold (at least) to all orders in string loops.

Hawking-Page phase transition

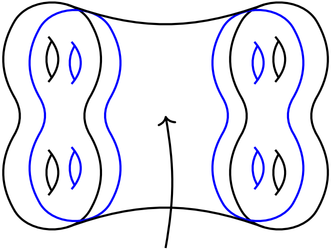


- Long strings on thermal AdS_3 can be equivalent to a Euclidean BTZ black hole (String/BH correspondence).

[Susskind '94; Horowitz, Polchinski '96; Giveon, Kutasov, Rabinovici, Sever '05]

A possible resolution of the factorization problem

- One can also consider the tensionless string on a Euclidean wormhole geometry



$\times \mathsf{S}^3 \times \mathbb{T}^4$

worldsheet contributions here cancel

- One can argue that the string partition function only receives contributions from worldsheets cover either the left or right boundary.

A possible resolution of the factorization problem (cont'd)

- The string partition function on the wormhole factorizes:

$$\mathfrak{Z}_{\mathsf{wormhole}} = \mathfrak{Z}_{\mathsf{L}} \times \mathfrak{Z}_{\mathsf{R}} \ .$$

- Since the worldsheets cannot enter the wormhole, the wormhole just looks like a disconnected geometry for the string.
- So there is no factorization problem.

Lessons

- The tensionless string is useful to explore quantum gravity at very short distances.
- The worldsheet theory simplifies enormously and is perturbatively under very good control.
- The natural ensemble of the duality is the grand canonical ensemble.
- The tensionless string is background independent and one does *not* need to sum over geometries.
- There does not seem to be a factorization problem for the tensionless string.