Structure Constants at Strong Coupling: From Hexagons to Minimal Surface

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Based on 1604.03575 with Y.Jiang, I.Kostov, D.Serban
1603.03164 with T.Nishimura, Y.Kazama

also based on works done with B.Basso, V.Goncalves, P.Vieira

Strings 2016

AdS/CFT correspondence

Emergence of Geometry from Quantum Mechanics

- d-dim CFT = d+1-dim Gravity
- Wilson Loop = Minimal Surface [Maldacena] [Rey]
- Entanglement Entropy = Minimal (Hyper-)surface

[Ryu-Takayanagi]

Difficulty:

Need to analyze the strong-coupling regime.

Even when it is possible,
it is often not clear what kind of physical/mathematical mechanisms are responsible for it.

Goal: Analyze the strong-coupling regime of planar N=4 SYM using integrability.

“Clustering” of magnons

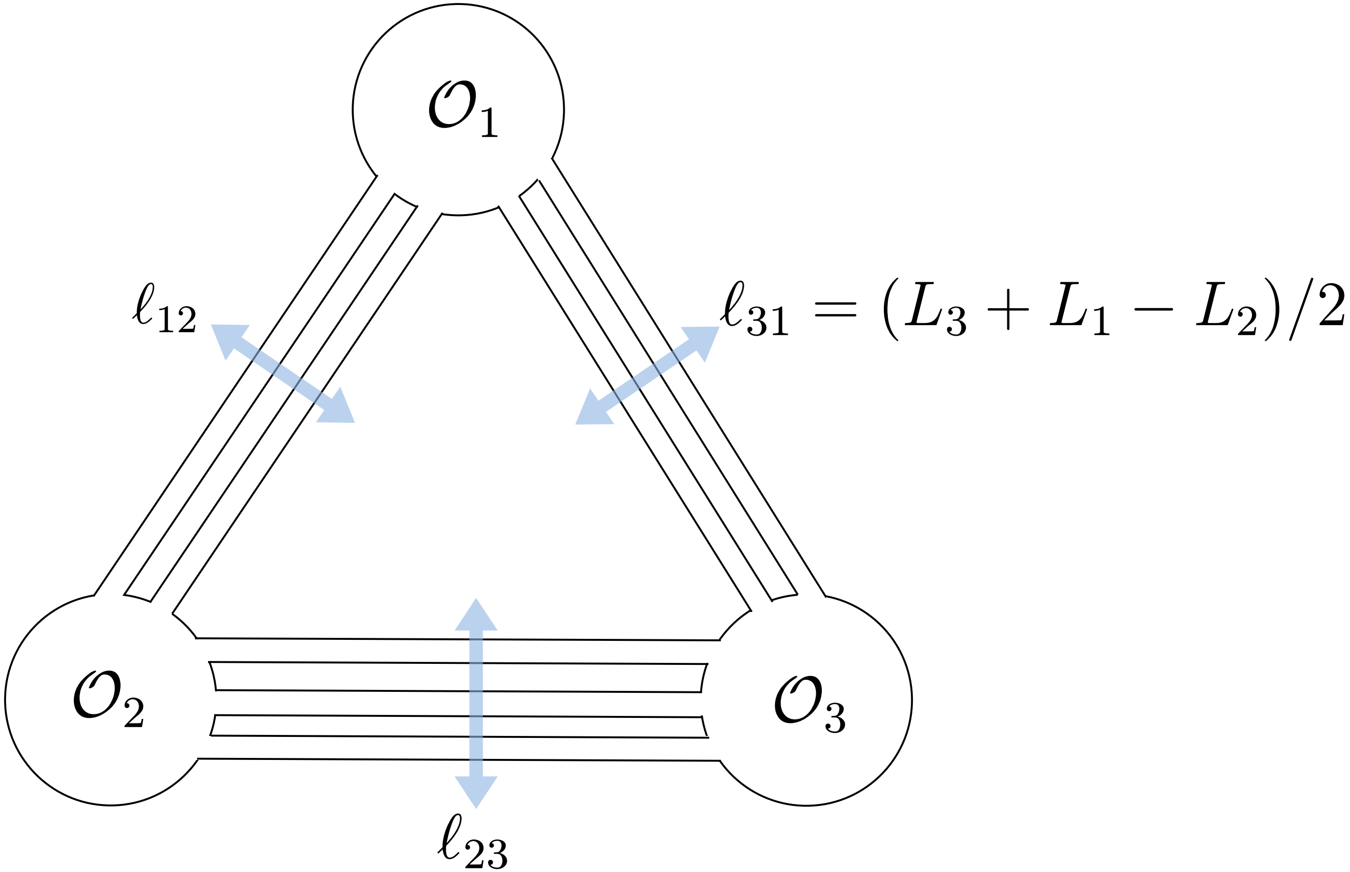
Similarity to NS limit of instanton partition functions
Relation to Fermi gas formalism for ABJM theory

Three-point Function of Single-Trace Ops.

$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle = \frac{C_{123}}{|x_1 - x_2|^{\Delta_1+\Delta_2-\Delta_3} |x_2 - x_3|^{\Delta_2+\Delta_3-\Delta_1} |x_3 - x_1|^{\Delta_3+\Delta_1-\Delta_2}}$$

Three-point Function of Single-Trace Ops.

At weak coupling (tree level):



Three-point Function of Single-Trace Ops.

At weak coupling (tree level):

$$\mathcal{O}_1 : \mathrm{Tr}\left(Z D_+ Z Z D_+ \cdots\right) \qquad \mathcal{O}_2 : \mathrm{Tr}\left(\bar{Z}^{L_2}\right) \qquad \mathcal{O}_3 : \mathrm{Tr}\left(\tilde{Z}^{L_3}\right)$$

Magnons

$$(Z = \phi_1 + i\phi_2, \tilde{Z} = \phi_1 + i\phi_4)$$

[Vieira, Wang]

$$C_{123} \sim \sum_{\alpha \cup \bar{\alpha} = \{u_1, u_2, \ldots\}} (\mathrm{Weight}) \times \mathcal{H}[\alpha] \times \mathcal{H}[\bar{\alpha}]$$

Rapidities/momenta of magnons

$$\mathcal{H}[\alpha] = \prod_{\substack{i<j \\ u_i, u_j \in \alpha}} h(u_i, u_j) \qquad h(u,v) = \frac{u-v}{u-v-i}$$

Three-point Function of Single-Trace Ops.

At finite coupling (Integrable Bootstrap):

[Basso, SK, Vieira]

(if $\ell_{ij} >> 1$)

$$C_{123} \sim \sum_{\alpha\cup\bar{\alpha}=\{u_1,u_2,\dots\}} (\text{Weight}) \times \mathcal{H}[\alpha] \times \mathcal{H}[\bar{\alpha}]$$

Asymptotic Part

$$g = \frac{\sqrt{\lambda}}{4\pi}$$

$$\mathcal{H}[\alpha] = \prod_{u_i\in\alpha} \sqrt{\mu(u_i)} \prod_{\substack{i<j \\ u_i,u_j\in\alpha}} h(u_i,u_j)$$

$$\mu(u) = \frac{\left(1-\frac{1}{x^+x^-}\right)^2}{\left(1-\frac{1}{(x^+)^2}\right)\left(1-\frac{1}{(x^-)^2}\right)}$$

$$h(u,v) = \frac{u-v}{u-v-i}\frac{\left(1-\frac{1}{x^+(u)x^-(v)}\right)^2}{\left(1-\frac{1}{x^+(u)x^+(v)}\right)\left(1-\frac{1}{x^-(u)x^-(v)}\right)}\frac{1}{\sigma(u,v)}$$

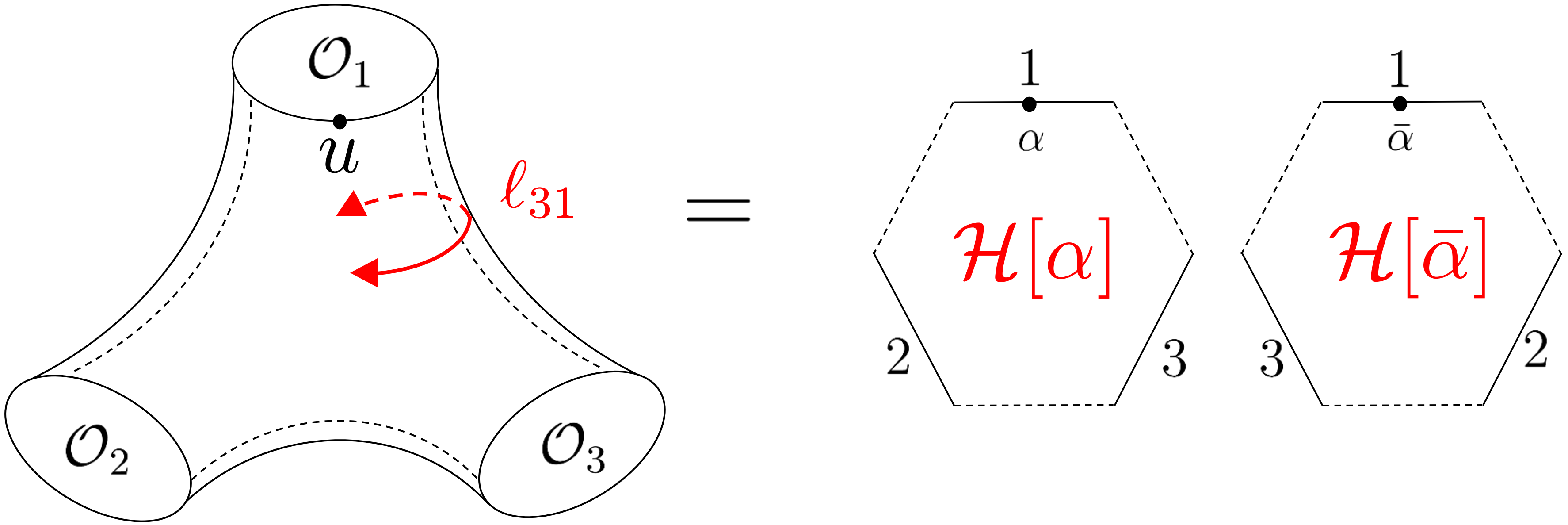
$$f^{\pm}(u) = f(u \pm \frac{i}{2})$$

$$x(u) = \frac{u+\sqrt{u^2-4g^2}}{2g}$$

Three-point Function of Single-Trace Ops.

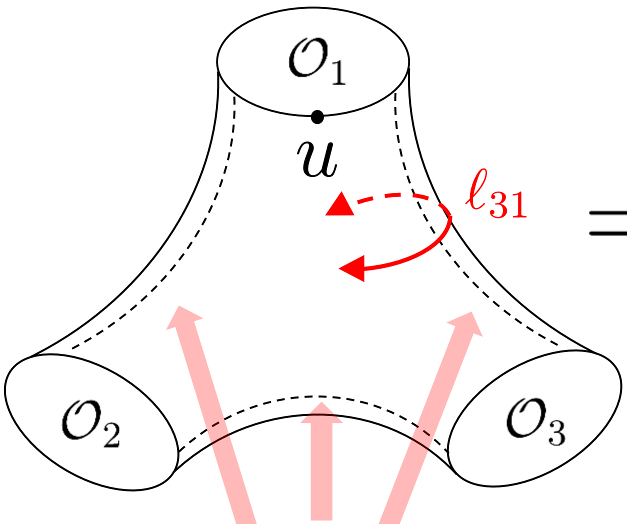
$$C_{123} \sim \sum_{\alpha \cup \bar{\alpha} = \{u_1, u_2, \ldots\}} (\text{Weight}) \times \mathcal{H}[\alpha] \times \mathcal{H}[\bar{\alpha}]$$

A pair of pants = Hexagon^2



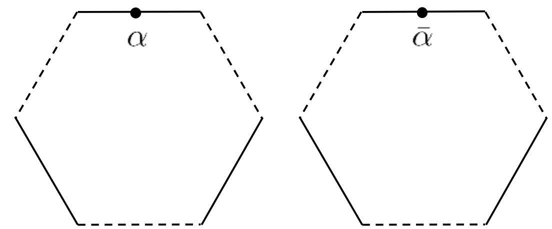
Three-point Function of Single-Trace Ops.

Finite size correction

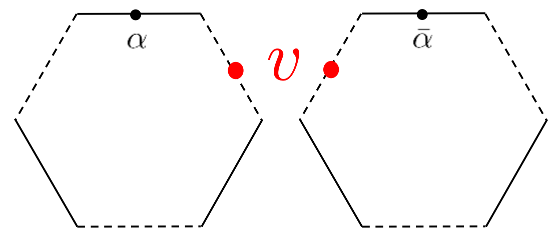


Insert a complete set of states

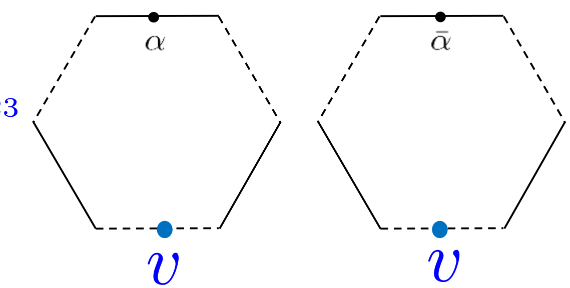
$=$



$$+\int dv\,\mu(v)e^{-E(v)\ell_{31}}$$



$$+\int dv\,\mu(v)e^{-E(v)\ell_{23}}$$

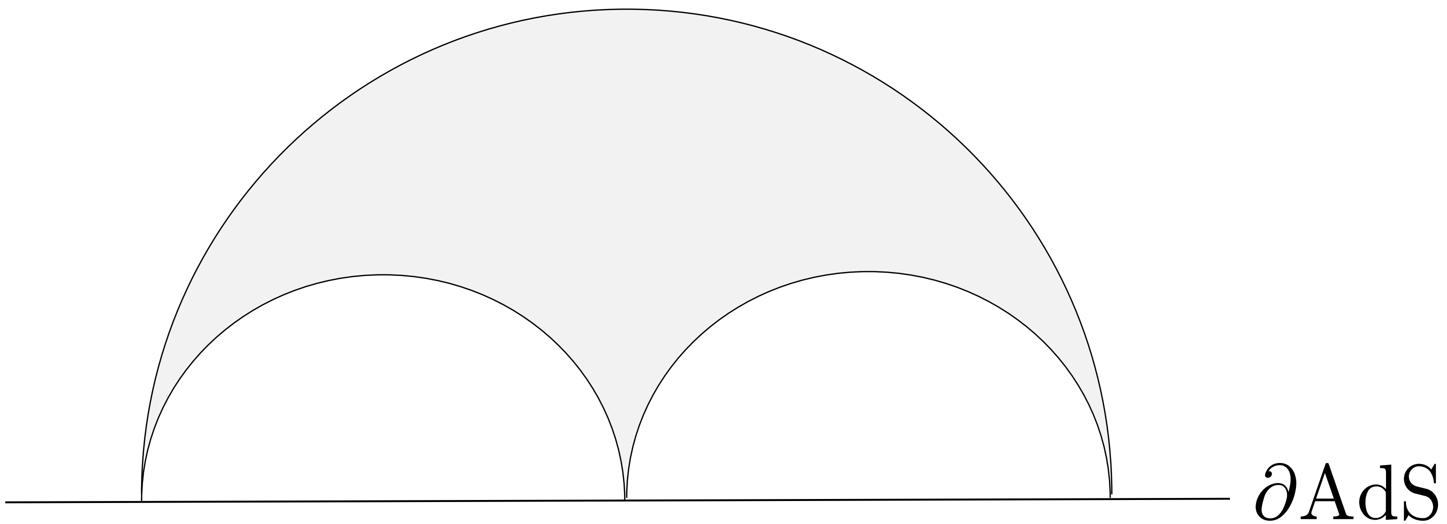


$$+\cdots$$

Checked up to 3 loops. [Eden, Sfondrini] [Basso, Goncalves, SK, Vieira]

Three-point Function of Single-Trace Ops.

At strong coupling: [Janik, Wereszczynski], [Kazama, SK], [Kazama, SK, Nishimura]



$$\log C_{123} \sim \text{Area}$$

Using integrability, one can compute the area without knowing the surface cf. [Alday, Maldacena]

Three-point Function of Single-Trace Ops.

At strong coupling:

$$\mathcal{O}_1 : \ \text{characterized by } p_1(u)$$

quasi-momentum
(= generating function of charges)

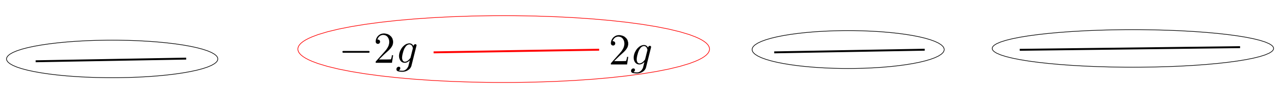
$-2g$ $2g$

Three-point Function of Single-Trace Ops.

At strong coupling: [Kazama, SK, Nishimura]

$$
\begin{aligned}
-\log C_{123} \sim & \oint_{\mathcal{C}} \frac{du}{2\pi} L[\hat{p}_1 + p_2 - p_3] \\
& + \oint_{\mathcal{U}} \frac{du}{2\pi} \left(L[\hat{p}_1 + p_2 + p_3] - L[p_1 + p_2 + p_3] \right) \\
& + \oint_{\mathcal{U}} \frac{du}{2\pi} \left(L[\hat{p}_1 + p_2 - p_3] - L[p_1 + p_2 - p_3] \right) \\
& + \oint_{\mathcal{U}} \frac{du}{2\pi} \left(L[\hat{p}_1 - p_2 + p_3] - L[p_1 - p_2 + p_3] \right) \\
& + \oint_{\mathcal{U}} \frac{du}{2\pi} \left(L[-\hat{p}_1 + p_2 + p_3] - L[-p_1 + p_2 + p_3] \right)
\end{aligned}
$$

$$L[f] := \mathrm{Li}_2 \left(e^{if} \right) \qquad p_i = \frac{L_i x(u)}{2g((x(u))^2 - 1)}$$



Can we reproduce the area
from Hexagons at strong coupling?

Step 1: From sum over partitions to integrals

$$C_{123} \sim \sum_{\alpha \cup \bar{\alpha} = \{u_1, u_2, \ldots\}} (\text{Weight}) \times \mathcal{H}[\alpha] \times \mathcal{H}[\bar{\alpha}]$$

Step 1: From sum over partitions to integrals

$$C_{123} \sim \sum_{\alpha \cup \bar{\alpha} = \{u_1, u_2, \dots\}} (\text{Weight}) \times \mathcal{H}[\alpha] \times \mathcal{H}[\bar{\alpha}]$$

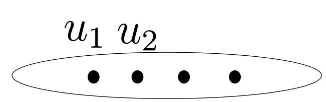
$$C_{123} \sim \sum_{\underline{n}=0}^{M} \frac{1}{n!} \oint_{\mathbf{u}} \prod_{j=1}^{n} \frac{dz_j}{2\pi} F(z_j) \prod_{i<j} \Delta(z_i, z_j)$$

$= |\bar{\alpha}|$

$$\Delta(u, v) = h(u, v) h(v, u) \qquad (\Delta(u, u) = 0)$$

$$F(z) = \frac{e^{-ip\ell_{31}} \mu(z)}{\prod_{j=1}^{M} h(z, u_j)}$$

Poles at u = u_j



Step 1: From sum over partitions to integrals

$$C_{123} \sim \sum_{\alpha \cup \bar{\alpha} = \{u_1, u_2, \ldots\}} (\text{Weight}) \times \mathcal{H}[\alpha] \times \mathcal{H}[\bar{\alpha}]$$

$$C_{123} \sim \sum_{n=0}^{\infty} \frac{1}{n!} \oint_{\mathbf{u}} \prod_{j=1}^{n} \frac{dz_j}{2\pi} F(z_j) \prod_{i<j} \Delta(z_i, z_j)$$

$$\Delta(u,v) \sim \frac{(u-v)^2}{(u-v)^2+1} \qquad (g \to \infty)$$

Step 1: From sum over partitions to integrals

$$C_{123} \sim \sum_{\alpha\cup\bar{\alpha}=\{u_1,u_2,\ldots\}} (\text{Weight}) \times \mathcal{H}[\alpha] \times \mathcal{H}[\bar{\alpha}]$$

$$C_{123} \sim \sum_{n=0}^{\infty} \frac{1}{n!} \oint_{\mathbf{u}} \prod_{j=1}^{n} \frac{dz_j}{2\pi} F(z_j) \, \det\left(\frac{i}{z_i - z_j + i}\right)$$

$$\Delta(u,v) \sim \frac{(u-v)^2}{(u-v)^2+1} \qquad (g \to \infty)$$

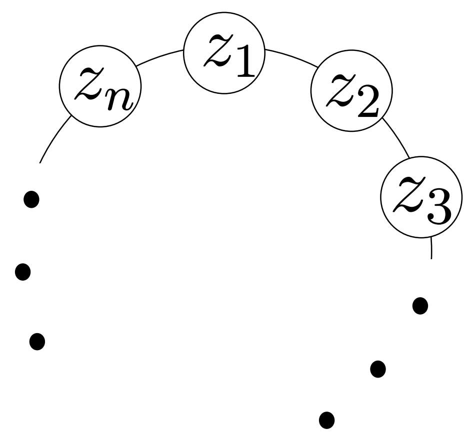
Same structure as the expansion of the Fredholm determinant

$$\det\left(\hat{I} + \hat{K}\right) \quad \left(= \exp\left[\operatorname{tr}\log\left(\hat{I}+\hat{K}\right)\right]\right)$$

Step 1: From sum over partitions to integrals

$$\log C_{123} = \sum_{n=1}^{\infty} \frac{I_n}{n}$$

$$I_n = \oint_{\mathbf{u}} \frac{dz_1}{2\pi i} \cdots \frac{dz_n}{2\pi i} \frac{F(z_1)}{z_1 - z_2 + i} \frac{F(z_2)}{z_2 - z_3 + i} \cdots \frac{F(z_n)}{z_n - z_1 + i}$$



Ring-like structure

Step 1: From sum over partitions to integrals

$$\log C_{123} = \sum_{n=1}^{\infty} \frac{I_n}{n}$$

$$I_n = \oint_{\mathbf{u}} \frac{dz_1}{2\pi i} \cdots \frac{dz_n}{2\pi i} \frac{F(z_1)}{z_1 - z_2 - i} \frac{F(z_2)}{z_2 - z_3 - i} \cdots \frac{F(z_n)}{z_n - z_1 - i}$$

Macroscopic string:

$$u_j, z_i \sim O(g) \qquad F(z_j) \sim O(1)$$

Naively…

$$I_1 = \int \frac{dz}{2\pi} F(z) \sim O(g)$$

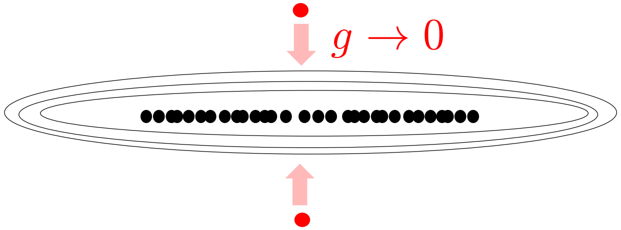
$$I_{n\geq 2} \sim O(1)$$

Step 2: Clustering of magnons

$z \to g z'$

$$I_n = \oint_{\mathbf{u}} \frac{dz'_1}{2\pi i} \cdots \frac{dz'_n}{2\pi i} \frac{F(gz'_1)}{z_1 - z_2 - i/g} \frac{F(gz'_2)}{z_2 - z_3 - i/g} \cdots \frac{F(gz'_n)}{z_n - z_1 - i/g}$$

[Nekrasov, Shatashvili]
[Meneghelli, Yang]



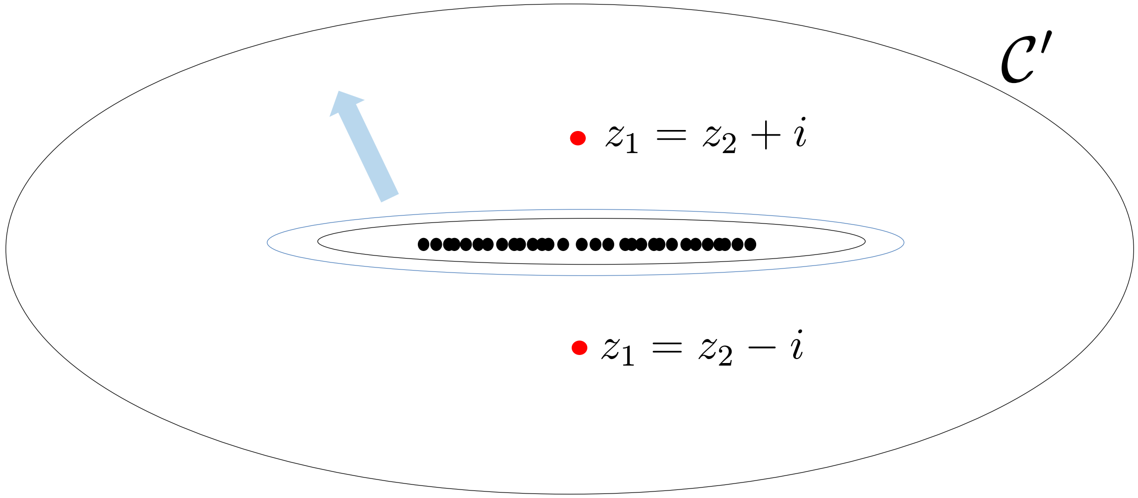
The contours get pinched by the poles and yield singular contributions.

Resolution:

Deform the contours s.t. they are well-separated from each other.
Take the limit afterwards.

Step 2: Clustering of magnons

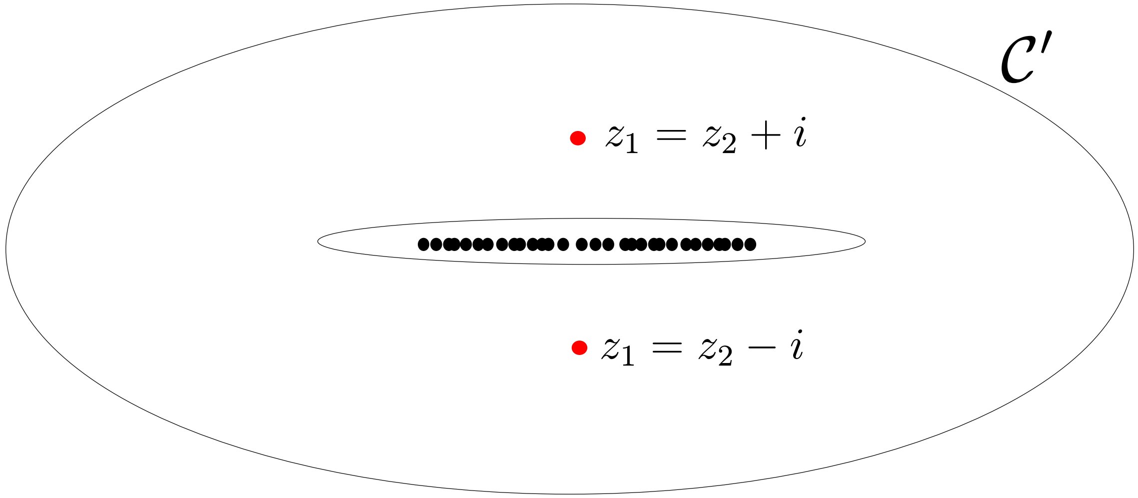
$$I_2 = \oint_{\mathbf{u}} \frac{dz_1}{2\pi i}\frac{dz_2}{2\pi i}\frac{F(z_1)}{z_1 - z_2 - i}\frac{F(z_2)}{z_2 - z_1 - i}$$



Step 2: Clustering of magnons

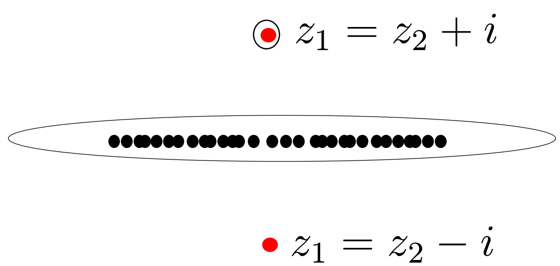
Double integral term: $$I_2 = \oint_{\mathcal{C}'} \frac{dz_1}{2\pi i} \oint_{\mathbf{u}} \frac{dz_2}{2\pi i} \frac{F(z_1)}{z_1 - z_2 - i} \frac{F(z_2)}{z_2 - z_1 - i}$$

$$\sim O(1)$$



Step 2: Clustering of magnons

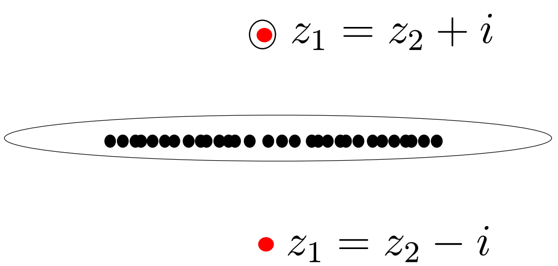
Pole term: $$I_2 = \oint_{\mathbf{u}} \frac{dz_1}{2\pi i} \frac{dz_2}{2\pi i} \frac{F(z_1)}{z_1 - z_2 - i} \frac{F(z_2)}{z_2 - z_1 - i}$$



Step 2: Clustering of magnons

Pole term: $$I_2 = \frac{1}{2}\oint_{\mathbf{u}} \frac{dz_2}{2\pi} F(z_2 + i) F(z_2)$$

$$\sim \frac{1}{2}\oint_{\mathbf{u}} \frac{dz_2}{2\pi} \left(F(z_2)\right)^2 \sim O(g)$$



Step 2: Clustering of magnons

More generally, pole terms yield

$$I_n \sim \frac{1}{n} \oint_{\mathbf{u}} \frac{dz}{2\pi} \left(F(z)\right)^n$$

This yields

$$\log C_{123} = \sum_{n=1}^{\infty} \frac{I_n}{n} \sim \sum_{n=1}^{\infty} \frac{1}{n^2} \oint_{\mathbf{u}} \frac{dz}{2\pi} \left(F(z)\right)^n$$

$$\sim \oint_{\mathbf{u}} \frac{dz}{2\pi} \mathrm{Li}_2\left(F(z)\right)$$

Step 2: Clustering of magnons

$$\log C_{123} = \sum_{n=1}^{\infty} \frac{I_n}{n} \sim \sum_{n=1}^{\infty} \frac{1}{n^2} \oint_{\mathbf{u}} \frac{dz}{2\pi} \left(F(z)\right)^n$$
$$\sim \oint_{\mathbf{u}} \frac{dz}{2\pi} \mathrm{Li}_2 \left(F(z)\right)$$

This reproduces one of the terms in the area:

$$\begin{aligned}
\ln C_{123} \sim & \oint_{\mathcal{C}} \frac{du}{2\pi} L[\hat{p}_1 + p_2 - p_3] \\
& + \oint_{\mathcal{U}} \frac{du}{2\pi} \left(L[\hat{p}_1 + p_2 + p_3] - L[p_1 + p_2 + p_3]\right) \\
& + \oint_{\mathcal{U}} \frac{du}{2\pi} \left(L[\hat{p}_1 + p_2 - p_3] - L[p_1 + p_2 - p_3]\right) \\
& + \oint_{\mathcal{U}} \frac{du}{2\pi} \left(L[\hat{p}_1 - p_2 + p_3] - L[p_1 - p_2 + p_3]\right) \\
& + \oint_{\mathcal{U}} \frac{du}{2\pi} \left(L[-\hat{p}_1 + p_2 + p_3] - L[-p_1 + p_2 + p_3]\right)
\end{aligned}$$

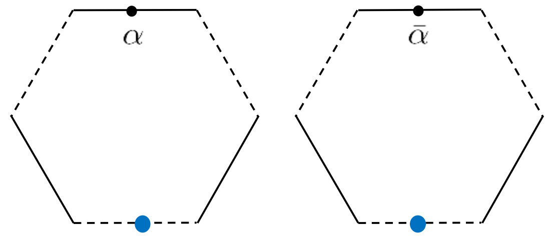
Remaining terms should come from the finite size corrections.

Finite size corrections

The contributions from the bottom channel factorizes:

[Basso, Goncalves, SK, Vieira]

$$\int dv \mu(v) e^{-E(v)\ell_{23}}$$



$$+ \cdots$$

$$\Rightarrow (\text{Bottom}) \times (\text{Asymptotic})$$

Only differences are

$$(\text{Bottom}) = 1 + \sum_{a=1}^{\infty} \int_{-\infty}^{\infty} \frac{du}{2\pi} \tilde{\mu}_a(u) e^{-\tilde{E}_a \ell_{23}} T_a(u) + \cdots$$

flavor sum

This can be regarded as the expansion of

$$\det\left(I + \sum_{a=1}^{\infty} \hat{K}_a\right)$$

cf. [Codesido, Grassi, Marino]

Finite size corrections

After clustering, it reproduces

$$\ln C_{123} \sim \oint_{\mathcal{C}} \frac{du}{2\pi} L[\hat{p}_1 + p_2 - p_3]$$
$$+ \oint_{\mathcal{U}} \frac{du}{2\pi} \left(L[\hat{p}_1 + p_2 + p_3] - L[p_1 + p_2 + p_3]\right)$$
$$+ \oint_{\mathcal{U}} \frac{du}{2\pi} \left(L[\hat{p}_1 + p_2 - p_3] - L[p_1 + p_2 - p_3]\right)$$
$$+ \oint_{\mathcal{U}} \frac{du}{2\pi} \left(L[\hat{p}_1 - p_2 + p_3] - L[p_1 - p_2 + p_3]\right)$$
$$+ \oint_{\mathcal{U}} \frac{du}{2\pi} \left(L[-\hat{p}_1 + p_2 + p_3] - L[-p_1 + p_2 + p_3]\right)$$

Conclusion

- Reproduced parts of the strong-coupling answer from hexagons.

- “Clustering” of magnons is important.

- Another test of hexagon conjecture.

Future directions

- Reproduce the other terms.

[in progress]

No factorization, poles in the integrand…

$$\ln C_{123} \sim \oint_{\mathcal{C}} \frac{du}{2\pi} L[\hat{p}_1 + p_2 - p_3]$$

Interaction of 3 channels

$$+ \oint_{\mathcal{U}} \frac{du}{2\pi} \left(L[\hat{p}_1 + p_2 + p_3] - L[p_1 + p_2 + p_3]\right)$$

Left channel

$$+ \oint_{\mathcal{U}} \frac{du}{2\pi} \left(L[\hat{p}_1 + p_2 - p_3] - L[p_1 + p_2 - p_3]\right)$$

Right channel

$$+ \oint_{\mathcal{U}} \frac{du}{2\pi} \left(L[\hat{p}_1 - p_2 + p_3] - L[p_1 - p_2 + p_3]\right)$$

$$+ \oint_{\mathcal{U}} \frac{du}{2\pi} \left(L[-\hat{p}_1 + p_2 + p_3] - L[-p_1 + p_2 + p_3]\right)$$

Future directions

- Reproduce the other terms. [in progress]

No factorization, poles in the integrand…

- Physical meaning of clustering?
- Resummation at finite coupling? Relation to TBA etc.
- Four-point function by OPE / other methods?

[Basso, Coronado, SK, Lam, Vieira, Zhong in progress]

[SK, Fleury in progress]

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谢谢

Back up slides

Three-point Function of Single-Trace Ops.

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Weight Factor :