Holography for Heavy Operators

Robert de Mello Koch

Mandlestam Institute for Theoretical Physics
University of the Witwatersrand

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The talk is based on work (arXiv:1608.00399) with David Gossman (MSc), Lwazi Nkumane (PhD) and Laila Tribelhorn (PhD).

Large N limits of matrix models are relevant for strong coupling limits of non-Abelian gauge theories and for AdS/CFT.

Difficult problem, but has been solved for the singlet sector of single matrix models.

Problem reduces to eigenvalue dynamics: HUGE reduction in degrees of freedom; non-interacting fermion dynamics; has found clear and convincing application in the 1/2 BPS LLM sector.
(Brezin, Izykson, Parisi, Zuber; Ginibre; Corley, Jevicki, Ramgoolam; Berenestein; Lin, Lunin, Maldacena;...)

In this talk we ask if there is a similar eigenvalue description for a two matrix model and if this has some AdS/CFT interpretation?

Naive answer: NO! We need to integrate over two random matrices and we can't simultaneously diagonalize them!

Perhaps there is a class of questions (generalizing the "singlet sector" of one matrix model) that can be approached using eigenvalue dynamics?

Single Complex Matrix Model:

Use the Schur decomposition $Z = U^\dagger D U$ with U a unitary matrix and D an upper triangular matrix, to explicitly change variables. A standard argument then maps eigenvalue dynamics to non-interacting fermion dynamics.
(Brezin, Izykson, Parisi, Zuber; Ginibre)

We could also construct a basis of operators that diagonalize the (free) inner product $\Rightarrow$ Schur polynomials $\Rightarrow$ non-interacting fermions

$$\langle \chi_R(Z)\chi_S(Z^\dagger)\rangle = f_R \delta_{RS}$$

$$\psi_{\rm FF} \propto \chi_R(Z)\Delta(Z)e^{-\frac{1}{2}\sum_i z_i\bar{z}_i}$$

(Corley, Jevicki, Ramgoolam; Berenestein)

Single Complex Matrix Model

Each row of the Schur polynomial can be identified with fermion $\leftrightarrow$ an eigenvalue.

The number of boxes in each row tells us how much each fermion was excited.

If we were to study a two matrix model, with many Zs and few Ys the rough outline of the above picture should still be visible.

Two Matrix Model: Restricted Schurs

For two matrices (Z, Y) we can diagonalize the free field inner product with restricted Schur polynomials

$$\langle \chi_{R,(r,s)ab}(Z, Y)\chi_{T,(t,u)cd}(Z^\dagger, Y^\dagger)\rangle$$

$$= f_R \frac{\text{hooks}_R}{\text{hooks}_r \text{hooks}_s} \delta_{RT}\delta_{rt}\delta_{su}\delta_{ac}\delta_{bd}$$

(Bhattacharrya,Collins,dMK;Collins,dMK,Stephanou)

Operator built using n Zs and m Ys. We have $r \vdash n$, $s \vdash m$, $R \vdash n+m$. The restricted Schur polynomials don't diagonalize the dilatation operator.

Two Matrix Model: Gauss Graph Operators

Using the restricted Schur polynomials as a basis, we can diagonalize the dilatation operator in a limit in which R has order 1 long rows and $m \ll n$.

Operators are labeled by graphs - one node for each row. Directed edges connect nodes - one Y for each edge.

(da Commarmond, dMK, Jefferies; dMK, Dessein, Giataganas, Mathwin; dMK, Ramgoolam)

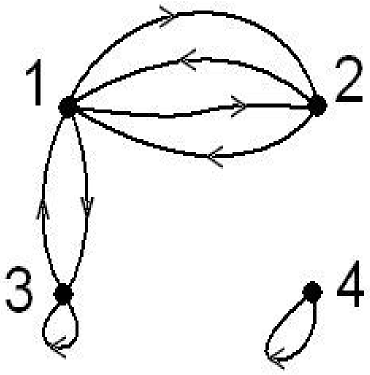

Figure: Each node is a fermion = a giant graviton and the edges stretching between nodes are open strings (each edge is a Y) attached to the giant graviton branes

Non-interacting fermion description ⇒ consider only operators labeled by Gauss graphs with no edges stretching between nodes.

These are the BPS operators.

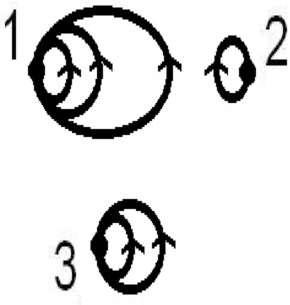

Figure: Non-interacting fermions are BPS operators

Non-interacting fermion description $\Rightarrow$ only Gauss graphs for BPS operators .

$\Rightarrow$ if there is a non-interacting fermion description at all, it will only describe BPS operators.

These are associated to sugra solutions of string theory: only one-particle states saturating the BPS bound in gravity are associated to massless particles and lie in the supergravity multiplet.

Eigenvalue dynamics as a dual to supergravity has been advocated (using different reasoning) also by Berenstein; (see: 0507203, 0605220, 0801.2739, 0805.4658, 1001.4509, 1404.7052). These works use a combination of numerical methods and physical arguments.

We obtain exact analytic statements about eigenvalue dynamics, from multi-matrix quantum mechanics. The results agree with Berenstein.

From direct change of variables:

$$\begin{aligned}
\langle \cdots \rangle &= \int [dZdZ^\dagger] e^{-\mathrm{Tr} ZZ^\dagger} \cdots \\
&= \int \prod_{i=1}^{N} dz_i d\bar{z}_i e^{-\sum_k z_z \bar{z}_k} \Delta(z) \Delta(\bar{z}) \cdots \\
&= \int \prod_{i=1}^{N} dz_i d\bar{z}_i |\psi_{\mathrm{gs}}(z, \bar{z})|^2 \cdots
\end{aligned}$$

$$\psi_{\mathrm{gs}}(z,\bar{z}) = \Delta(z) e^{-\frac{1}{2}\sum_k z_z \bar{z}_k}$$

$$\Delta(z) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_N \\ \vdots & \vdots & \vdots\ \vdots\ \vdots & \vdots \\ z_1^{N-1} & z_2^{N-1} & \cdots & z_N^{N-1} \end{vmatrix}$$

We replaced $dZ \to \prod_{i=1}^N dz_i \ \Delta(z)$. Four important observations:

- identical particles ($Z \to UZU^\dagger$)
- correct dimension (if $[Z] = L$, need $[\psi_{\mathrm{gs}}] = L^{N(N-1)}$)
- correct scaling under $Z \to e^{i\theta} Z$
- evenly spaced dimensions $\Rightarrow$ oscillator

Wave function for two matrices must obey:

- N identical particles ($Z \to UZU^\dagger$ and $Y \to UYU^\dagger$)
- correct dimension (if $[Z] = L = [Y]$, need $[\Psi_{\rm gs}] = L^{2N(N-1)}$)
- correct scaling under $Z \to e^{i\theta} Z$, $Y \to Y$ and $Z \to Z$, $Y \to e^{i\theta} Y$
- evenly spaced dimensions $\Rightarrow$ oscillator
- wave function must have $SO(4)$ symmetry which preserves $|z_i|^2 + |y_i|^2$

Guess for the wave function satisfying these properties:

$$\Psi(z, y) = \Delta(z, y) e^{-\frac{1}{2}\sum_k z_k \bar{z}_k - \frac{1}{2}\sum_k y_k \bar{y}_k}$$

$$\Delta(z, y) = \begin{vmatrix} y_1^{N-1} & y_2^{N-1} & \cdots & y_N^{N-1} \\ z_1 y_1^{N-2} & z_2 y_2^{N-2} & \cdots & z_N y_N^{N-2} \\ \vdots & \vdots & \vdots\ \vdots\ \vdots & \vdots \\ z_1^{N-2} y_1 & z_2^{N-1} y_2 & \cdots & z_N^{N-1} y_N \\ z_1^{N-1} & z_2^{N-1} & \cdots & z_N^{N-1} \end{vmatrix}$$

Matrix model computation:

$$\langle \mathrm{Tr}(Z^J)\mathrm{Tr}(Z^{\dagger J})\rangle = \frac{1}{J+1}\left[\frac{(J+N)!}{(N-1)!} - \frac{N!}{(N-J-1)!}\right]$$

if $J < N$ and

$$\langle \mathrm{Tr}(Z^J)\mathrm{Tr}(Z^{\dagger J})\rangle = \frac{1}{J+1}\frac{(J+N)!}{(N-1)!} \qquad \text{if} \qquad J \geq N$$

Eigenvalue computation:

$$\int \prod_{i=1}^{N} dz_i d\bar{z}_i dy_i d\bar{y}_i |\Psi(z_i, y_i)|^2 \sum_i z_i^J \sum_j \bar{z}_j^J = \frac{1}{J+1}\frac{(J+N)!}{(N-1)!}$$

$\Rightarrow$ we correctly reproduce (to all orders in $1/N$) all single traces of dimension N or greater.

We know that we will reproduce both $\mathrm{Tr}(Z^J)$, $\mathrm{Tr}(Y^J)$ for $J \geq N$.

There are trace relations implying

$$\mathrm{Tr}(Z^J) = \sum_{i,j,\ldots,k} c_{ij\ldots k} \mathrm{Tr}(Z^i)\mathrm{Tr}(Z^j)\cdots\mathrm{Tr}(Z^k)$$

$i, j, \ldots, k \leq N$ and $i + j + \cdots + k = J$ so that we also start to reproduce sums of products of traces of less than N fields.

We can build BPS operators using both Y and Z fields - symmetrized traces. Are these correlators correctly reproduced?

$$\mathcal{O}_{J,K} = \frac{J!}{(J+K)!}\mathrm{Tr}\left(Y\frac{\partial}{\partial Z}\right)^K \mathrm{Tr}(Z^{J+K}) \leftrightarrow \sum_i z_i^J y_i^K$$

Matrix model computation:

$$\langle \mathcal{O}_{J,K}\mathcal{O}^{\dagger}_{J,K}\rangle = \frac{J!K!}{(J+K+1)!}\left[\frac{(J+K+N)!}{(N-1)!} - \frac{N!}{(N-J-K-1)!}\right]$$

if $J+K<N$ and

$$\langle \mathcal{O}_{J,K}\mathcal{O}^{\dagger}_{J,K}\rangle = \frac{J!K!}{(J+K+1)!}\frac{(J+K+N)!}{(N-1)!} \qquad \text{if} \qquad J+K \geq N$$

Eigenvalue computation:

$$\int \prod_{i=1}^{N} dz_i d\bar{z}_i dy_i d\bar{y}_i |\Psi(z_i, y_i)|^2 \sum_i z_i^J y_i^K \sum_j \bar{z}_j^J \bar{y}_j^K$$

$$= \frac{J!K!}{(J+K+1)!}\frac{(J+K+N)!}{(N-1)!}$$

$\Rightarrow$ we again correctly reproduce (to all orders in $1/N$) all single traces of a dimension N or greater.

Multitrace correlators are also reproduced exactly, as long as they have a dimension $J + K \geq N$

$$\langle O_{J_1,K_1} O_{J_2,K_2} \cdots O_{J_n,K_n} O^{\dagger}_{J,K} \rangle$$

$$= \frac{J!K!}{(J+K+1)!} \frac{(J+K+N)!}{(N-1)!} \delta_{J_1+\cdots+J_n,J} \delta_{K_1+\cdots+K_n,K}$$

$$= \int \prod_{i=1}^{N} dz_i d\bar{z}_i dy_i d\bar{y}_i |\Psi(z_i, y_i)|^2 \sum_{i_1} z_{i_1}^{J_1} y_{i_1}^{K_1} \cdots \sum_{i_n} z_{i_n}^{J_n} y_{i_n}^{K_n}$$

$$\times \sum_{j} \bar{z}_j^{J} \bar{y}_j^{K}$$

Operator with a dimension of order N^2: Young diagram R with N rows and M columns

$$\chi_R(Z) = (\det Z)^M = z_1^M z_2^M \cdots z_N^M$$

$$\chi_R(Z^\dagger) = (\det Z^\dagger)^M = \bar{z}_1^M \bar{z}_2^M \cdots \bar{z}_N^M$$

The two point correlator of this Schur polynomial is

$$\langle \chi_R \chi_R^\dagger \rangle = \int \prod_{i=1}^{N} dz_i d\bar{z}_i dy_i d\bar{y}_i |\Psi|^2 |z_1|^{2M} |z_2|^{2M} \cdots |z_N|^{2M}$$

$$= \prod_{i=1}^{N} \frac{(i-1+M)!}{(i-1)!}$$

which is again the exact answer for this correlator. Not all Schur polynomials are correctly reproduced!

Local supersymmetric geometries with $SO(4) \times U(1)$ isometries have the form

$$ds_{10}^2 = -h^{-2}(dt + \omega)^2 + h^2 \left[\frac{2}{Z + \frac{1}{2}} \partial_a \bar{\partial}_b K dz^a d\bar{z}^b + dy^2 \right]$$

$$+ y(e^G d\Omega_3^2 + e^{-G} d\psi^2)$$

y is the product of warp factors for S^3 and $S^1 \Rightarrow$ must impose boundary conditions at $y = 0$ hypersurface to avoid singularities: when S^3 contracts to zero, we need $Z = -\frac{1}{2}$ and when ψ-circle collapses we need $Z = \frac{1}{2}$.
(Donos; Lunin; Chen, Cremonini, Donos, Lin, Lin, Liu, Vaman, Wen)

There is a surface separating these two regions, and hence, defining the supergravity solution. Is this surface visible in the eigenvalue dynamics?

At large N we expect a definite eigenvalue distribution. These eigenvalues will trace out a surface specified by where the single fermion probability peaks

$$\rho(z_1, \bar{z}_1, y_1, \bar{y}_1) = \int \prod_{i=2}^{N} dz_i d\bar{z}_i dy_i d\bar{y}_i |\Psi(z_1, \cdots, \bar{y}_N)|^2 \quad (1)$$

Denote the points lying on this surface by z^{MM}, y^{MM}.
For the $AdS_5 \times S^5$ solution, the supergravity boundary condition is

$$|z^1|^2 + |z^2|^2 = N - 1 \quad (2)$$

The surfaces (1) and (2) are mapped into each other by identifying $z^1 = z^{MM}$, $z^2 = y^{MM}$.

For an LLM annulus geometry (R has N rows and M columns)

$$\Psi(z,y) = \Delta(z,y)\chi_R(Z)e^{-\frac{1}{2}\sum_k z_z \bar{z}_k - \frac{1}{2}\sum_k y_z \bar{y}_k}$$

determines $\rho(z_1, \bar{z}_1, y_1, \bar{y}_1)$.

The supergravity boundary condition is

$$|z^2|^2 = \frac{M+N-1-z^1\bar{z}^1}{z^1\bar{z}^1 - M + 1}$$

The map between the two surfaces is $z^2 = \frac{y^{MM}}{\sqrt{|z^{MM}|^2 - M+1}}$, $z^1 = z^{MM}$.

$\Rightarrow$ eigenvalues again condense on the surface that defines the wall between the two boundary conditions, agreeing with a proposal by Berenstein

Conclusions and questions:

- There is a sector of the two matrix model described (exactly) by eigenvalue dynamics. In the dual gravity these states are supergravity states corresponding to classical geometries.
- The supergravity boundary conditions appear to match the large N eigenvalue description.
- How could one derive these results? How could one argue for the relevance/applicability of eigenvalue dynamics?
- Can this be extended to more matrices?
- Can the eigenvalue dynamics be perturbed by off diagonal elements to start including stringy degrees of freedom?

Thanks for your attention!