Elliptic Virasoro Conformal Blocks

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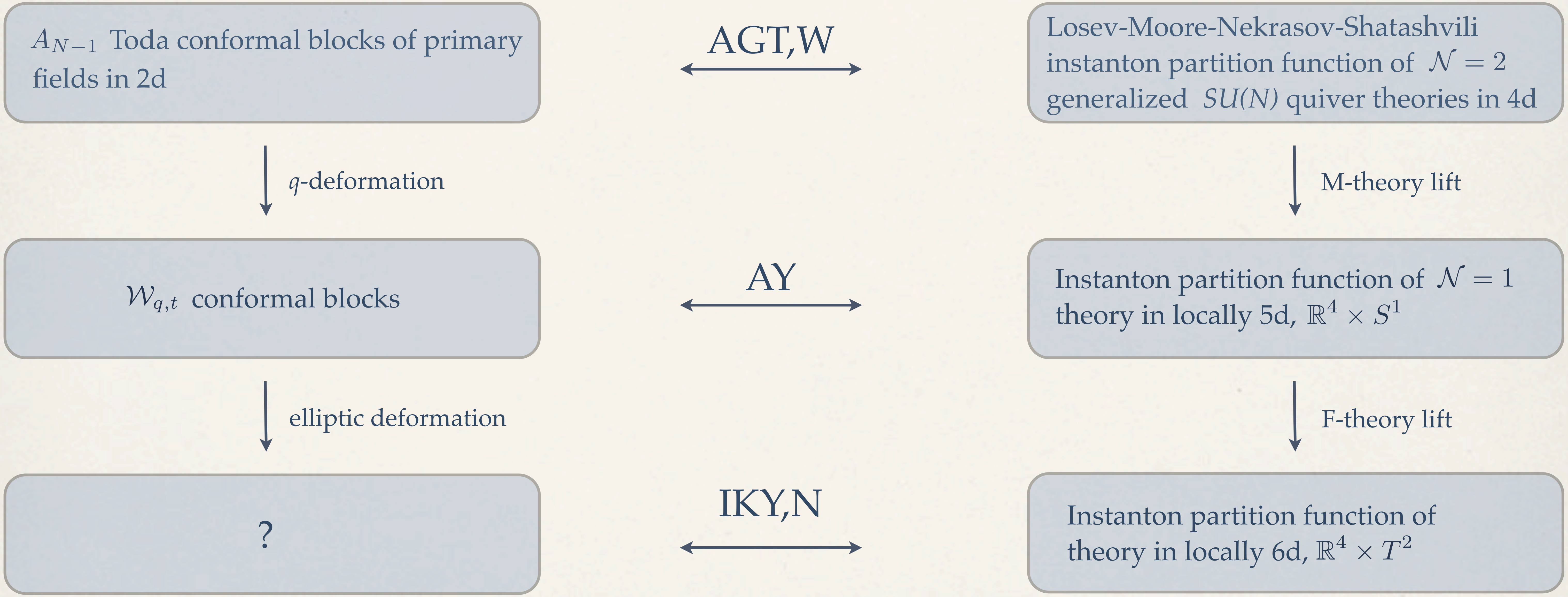
Motivation


A_{N-1} Toda conformal blocks of primary fields in 2d
AGT,W
Losev-Moore-Nekrasov-Shatashvili instanton partition function of $\mathcal{N}=2$ generalized $SU(N)$ quiver theories in 4d
q-deformation
M-theory lift
$\mathcal{W}_{q,t}$ conformal blocks
AY
Instanton partition function of $\mathcal{N}=1$ theory in locally 5d, $\mathbb{R}^4 \times S^1$
elliptic deformation
F-theory lift
?
IKY,N
Instanton partition function of theory in locally 6d, $\mathbb{R}^4 \times T^2$

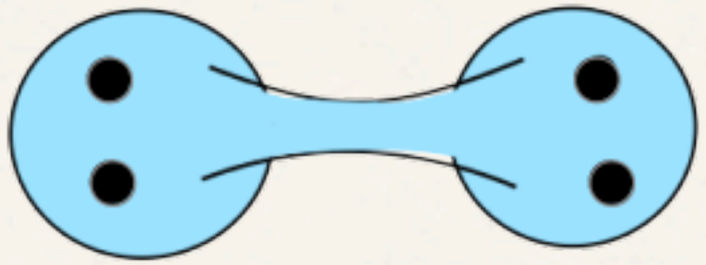
Outline

- Alday-Gaiotto-Tachikawa Relation
- 5d Alday-Gaiotto-Tachikawa Relation & Triality
- Elliptic Virasoro Conformal Blocks

AGT Relation

Gaiotto constructed generalized quiver theories by compactification of 6d *(2,0)* SCFT's of A_{N-1} type on punctured Riemann surfaces Σ.

For example:



$SU(2)$ with $N_f = 4$

Alday, Gaiotto and Tachikawa proposed a precise relationship between 4d $\mathcal{N} = 2$ $SU(2)$ quiver gauge theories (subject to Ω-deformation) and Liouville conformal theory on Riemann surfaces Σ.

AGT connects the perturbative and instanton part of the prepotential of Seiberg-Witten theory with the 3-point function and conformal blocks of Liouville theory, respectively.

AGT Relation

More precisely, AGT proposed the following "dictionary" for the Losev-Moore-Nekrasov-Shatashvilli partition function Z of 4d SCFT and 2d Liouville conformal theory:

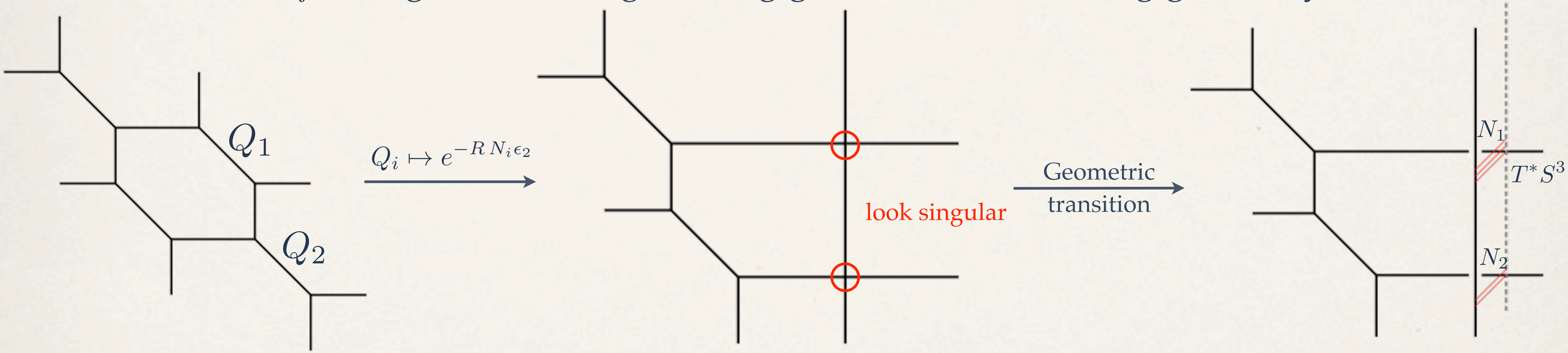
Gauge Theory	Liouville Theory
Instanton partition function Z_{inst}	Chiral conformal block $\mathcal{B}$
1-loop perturbative part Z_{pert}	DOZZ 3-point functions
Mass of hypermultiplets m_i	External momenta α_i
Coulomb branch parameters a	Internal momenta σ
Exponentiated UV coupling constant	Sewing parameter Λ
Equivariant localization parameters $\epsilon_{1,2}$	Liouville parameters b and b^{-1}

5d AGT Relation & Triality

Awata and Yamada extended the AGT relation by considering locally 5d gauge theories with $\mathcal{N}=1$ supersymmetry on $\mathbb{R}^4 \times S^1$ and conformal blocks of the q-deformed Virasoro algebra..

Together with Aganagic, Haouzi and Shakirov, we used the M-theory lift and topological string theory to extend the duality into a "triality" and give a physical interpretation for this relation.

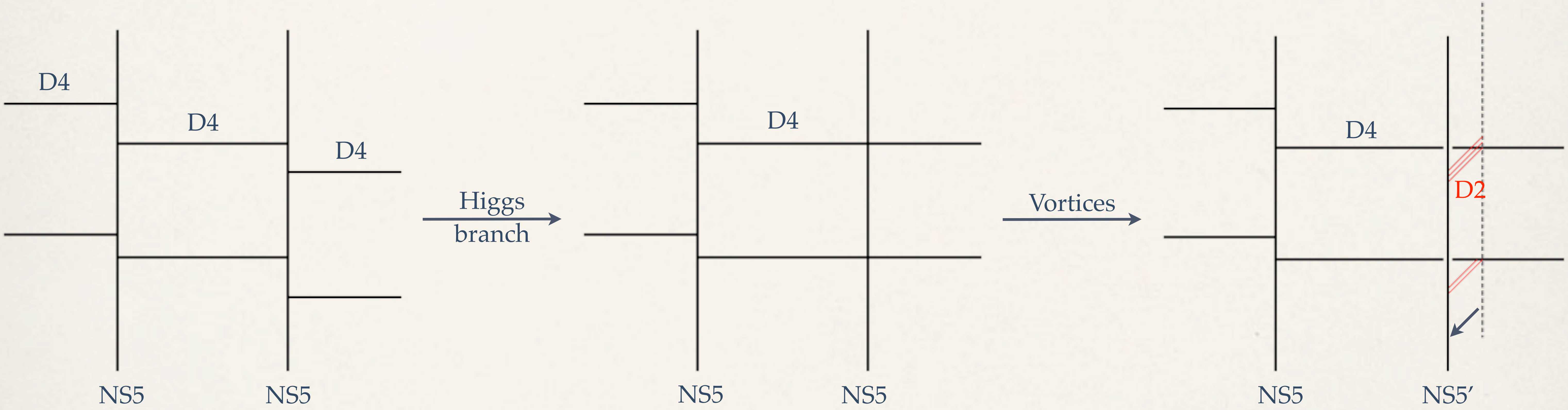
For $SU(2)$ with $N_f = 4$, geometric engineering gives us the following geometry:



local $\mathbb{P}^1 \times \mathbb{P}^1$ blown-up at four point

5d AGT Relation & Triality

The above limit is familiar in the context of vortex string in type IIA string theory



The Higgs branch meets the Coulomb branch when $\vec{a} = \vec{m}_F - \epsilon_2 \vec{N}$.

We can study the same setup in one dimension higher, with a circle fibration!

5d AGT Relation & Triality

Geometric transition implies that the open and closed topological string partition functions coincide:

$$Z_{closed} = Z_{open}$$

↙ ↘

LMNS partition function = Vortex partition function

Both sides can be computed by localization, and they reduce to matrix models!

The matrix model for vortex partition function *is* the Dotsenko-Fateev Coulomb gas representation of the *q*-deformed Liouville theory.

5d AGT Relation & Triality

The action of the Liouville theory is given by

$$S = \int dz d\bar{z} \sqrt{g} \left(g^{z\bar{z}} \partial_z \phi \, \partial_{\bar{z}} \phi + Q \phi R + e^{2b\phi} \right)$$

The Liouville $\phi(z)$ field has a free field representation

$$b\phi(z) = \phi_0 + h_0 \log z + \sum_{n \neq 0} h_n \frac{z^{-n}}{n}$$

where h_n's satisfy the Heisenberg algebra

$$[h_n, h_m] = -\frac{b^2}{2} n \, \delta_{n+m,0}$$

5d AGT Relation & Triality

The conformal blocks have the Dotsenko-Fateev integral reprsentation

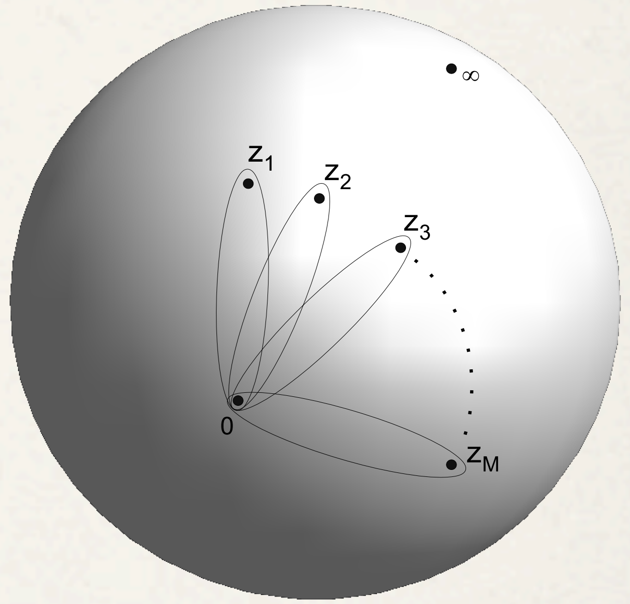
$$\mathcal{B}_{\alpha_0,\ldots,\alpha_{M+1}}(z_1,\ldots,z_M) = \oint_{\mathcal{C}_1} dy_1 \ldots \oint_{\mathcal{C}_n} dy_n \langle V_{\alpha_0}(0)\ldots V_{\alpha_M}(z_M)V_{\alpha_{M+1}}(\infty)S(y_1)\ldots S(y_n)\rangle$$

The primary field with momentum α inserted at z is realized by the vertex operator $V_\alpha(z)$:

$$V_\alpha(z) = :\exp\left(-\frac{\alpha}{b}\phi(z)\right):$$

The screening charges are represented by

$$S(y) = :\exp\left(2b\phi(y)\right):$$



The choices of contours

5d AGT Relation & Triality

In this free field representation, the Dotsenko-Fateev integrals takes the following form

$$\mathcal{B}_{\alpha_0,\ldots,\alpha_{M+1}}(z_1,\ldots,z_M) = \frac{C}{\prod_{a=1}^{M} N_a!}\oint d^N y\, \Delta_\beta^2(y) \prod_{a=0}^{M} V_a(y)$$

$$= \frac{C}{\prod_{a=1}^{M} N_a!}\oint d^N y \prod_{1\le i<j\le N} (y_i - y_j)^{2\beta} \prod_{a=0}^{M}\prod_{i=1}^{N_a}(y_i - z_a)^{\alpha_a}$$

The last integral can be obtained by Wick's theorem of the vertex operators, and they can be algebraically q-deformed: the Heisenberg algebra becomes

$$[h_n, h_m] = -\frac{b^2}{2} n\, \delta_{n+m,0} \xrightarrow{q\text{-deformation}} [\mathbf{h}_n, \mathbf{h}_m] = -\frac{1}{1+(q/t)^n}\frac{1-t^n}{1-q^n} n\delta_{n+m,0}, \text{ with } t = q^\beta, \text{ and } \beta = -b^2$$

5d AGT Relation & Triality

The primary fields and screening charges are deformed as well

$$V_\alpha(z) = :\exp\left(-\frac{\alpha}{b}\sum_{n\neq 0} h_n \frac{z^{-n}}{n}\right): \qquad \xrightarrow{q\text{-deformation}} \qquad \mathbf{V}_\alpha(z) = :\exp\left(-\frac{\alpha}{b}\sum_{n\neq 0} \frac{1}{n}\frac{1-q^{\alpha n}}{1-t^n}\mathbf{h}_n z^n\right):$$

$$S(y) = :\exp\left(2b\sum_{n\neq 0} h_n \frac{y^{-n}}{n}\right): \qquad \mathbf{S}(y) = :\exp\left(\sum_{n\neq 0} \frac{1+(q/t)^n}{n}\mathbf{h}_n \frac{y^{-n}}{n}\right):$$

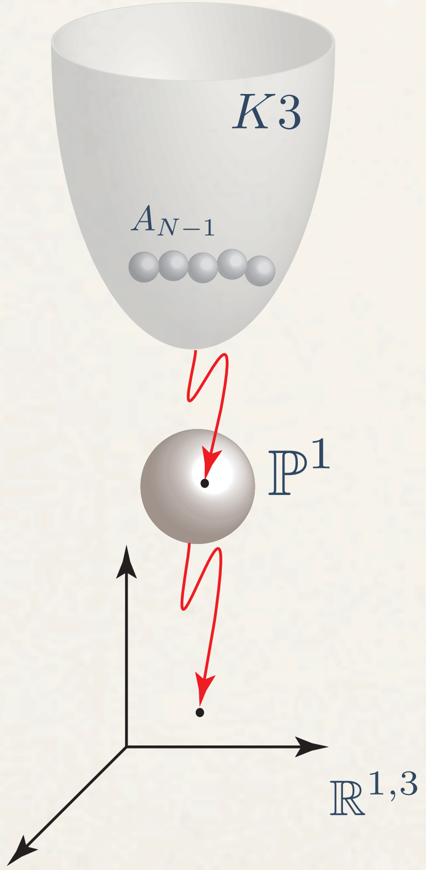
The integrand can be computed again by Wick's theorem

$$\mathcal{B}_{\alpha_0,\ldots,\alpha_{M+1}}(z_1,\ldots,z_M) = \frac{C}{\prod_{a=1}^M N_a!}\oint d^N y \prod_{1\leq i\neq j\leq N} \frac{(y_i/y_j;q)_\infty}{(ty_i/y_j;q)_\infty} \prod_{a=0}^{M}\prod_{i=1}^{N_a} \frac{(q^{\alpha_a} z_a/y_i;q)_\infty}{(z_a/y_i;q)_\infty}$$

5d AGT Relation & Triality

The last integral is the same matrix integral after localization of the vortex partition function on $(\mathbb{C}\times S^1)_q$, so we established:

$$Z^{5d}_{LMNS}\big|_{\vec{a}=\vec{m}_F-\epsilon_2\vec{N}} = Z^{3d}_{vortex} = \mathcal{B}_{q-deformed}$$

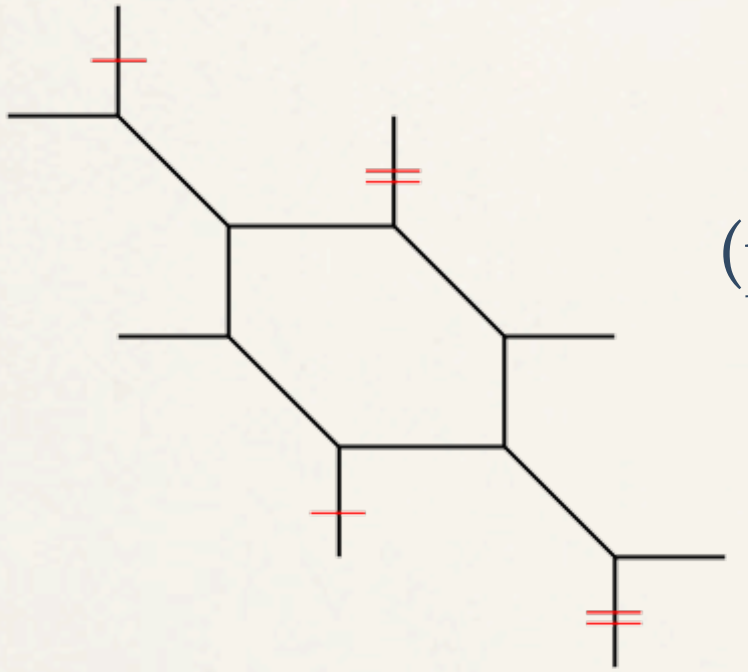


Remark: One important difference from AGT is that the 5d theory is **not** the usual '*AGT dual*' but rather its *fiber-base dual*!

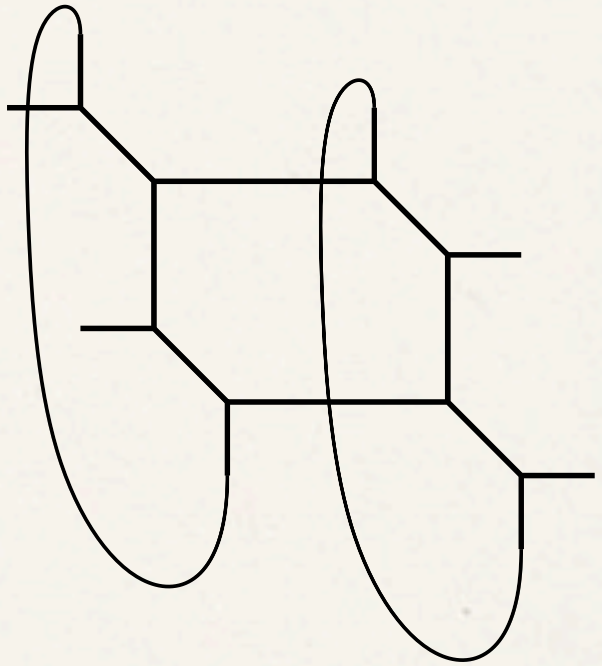
For example: $SU(2)\times SU(2)\mapsto SU(3)$ with proper matter content.

Elliptic Virasoro Conformal Blocks

We can lift to F-theory by introducing an elliptic fibration in the toric geometry:



(p,q) web lives on a cylinder



Rational $\xrightarrow{q\text{-deformation}}$ Trigonometric $\xrightarrow{\text{Elliptic fibration}}$ Elliptic

Elliptic Virasoro Conformal Blocks

Saito constructed the elliptic deformations of Ding-Iohara-Miki algebra and the vertex operators:

$$X(z) = \exp\left(\sum_{n>0} X^-_{-n}\mathbf{h}_{-n}z^n\right)\exp\left(\sum_{n>0} X^+_n\mathbf{h}_n z^{-n}\right)$$

In addition to a further deformation of the Heisenberg algebra, a new set of generators are needed:

$$[a_n, a_m] = n(1-p^{|n|})\frac{1-q^n}{1-t^n}\delta_{n+m,0}, \qquad [b_n, b_m] = n\frac{1-p^{|n|}}{(qt^{-1}p)^n}\frac{1-q^n}{1-t^n}\delta_{n+m,0}$$

$$[a_n, b_m] = 0$$

Elliptic Virasoro Conformal Blocks

The elliptic version of the vertex operator $X(z)$

$$\mathcal{X}(p;z) \equiv X_a(p;z)X_b(p;z)$$

with

$$X_a(p;z) \equiv \exp\left(\sum_{n>0} \frac{1}{1-p^n} X^-_{-n} a_{-n} z^n\right) \exp\left(\sum_{n>0} \frac{1}{1-p^n} X^+_n a_n z^{-n}\right)$$

$$X_b(p;z) \equiv \exp\left(\sum_{n>0} \frac{p^n}{1-p^n} X^-_n b_{-n} z^{-n}\right) \exp\left(\sum_{n>0} \frac{p^n}{1-p^n} X^+_{-n} a_n z^n\right)$$

This deformation replaces $(z;q)_\infty$ with $\Gamma_{p,q}(z)^{-1}$ in the Dotsenko-Fateev integral. Nieri established analogous results independently.

Elliptic Virasoro Conformal Blocks

The elliptic Dotsenko-Fateev integral can be again performed by residues and it matches the instanton partition function of 6d theory which can computed by refined topological vertex.

Moreover, one can further discuss the geometric meaning of the elliptic deformation in terms of open Gromov-Witten theory. We can define some vectors of the form $|\lambda\rangle = \exp(\ldots)|\emptyset\rangle$.

Thank you!