The Exact Renormalization Group and Higher Spin Holography

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Introduction

- An appealing aspect of holography is its interpretation in terms of the renormalization group of quantum field theories — the 'radial coordinate' is a geometrization of the renormalization scale — Hamilton-Jacobi theory of the radial quantization is expected to play a central role.

  e.g., [de Boer, Verlinde^2 '99, Skenderis '02, Heemskerk & Polchinski '10, Faulkner, Liu & Rangamani '10 ...]
- usually this is studied from the bulk side, as the QFT is typically strongly coupled
- here, we will approach the problem directly from the field theory side, using the Wilson-Polchinski exact renormalization group around (initially free) field theories [Douglas, Mazzucato & Razamat '10]
- of course, we can't possibly expect to find a purely gravitational dual
  - but there is some hope given the conjectured dualities between higher spin theories and vector models (for example).

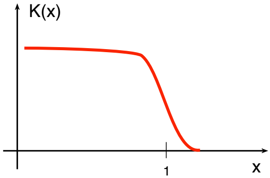
[Klebanov & Polyakov '02, Sezgin & Sundell '02, Leigh & Petkou '03] [Vasiliev '96, '99, '12] [de Mello Koch, et al]

The Exact Renormalization Group (ERG)

- Polchinski '84: formulated field theory path integral by introducing a regulator given by a cutoff function accompanying the fixed point action (i.e., the kinetic term).

$$Z = \int [d\phi] e^{-\int \phi K_F^{-1}(-\Box/M^2)\Box\phi - S_{int}[\phi]}$$

$$M\frac{\partial S_{int}}{\partial M} = -\frac{1}{2}\int M\frac{\partial K_F}{\partial M}\Box^{-1}\left[\frac{\delta S_{int}}{\delta\phi}\frac{\delta S_{int}}{\delta\phi} + \frac{\delta^2 S_{int}}{\delta\phi^2}\right]$$



- this equation describes how the couplings must depend on the RG scale in order that the partition function be independent of the cutoff.
- can apply similar methods to correlation functions, and thus obtain exact Callan-Symanzik equations as well

The ERG and Holography

- in this form, the ERG equations will be inconvenient — instead of moving the cutoff, we would like to fix the cutoff and move a renormalization scale (z)
- the ERG equations are first order equations, while bulk EOM are often second order
- solutions of such equations though are interpreted in terms of sources and vevs — the expected H-J structure implies that these should be thought of as canonically conjugate in radial quantization
- thus, we anticipate that the ERG equations for sources and vevs should be thought of as first-order Hamilton equations in the bulk

Locality is Over-Rated

- higher spin theories possess a huge gauge symmetry
- if the theory is really holographic, we expect to be able to identify this symmetry within the dual field theory
- unbroken higher spin symmetry implies an infinite number of conserved currents — one can hardly expect to find a local theory
- indeed, free field theories have a huge non-local symmetry
- e.g., N Majoranas in $2+1$

$$S_0 = \int_{x,y} \widetilde{\psi}^m(x)\gamma^\mu P_{F;\mu}(x,y)\psi^m(y) \equiv \int \widetilde{\psi}^m \cdot \gamma^\mu P_{F;\mu} \cdot \psi^m$$

$$P_{F;\mu}(x,y) = K_F^{-1}(-\Box/M^2)\partial_\mu^{(x)}\delta(x-y)$$

- we also include sources for 'single-trace' operators

$$S_{int} = U + \frac{1}{2}\int_{x,y} \widetilde{\psi}^m(x)\Big(A(x,y) + \gamma^\mu W_\mu(x,y)\Big)\psi^m(y)$$

The $O(L_2(\mathbb{R}^d))$ Symmetry

- Bi-local sources collect together infinite sets of local operators, obtained by expanding near $x \to y$

$$A(x,y) = \sum_{s=0}^{\infty} A^{a_1\cdots a_s}(x)\partial_{a_1}^{(x)}\cdots\partial_{a_s}^{(x)}\delta(x-y)$$

- Now we consider the following bi-local map of elementary fields

$$\psi^m(x) \mapsto \int_y \mathcal{L}(x,y)\psi^m(y) = \mathcal{L}\cdot\psi^m(x)$$

- We look at the action

$$\begin{aligned} S &\to \widetilde{\psi}^m \cdot \mathcal{L}^T \cdot \left[\gamma^\mu(P_{F;\mu} + W_\mu) + A\right] \cdot \mathcal{L} \cdot \psi^m \\ &= \widetilde{\psi}^m \cdot \gamma^\mu \mathcal{L}^T \cdot \mathcal{L} \cdot P_{F;\mu} \cdot \psi^m \\ &\quad + \widetilde{\psi}^m \cdot \left[\gamma^\mu(\mathcal{L}^T \cdot [P_{F;\mu}, \mathcal{L}] + \mathcal{L}^T \cdot W_\mu \cdot \mathcal{L}) + \mathcal{L}^T \cdot A \cdot \mathcal{L}\right] \cdot \psi^m \end{aligned}$$

The $O(L_2)$ Symmetry

- Thus, if we take $\mathcal{L}$ to be orthogonal,

$$\mathcal{L}^T \cdot \mathcal{L}(x,y) = \int_z \mathcal{L}(z,x)\mathcal{L}(z,y) = \delta(x,y),$$

the kinetic term is invariant, while the sources transform as

$O(L_2)$ gauge symmetry

$$\begin{aligned} W_\mu &\mapsto \mathcal{L}^{-1} \cdot W_\mu \cdot \mathcal{L} + \mathcal{L}^{-1} \cdot \left[P_{F;\mu}, \mathcal{L}\right] \\ A &\mapsto \mathcal{L}^{-1} \cdot A \cdot \mathcal{L} \end{aligned}$$

- We interpret this to mean that the source $W_\mu(x,y)$ is the $O(L_2)$ connection, with the regulated derivative $P_{F;\mu}$ playing the role of derivative

The $O(L_2)$ Ward Identity

- But this was a trivial operation from the path integral point of view, and so we conclude that there is an exact Ward identity

$$Z[M, g_{(0)}, W_\mu, A] = Z[M, g_{(0)}, \mathcal{L}^{-1} \cdot W_\mu \cdot \mathcal{L} + \mathcal{L}^{-1} \cdot P_{F;\mu} \cdot \mathcal{L}, \mathcal{L}^{-1} \cdot A \cdot \mathcal{L}]$$

- this is the usual notion of a background symmetry: a transformation of the elementary fields is compensated by a change in background
- more generally, we can turn on sources for arbitrary multi-local multi-trace operators — the sources will generally transform tensorially under $O(L_2)$

The $O(L_2)$ Symmetry

- Punch line: the $O(L_2)$ transformation leaves the (regulated) fixed point action invariant. $D_\mu = P_{F;\mu} + W_\mu$ plays the role of covariant derivative.
- More precisely, the free fixed point corresponds to any configuration
$$(A, W_\mu) = (0, W_\mu^{(0)})$$
where $W^{(0)}$ is any flat connection, $dW^{(0)} + W^{(0)} \wedge W^{(0)} = 0$
- It is therefore useful to split the full connection as
$$W_\mu = W_\mu^{(0)} + \widehat{W}_\mu$$
- will choose it to be invariant under the conformal algebra
  - $W^{(0)}$ is a flat connection associated with the fixed point
  - A, $\widehat{W}$ are operator sources, transforming tensorially under $O(L_2)$

The $CO(L_2)$ symmetry

- We generalize $O(L_2)$ to include scale transformations

$$\int_z \mathcal{L}(z,x)\mathcal{L}(z,y) = \lambda^{2\Delta_\psi}\delta(x-y)$$

- This is a symmetry (in the previous sense) provided we also transform the metric, the cutoff and the sources

$$g_{(0)} \mapsto \lambda^2 g_{(0)}, \quad M \mapsto \lambda^{-1} M$$
$$A \mapsto \mathcal{L}^{-1}\cdot A\cdot \mathcal{L}$$
$$W_\mu \mapsto \mathcal{L}^{-1}\cdot W_\mu \cdot \mathcal{L} + \mathcal{L}^{-1}\cdot \left[P_{F;\mu}, \mathcal{L}\right].$$

- A convenient way to keep track of the scale is to introduce the conformal factor $g_{(0)} = \frac{1}{z^2}\eta$. Then $z \mapsto \lambda^{-1} z$. This z should be thought of as the renormalization scale.

The Renormalization group

- To study RG systematically, we proceed in two steps:

Step 1: Lower the cutoff $M \mapsto \lambda M$, by integrating out the “fast modes”

$$Z[M, z, A, W] = Z[\lambda M, z, \widetilde{A}, \widetilde{W}] \qquad (\textit{Polchinski})$$

Step 2: Perform a $CO(L_2)$ transformation to bring the cutoff back to M, but in the process changing $z \mapsto \lambda^{-1} z$

$$Z[\lambda M, z, \widetilde{A}, \widetilde{W}] = Z[M, \lambda^{-1} z, \mathcal{L}^{-1} \cdot \widetilde{A} \cdot \mathcal{L}, \mathcal{L}^{-1} \cdot \widetilde{W} \cdot \mathcal{L} + \mathcal{L}^{-1} \cdot [P_F, \mathcal{L}]]$$

- We can now compare the sources at the same cutoff, but different z. Thus, z becomes the natural flow parameter, and we can think of the sources as being z-dependent. (Thus we have the Wilson-Polchinski formalism extended to include both a cutoff and an RG scale — required for a holographic interpretation).

Infinitesimal version: RG equations

- Infinitesimally, we parametrize the $CO(L_2)$ transformation as

$$\mathcal{L} = \mathbf{1} + \varepsilon z W_z$$

- should be thought of as the z-component of the connection.
- The RG equations become

$$A(z + \varepsilon z) = A(z) + \varepsilon z\,[W_z, A] + \varepsilon z \beta^{(A)} + O(\varepsilon^2)$$

$$W_\mu(z + \varepsilon z) = W_\mu(z) + \varepsilon z\left[P_{F;\mu} + W_\mu, W_z\right] + \varepsilon z \beta^{(W)}_\mu + O(\varepsilon^2)$$

- The beta functions are *tensorial*, and quadratic in A and $\widehat{W}$.
- Thus, RG extends the sources A and W to bulk fields $\mathcal{A}$ and $\mathcal{W}$.

RG equations

- Comparing terms linear in ε gives

$$\partial_z \mathcal{W}_\mu^{(0)} - [P_{F;\mu}, \mathcal{W}_z^{(0)}] + [\mathcal{W}_z^{(0)}, \mathcal{W}_\mu^{(0)}] = 0$$

$$\partial_z \mathcal{A} + [\mathcal{W}_z, \mathcal{A}] = \beta^{(\mathcal{A})}$$

$$\partial_z \mathcal{W}_\mu - [P_{F;\mu}, \mathcal{W}_z] + [\mathcal{W}_z, \mathcal{W}_\mu] = \beta_\mu^{(\mathcal{W})}$$

- These equations are naturally thought of as being part of fully covariant equations (e.g., the first is the $z\mu$ component of a bulk 2-form equation, where $d \equiv dx^\mu P_{F,\mu} + dz\partial_z$.)

$$d\mathcal{W}^{(0)} + \mathcal{W}^{(0)} \wedge \mathcal{W}^{(0)} = 0$$

$$d\mathcal{A} + [\mathcal{W}, \mathcal{A}] = \boldsymbol{\beta}^{(\mathcal{A})}$$

$$d\mathcal{W} + \mathcal{W} \wedge \mathcal{W} = \boldsymbol{\beta}^{(\mathcal{W})}$$

$$\mathcal{D}\boldsymbol{\beta}^{(\mathcal{A})} = \left[\boldsymbol{\beta}^{(\mathcal{W})}, \mathcal{A}\right], \quad \mathcal{D}\boldsymbol{\beta}^{(\mathcal{W})} = 0$$

- The resulting equations are then diff invariant in the bulk.

Hamilton-Jacobi Structure

- Similarly, one can extract exact Callan-Symanzik equations for the z-dependence of $\Pi(x,y) = \langle\tilde{\psi}(x)\psi(y)\rangle$, $\Pi^\mu(x,y) = \langle\tilde{\psi}(x)\gamma^\mu\psi(y)\rangle$. These extend to bulk fields $\mathcal{P}, \mathcal{P}^A$.
- The full set of equations then give rise to a phase space formulation of a dynamical system — $(\mathcal{A}, \mathcal{P})$ and $(\mathcal{W}_A, \mathcal{P}^A)$ are canonically conjugate pairs from the point of view of the bulk.
- If we identify $Z = e^{iS_{HJ}}$, then a fundamental relation in H-J theory is

$$\frac{\partial}{\partial z} S_{HJ} = -\mathcal{H}$$

- We can thus read off this Hamiltonian — it can be thought of as the output of the ERG analysis
- there is a corresponding action S_{HJ} for this higher spin theory, written in terms of phase space variables

Hamilton-Jacobi Structure

- We interpret this phase space theory as the higher spin gauge theory
- this theory is written as a gauge theory on a spacetime, topology $\sim \mathbb{R}^d \times \mathbb{R}^+$
- we've identified a specific flat connection $\mathcal{W}^{(0)}$ representing the free fixed point

$$\mathcal{W}^{(0)}(x,y) = -\frac{dz}{z} D(x,y) + \frac{dx^\mu}{z} P_\mu(x,y)$$

where $P_\mu(x,y) = \partial_\mu^{(x)} \delta(x-y)$ and $D(x,y) = (x^\mu \partial_\mu^{(x)} + \Delta)\delta(x-y)$.

- This connection is equivalent to the vielbein and spin connection of AdS_{d+1}.
  - $\mathcal{W}^{(0)}$ is invariant under the conformal algebra $o(2,d) \subset co(L_2)$

Geometry: The Infinite Jet bundle

- we can put the non-local transformation $\psi(x) \mapsto \int_y \mathcal{L}(x,y)\psi(y)$ in more familiar terms by introducing the notion of a jet bundle
- The simple idea is that we can think of a differential operator $\mathcal{L}(x,y)$ as a matrix by "prolongating" the field

$$\psi^m(x) \mapsto \left(\psi^m(x), \frac{\partial\psi^m}{\partial x^\mu}(x), \frac{\partial^2\psi^m}{\partial x^\mu \partial x^\nu}(x)\cdots\right) \quad \text{"}jet\text{"}$$

- Then, differential operators, such as $P_\mu(x,y) = \partial_\mu^{(x)}\delta(x-y)$ are interpreted as matrices $\mathbb{P}_\mu$ that act on these vectors
- The bi-local transformations can be thought of as local gauge transformations of the jet bundle.
- The gauge field W is a connection 1-form on the jet bundle, while A is a section of its endomorphism bundle.

Other Examples

- the $2+1$ Majorana model is presumably equivalent to the Vasiliev B-model
- extensions to higher dimensions require additional sources for $\tilde{\psi}\gamma^{ab}\psi$, ...
- N complex bosons: construct in similar terms

$$S = \int \tilde{\phi}_m \cdot \Big(\big[D_{F;\mu} + W_\mu\big]^2 + B \Big) \cdot \phi^m$$

- The ERG equations give rise to an 'A-model' in any dimension.

[RGL, O. Parrikar, A.B. Weiss, to appear.]

- Here though there is an extra background symmetry

$$Z[M, z, B, W_\mu^{(0)}, \widehat{W}_\mu + \Lambda_\mu] = Z[M, z, B + \{\Lambda^\mu, D_\mu\} + \Lambda_\mu \cdot \Lambda^\mu, W_\mu^{(0)}, \widehat{W}_\mu]$$

- this background symmetry allows for fixing $W_\mu \to W_\mu^{(0)}$, and the corresponding transformed B sources all single-trace currents.

The Bulk Action and Correlation Functions

- For the bosonic theory, the bulk phase space action is

$$I = \int dz \, \mathrm{Tr} \left\{ \mathcal{P}^I \cdot \left(\mathcal{D}_I \mathfrak{B} - \beta_I^{(\mathfrak{B})} \right) + \mathcal{P}^{IJ} \cdot \mathcal{F}_{IJ}^{(0)} + N \, \Delta_B \cdot \mathfrak{B} \right\}$$

- Here Δ_B is a derivative with respect to M of the cutoff function.
- As in any holographic theory, we solve the bulk equations of motion in terms of boundary data, and obtain the on-shell action, which encodes the correlation functions of the field theory.
- It is straightforward to carry this out exactly for the free fixed point.
- Here we have

$$I_{o.s.} = N \int \Delta_B \cdot \mathfrak{B}_{o.s.}$$

where now $\mathfrak{B}_{o.s.}$ is the bulk solution

The Bulk Action and Correlation Functions

- The RG equation

$$\left[\mathcal{D}_z^{(0)}, \mathfrak{B}\right] = \beta_z^{(\mathfrak{B})} = \mathfrak{B} \cdot \Delta_B \cdot \mathfrak{B}$$

can be solved iteratively

$$\mathfrak{B} = \alpha \mathfrak{B}_{(1)} + \alpha^2 \mathfrak{B}_{(2)} + \ldots,$$

$$\begin{aligned}
\left[\mathcal{D}_z^{(0)}, \mathfrak{B}_{(1)}\right] &= 0 \\
\left[\mathcal{D}_z^{(0)}, \mathfrak{B}_{(2)}\right] &= \mathfrak{B}_{(1)} \cdot \Delta_B \cdot \mathfrak{B}_{(1)} \\
\left[\mathcal{D}_z^{(0)}, \mathfrak{B}_{(3)}\right] &= \mathfrak{B}_{(2)} \cdot \Delta_B \cdot \mathfrak{B}_{(1)} + \mathfrak{B}_{(1)} \cdot \Delta_B \cdot \mathfrak{B}_{(2)} \\
&\vdots
\end{aligned}$$

The Bulk Action and Correlation Functions

- The first equation (1) is homogeneous and has the solution

$$\mathfrak{B}_{(1)}(z;x,y) = \int_{x',y'} K^{-1}(z;x,x')b_{(0)}(x',y')K(z;y',y)$$

where we have defined the boundary-to-bulk Wilson line

$$K(z) = P.\exp\int_{\epsilon}^{z} dz'\,\mathcal{W}_z^{(0)}(z')$$

with the boundary being placed at $z=\epsilon$.

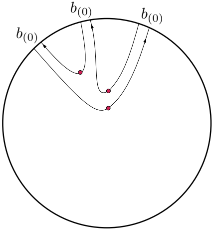
- $b_{(0)}$ has the interpretation of a boundary source
- this can then be inserted into the second order equation and the whole system solved iteratively

The Bulk Action and Correlation Functions

- At k^{th} order, one finds a contribution to the on-shell action

$$\begin{aligned} I^{(k)}_{o.s.} &= N \int_\epsilon^\infty dz_1 \int_\epsilon^{z_1} dz_2 ... \int_\epsilon^{z_{k-1}} dz_k \\ &\times \mathrm{Tr}\, H(z_1)\cdot b_{(0)} \cdot H(z_2) \cdot b_{(0)} \cdot ... \cdot H(z_k) \cdot b_{(0)} \\ &+ permutations \end{aligned}$$

where $H(z) \equiv K^{-1}(z) \cdot \Delta_B(z) \cdot K(z) = \partial_z g(z)$



The Witten diagram for the bulk on-shell action at third order.

The Bulk Action and Correlation Functions

- The z-integrals can be performed trivially, resulting in

$$I^{(k)}_{o.s.} = \frac{N}{k} \mathrm{Tr} \left(g_{(0)} \cdot b_{(0)} \right)^k$$

where $g_{(0)} = g(\infty)$ is the boundary free scalar propagator

- These can be resummed, resulting in

$$Z[b_{(0)}] = \det{}^{-N} \left(1 - g_{(0)} b_{(0)} \right)$$

which is the exact generating functional for the free fixed point.

- Thus, this holographic theory does everything that it can for us.

Interactions and Non-trivial fixed points

- really, this analysis should be thought of within a larger system, in which field theory interactions are turned on
- for example, if we turn on all multi-local multi-trace interactions, we obtain an infinite set of ERG equations – the bulk theory now contains an infinite number of conjugate pairs
- the Gaussian theory is a consistent truncation of this more general theory, in which the higher spin gauge symmetry remains unbroken
- we expect that there are other solutions of the full ERG equations with other boundary data specified (such as a 4-point coupling), corresponding to other fixed points
- an example is the W-F large N critical point