Three-Dimensional CFTs and Emergent Supersymmetry

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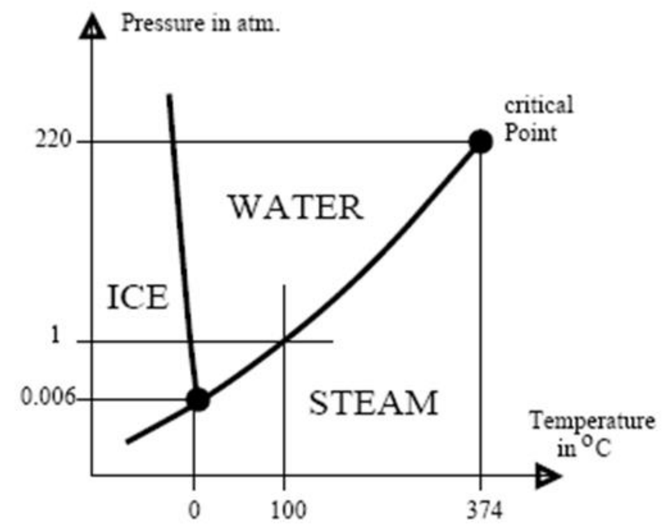
Talk mostly based on

- S. Giombi, IK, arXiv:1409.1937
- L. Fei, S. Giombi, IK, G. Tarnopolsky, arXiv:1502.07271
- L. Fei, S. Giombi, IK, G. Tarnopolsky, arXiv:1507.01960
- L. Fei, S. Giombi, IK, G. Tarnopolsky, arXiv:1607.05316

In Praise of 3-d CFTs

- They describe second-order phase transitions that occur in many 3-d statistical systems.
- A famous example is the critical 3-d Ising model described by the long-distance limit of the 3-d Euclidean ϕ^4 QFT.
- The critical point of the water phase diagram.
- Quantum criticality in two spatial dimensions.



Emergent Global Symmetries

- Renormalization Group flow can lead to IR fixed points with enhanced symmetry.
- The minimal 3-d Yukawa theory for one Majorana fermion and one real pseudo-scalar was conjectured to have "emergent supersymmetry."
  Scott Thomas, unpublished seminar at KITP.
- The fermion mass is forbidden by the time reversal symmetry.
- After tuning the pseudo-scalar mass to zero, the theory is conjectured to flow to a $\mathcal{N}$=1 supersymmetric 3-d CFT.

Superconformal Theory

- The UV lagrangian may be taken as

$$\mathcal{L}_{\mathcal{N}=1} = \frac{1}{2}(\partial_\mu \sigma)^2 + \frac{1}{2}\bar{\psi}\not{\partial}\psi + \frac{\lambda}{2}\sigma\bar{\psi}\psi + \frac{\lambda^2}{8}\sigma^4$$

- Has cubic superpotential $W \sim \lambda\Sigma^3$ in terms of the superfield $\Sigma = \sigma + \bar{\theta}\psi + \frac{1}{2}\bar{\theta}\theta f$
- Some evidence for its existence from the conformal bootstrap (but requires tuning of some operator dimensions). Iliesiu, Kos, Poland, Pufu, Simmons-Duffin, Yacoby; Bashkirov
- Condensed matter realization has been proposed: emergent SUSY may arise at the boundary of a topological superconductor. Grover, Sheng, Vishwanath

The Gross-Neveu Model

$$\mathcal{L}_{\rm GN} = \bar{\psi}_j \partial\!\!\!/ \psi^j + \frac{g}{2}(\bar{\psi}_j \psi^j)^2 \qquad j = 1, \dots N_f$$

- In 2 dimensions it has some similarities with the 4-dimensional QCD.
- It is asymptotically free and exhibits dynamical mass generation.
- Similar physics in the 2-d O(N) non-linear sigma model with N>2.
- In dimensions slightly above 2 both the O(N) and GN models have weakly coupled UV fixed points.

2+ ε expansion

- The beta function and fixed-point coupling are

$$\beta = \epsilon g - (N-2)\frac{g^2}{2\pi} + (N-2)\frac{g^3}{4\pi^2} + (N-2)(N-7)\frac{g^4}{32\pi^3} + \mathcal{O}(g^5)$$

$$g_* = \frac{2\pi}{N-2}\epsilon + \frac{2\pi}{(N-2)^2}\epsilon^2 + \frac{(N+1)\pi}{2(N-2)^3}\epsilon^3 + \mathcal{O}(\epsilon^4)\,,$$

- $N = N_f \mathrm{tr}\mathbf{1} = 4N_f$ is the number of 2-component Majorana fermions.
- Can develop 2+ε expansions for operator scaling dimensions, e.g. Gracey; Kivel, Stepanenko, Vasiliev

$$\Delta_\psi = \frac{1}{2} + \frac{1}{2}\epsilon + \frac{N-1}{4(N-2)^2}\epsilon^2 - \frac{(N-1)(N-6)}{8(N-2)^3}\epsilon^3 + \mathcal{O}(\epsilon^4)\,,$$

$$\Delta_\sigma = 1 - \frac{1}{N-2}\epsilon - \frac{N-1}{2(N-2)^2}\epsilon^2 + \frac{N(N-1)}{4(N-2)^3}\epsilon^3 + \mathcal{O}(\epsilon^4)\,, \qquad \sigma \sim \bar{\psi}\psi$$

- Similar expansions in the O(N) sigma model with N>2.
  Brezin, Zinn-Justin

4-ε expansion

- The O(N) sigma model is in the same universality class as the O(N) model:

$$S = \int d^d x \left(\frac{1}{2} (\partial \phi^i)^2 + \frac{\lambda}{4} (\phi^i \phi^i)^2 \right)$$

- It has a weakly coupled Wilson-Fisher IR fixed point in 4-ε dimensions.
- Using the two ε expansions, the scalar CFTs with various N may be studied in the range 2<d<4. This is an excellent practical tool for CFTs in d=3.

The Gross-Neveu-Yukawa Model

- The GN model is in the same universality class as the GNY model Zinn-Justin; Hasenfratz, Hasenfratz, Jansen, Kuti, Shen

$$\mathcal{L}_{\rm GNY} = \frac{1}{2}(\partial_\mu \sigma)^2 + \bar{\psi}_j \partial\!\!\!/ \psi^j + g_1 \sigma \bar{\psi}_j \psi^j + \frac{1}{24} g_2 \sigma^4$$

- IR stable fixed point in 4-ε dimensions

$$\beta_{g_1} = -\frac{\epsilon}{2} g_1 + \frac{N+6}{2(4\pi)^2} g_1^3 + \frac{1}{(4\pi)^4}\Big(-\frac{3}{4}(4N+3) g_1^5 - 2 g_1^3 g_2 + \frac{g_1 g_2^2}{12}\Big)$$

$$\beta_{g_2} = -\epsilon g_2 + \frac{1}{(4\pi)^2}\Big(3 g_2^2 + 2N g_1^2 g_2 - 12 N g_1^4\Big) + \frac{1}{(4\pi)^4}\Big(96 N g_1^6 + 7 N g_1^4 g_2 - 3 N g_1^2 g_2^2 - \frac{17 g_2^3}{3}\Big)$$

$$\frac{(g_1^*)^2}{(4\pi)^2} = \frac{1}{N+6}\epsilon + \frac{(N+66)\sqrt{N^2+132N+36} - N^2 + 516N + 882}{108(N+6)^3}\epsilon^2$$

$$\frac{g_2^*}{(4\pi)^2} = \frac{-N+6+\sqrt{N^2+132N+36}}{6(N+6)}\epsilon$$

- Operator scaling dimensions

$$\Delta_\sigma = 1 - \frac{3}{N+6}\epsilon + \frac{52N^2 - 57N + 36 + (11N+6)\sqrt{N^2+132N+36}}{36(N+6)^3}\epsilon^2$$

$$\Delta_\psi = \frac{3}{2} - \frac{N+5}{2(N+6)}\epsilon + \frac{-82N^2 + 3N + 720 + (N+66)\sqrt{N^2+132N+36}}{216(N+6)^3}\epsilon^2$$

$$\Delta_{\sigma^2} = d - 2 + \gamma_{\sigma^2} = 2 + \frac{\sqrt{N^2+132N+36} - N - 30}{6(N+6)}\epsilon$$

- Using the two ε expansions, we can study the Gross-Neveu CFTs in the range 2<d<4.

Sphere Free Energy in Continuous d

- A natural quantity to consider is Giombi, IK

$$\tilde{F} = \sin(\pi d/2) \log Z_{S^d} = -\sin(\pi d/2) F$$

- In odd d, this reduces to IK, Pufu, Safdi

$$\tilde{F} = (-1)^{\frac{d+1}{2}} F = (-1)^{\frac{d-1}{2}} \log Z_{S^d}$$

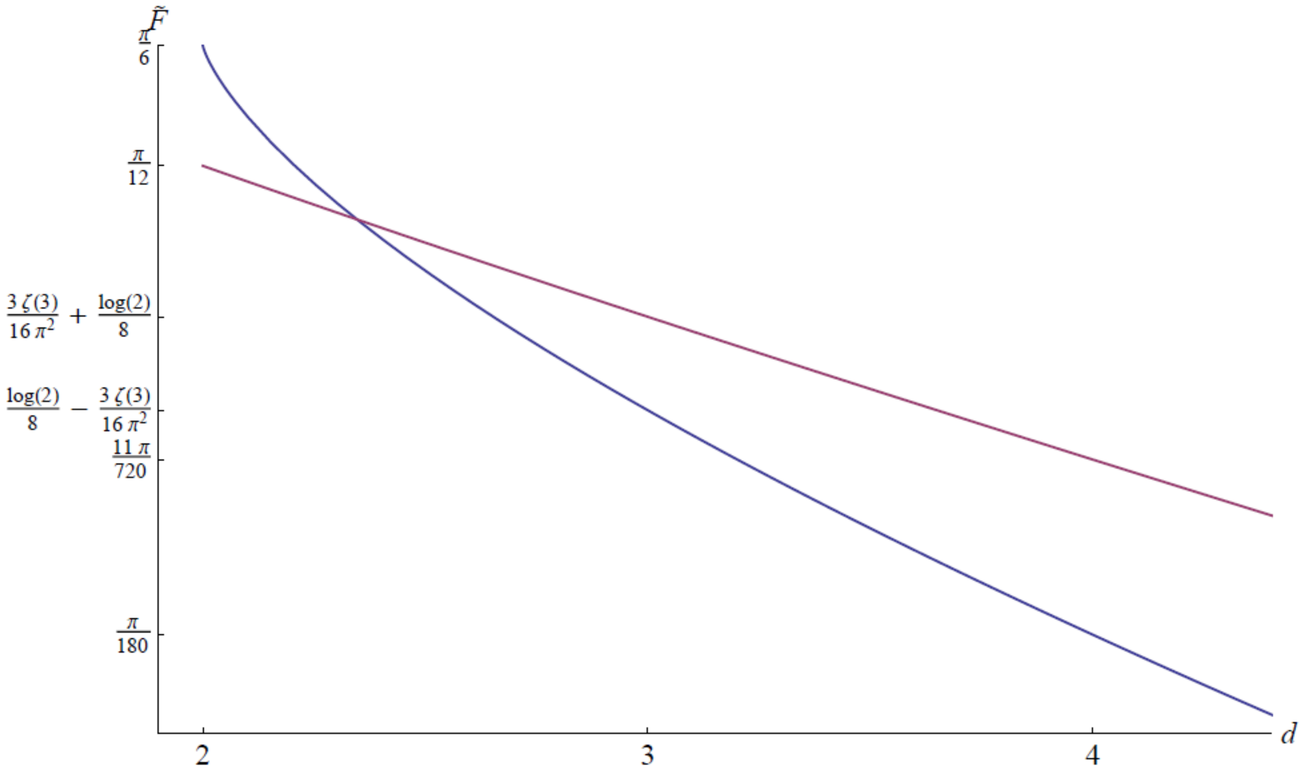
- In even d, -log Z has a pole in dimensional regularization whose coefficient is the Weyl *a*-anomaly. The multiplication by $\sin(\pi d/2)$ removes it.
- $\tilde{F}$ smoothly interpolates between *a*-anomaly coefficients in even and ``F-values" in odd d.
- Gives the universal entanglement entropy across d-2 dimensional sphere. Casini, Huerta, Myers

Free Conformal Scalar and Fermion

$$\tilde{F}_s = \frac{1}{\Gamma(1+d)} \int_0^1 du\, u \sin \pi u\, \Gamma\left(\frac{d}{2}+u\right) \Gamma\left(\frac{d}{2}-u\right) ,$$

$$\tilde{F}_f = \frac{1}{\Gamma(1+d)} \int_0^1 du\, \cos\left(\frac{\pi u}{2}\right) \Gamma\left(\frac{1+d+u}{2}\right) \Gamma\left(\frac{1+d-u}{2}\right)$$

- Smooth and positive for all d.



Sphere Free Energy for the O(N) Model

- At the Wilson-Fisher fixed point it is necessary to include the curvature terms in the Lagrangian *Fei, Giombi, IK, Tarnopolsky*

$$\frac{\eta_0}{2}\mathcal{R}\sigma^2 + a_0 W^2 + b_0 E + c_0 \mathcal{R}^2$$

$$E = \mathcal{R}_{\mu\nu\lambda\rho}\mathcal{R}^{\mu\nu\lambda\rho} - 4\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + \mathcal{R}^2$$

- The 4-ε expansion then gives

$$\begin{aligned}\tilde{F}_{\rm IR} =& N\tilde{F}_s(\epsilon) - \frac{\pi N(N+2)\epsilon^3}{576(N+8)^2} - \frac{\pi N(N+2)(13N^2+370N+1588)\epsilon^4}{6912(N+8)^4} \\ &+ \frac{\pi N(N+2)}{414720(N+8)^6}\big(10368(N+8)(5N+22)\zeta(3) - 647N^4 - 32152N^3 \\ &-606576N^2 - 3939520N + 30\pi^2(N+8)^4 - 8451008\big)\,\epsilon^5 + \mathcal{O}(\epsilon^6)\end{aligned}$$

- The 2+ε expansion in the O(N) sigma model is plagued by IR divergences. It has not been developed yet, but we know the value in d=2 and can use it in the Pade extrapolations.

Sphere Free Energy for the GN CFT

- The 4-ε expansion

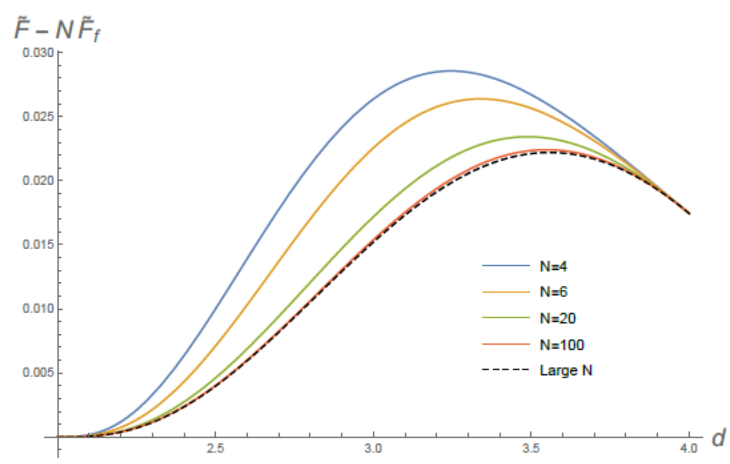
$$\tilde{F} = N\tilde{F}_f + \tilde{F}_s - \frac{N\pi\epsilon^2}{96(N+6)} - \frac{1}{31104(N+6)^3}\Big(161N^3 + 3690N^2 + 11880N + 216 + (N^2+132N+36)\sqrt{N^2+132N+36}\Big)\pi\epsilon^3 + \mathcal{O}(\epsilon^4)$$

- The 2+ε expansion is under good control; no IR divergences:

$$\tilde{F} = N\tilde{F}_f + \frac{N(N-1)\pi\epsilon^3}{48(N-2)^2} - \frac{N(N-1)(N-3)\pi\epsilon^4}{32(N-2)^3} + \mathcal{O}(\epsilon^5)$$

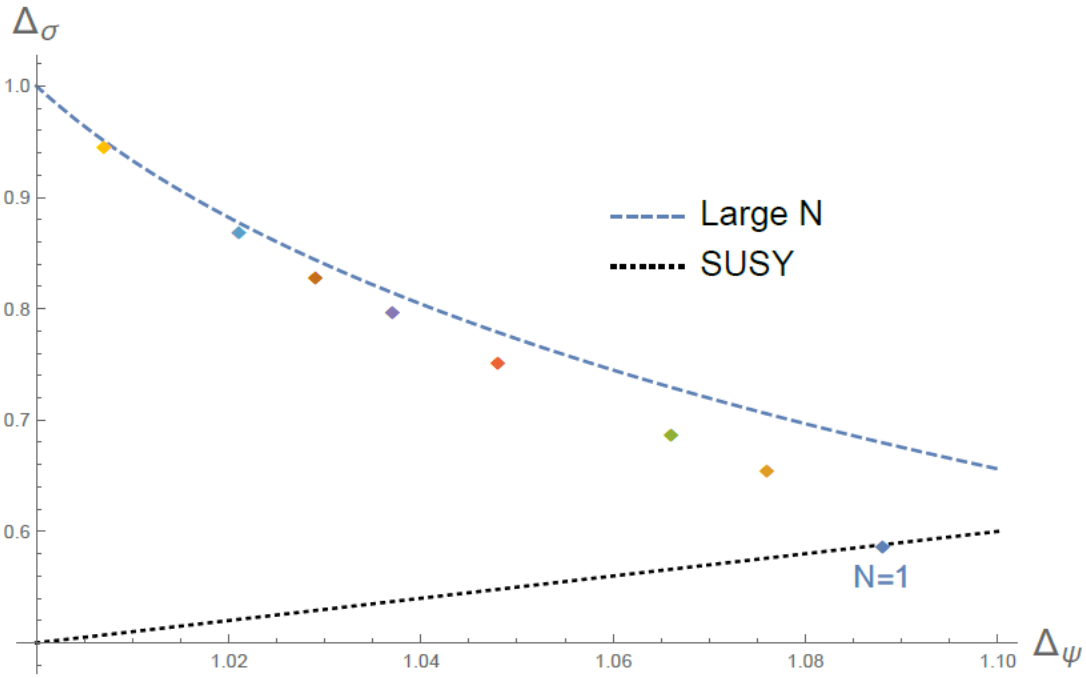
- It is a pleasure to Pade.
- Once again,

$$\tilde{F}_{UV} > \tilde{F}_{IR}$$



Summary for the 3-d GN CFTs

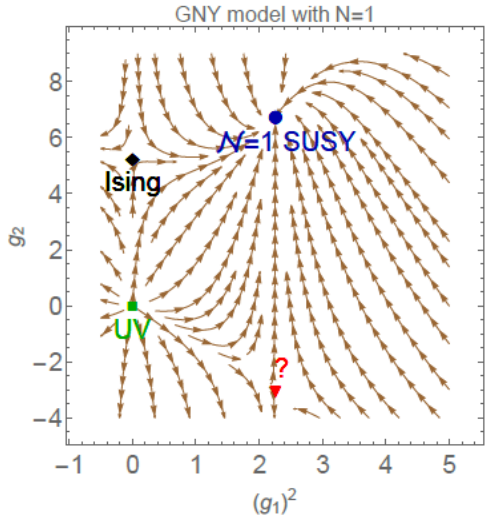
N	3	4	5	6	8	20	100
Δ_ψ ($\mathrm{Pade}_{[4,2]}$)	1.066	1.048	1.037	1.029	1.021	1.007	1.0013
Δ_σ ($\mathrm{Pade}_{[4,2]}$)	0.688	0.753	0.798	0.829	0.87	0.946	0.989
Δ_{σ^2} ($\mathrm{Pade}_{[1,5]}$)	2.285	2.148	2.099	2.075	2.052	2.025	2.008
$F/(NF_f)$ ($\mathrm{Pade}_{[4,4]}$)	1.091	1.060	1.044	1.034	1.024	1.008	1.0014



The Minimal Case: N=1

- For a single Majorana doublet the GN quartic interaction vanishes. Cannot use the 2+ε expansion to describe an interacting CFT.
- We have developed the 4-ε expansion by continuing the GNY model to N=1.
- $\sqrt{N^2+132N+36}$ equals 13.

$$\frac{(g_1^*)^2}{(4\pi)^2} = \frac{1}{7}\epsilon + \frac{3}{49}\epsilon^2 + \mathcal{O}(\epsilon^3)$$
$$\frac{g_2^*}{(4\pi)^2} = \frac{3}{7}\epsilon + \frac{9}{49}\epsilon^2 + \mathcal{O}(\epsilon^3)$$



- Consistent with the emergent SUSY relation!

$$3g_1^2 = g_2 = 3\lambda^2$$

More Evidence of SUSY for N=1

$$\Delta_\sigma = 1 - \frac{3}{7}\epsilon + \frac{1}{49}\epsilon^2 + \mathcal{O}(\epsilon^3)$$
$$\Delta_\psi = \frac{3}{2} - \frac{3}{7}\epsilon + \frac{1}{49}\epsilon^2 + \mathcal{O}(\epsilon^3)$$
$$\Delta_{\sigma^2} = 2 - \frac{3}{7}\epsilon + \frac{1}{49}\epsilon^2 + \mathcal{O}(\epsilon^3)$$

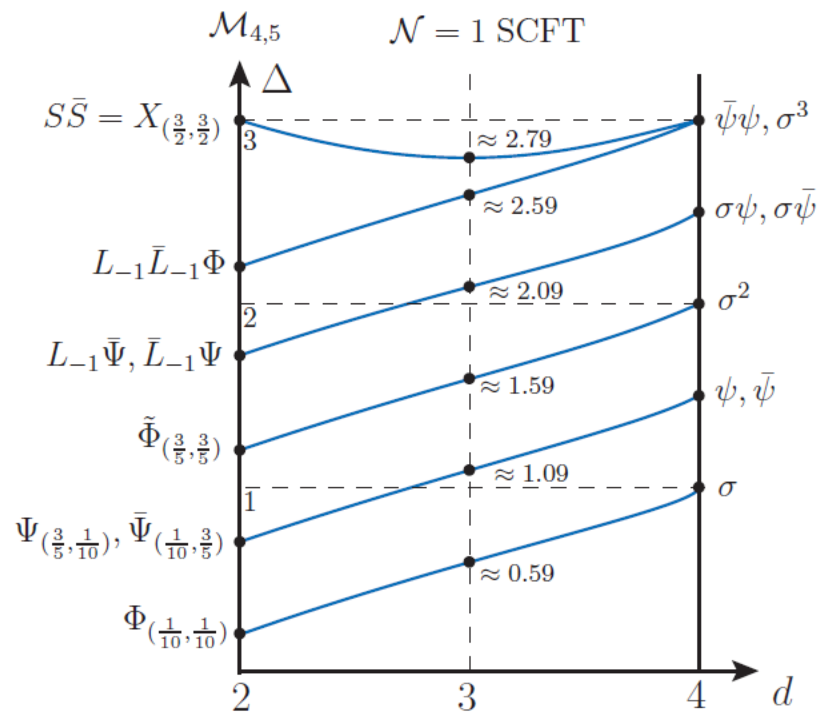
- Consistent with the SUSY relation

$$\Delta_{\sigma^2} = \Delta_\psi + \frac{1}{2} = \Delta_\sigma + 1$$

- We conjecture that it holds exactly for d< 4.
- May be tested at higher orders in ε. This requires doing Yukawa theory at 3 loops and beyond.
- Pade to d=3 gives $\Delta_\sigma \approx 0.588$ which seems close to the bootstrap result. Iliesiu, Kos, Poland, Pufu, Simmons-Duffin, Yacoby

Continuation to d=2

- Gives an interacting superconformal theory.
- Likely the tri-critical Ising model with c=7/10.
- Pade extrapolation gives $\Delta_\sigma \approx 0.217$, close to dimension 1/5 of the energy operator in the (4,5) minimal model.
- Pade also gives $\tilde{F}/\tilde{F}_s \approx 0.68$, close to c=0.7.



Models with U(1) Chiral Symmetry

- The Nambu-Jona-Lasinio (also known as the chiral Gross-Neveu) model

$$\mathcal{L}_{\rm NJL} = \bar{\psi}_j \partial\!\!\!/ \psi^j + \frac{g}{2}\left((\bar{\psi}_j \psi^j)^2 - (\bar{\psi}_j \gamma_5 \psi^j)^2\right)$$

  has U(1) chiral symmetry $\psi_j \to e^{i\alpha\gamma_5}\psi_j$

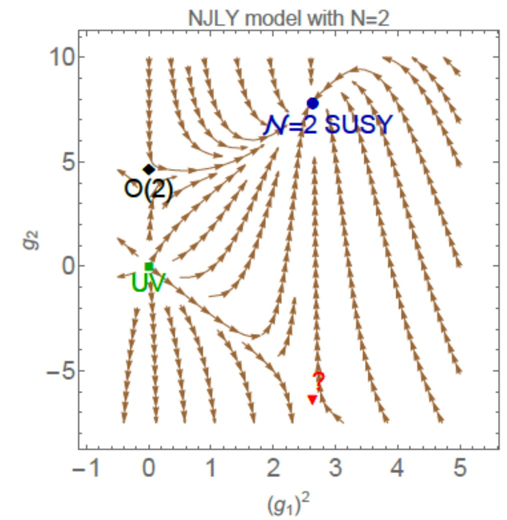
- Its UV completion in 2<d<4 is the Nambu-Jona-Lasinio-Yukawa model. Zinn-Justin
- It contains a complex scalar $\phi = \phi_1 + i\phi_2$

$$\mathcal{L}_{\rm NJLY} = \frac{1}{2}(\partial_\mu \phi_1)^2 + \frac{1}{2}(\partial_\mu \phi_2)^2 + \bar{\psi}_j \partial\!\!\!/ \psi^j + g_1 \bar{\psi}_j(\phi_1 + i\gamma_5 \phi_2)\psi^j + \frac{1}{24} g_2 (\phi\bar{\phi})^2$$

- Under the U(1) $\phi \to e^{-2i\alpha}\phi$

Emergent $\mathcal{N}$=2 Supersymmetry

- For N=2, which corresponds to one 4-component Majorana fermion, the NJLY model flows to the Wess-Zumino model. S.Thomas; S.-S. Lee
- The U(1) turns into R-symmetry.
- The sphere free energy in d=4-ε, including the curvature terms,



$$\tilde{F}_{N=2} = 2\tilde{F}_s + 2\tilde{F}_f - \frac{\pi\epsilon^2}{144} - \frac{\pi\epsilon^3}{162} + \mathcal{O}(\epsilon^4)$$

agrees with the result for the WZ model in d dimensions based on SUSY localization. Giombi, IK

Above 4 Dimensions

- Both O(N) and GN models make sense at least to all orders in the 1/N expansion.
- Interesting weak coupling expansions near even dimensions.
- For example, in 6-ε dimensions find the cubic O(N) symmetric theory

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi^i)^2 + \frac{1}{2}(\partial_\mu \sigma)^2 + \frac{g_1}{2}\sigma(\phi^i\phi^i) + \frac{g_2}{6}\sigma^3$$

- It has an IR fixed point for sufficiently large N. Results there agree with the 1/N corrections found for O(N) model as a function of d. Fei, Giombi, IK, Tarnopolsky; Gracey

Higher Spin AdS/CFT

- When N is large, the O(N) and GN models have an infinite number of higher spin currents whose anomalous dimensions are of order 1/N.
- Their singlet sectors have been conjectured to be dual to the Vasiliev interacting higher-spin theories in d+1 dimensional AdS space.
- One passes from the dual of the free to that of the interacting large N theory by changing boundary conditions at AdS infinity. IK, Polyakov; Leigh, Petkou; Sezgin, Sundel; for a recent review, see Giombi's TASI lectures

Higher-Spin dS/CFT

- To construct non-unitary CFTs dual to higher spin theory in de Sitter space, replace the commuting scalar fields by anti-commuting ones. Anninos, Hartman, Strominger
- The conjectured dual to minimal Vasiliev theory in dS_4 is the interacting Sp(N) model introduced earlier LeClair, Neubert

$$S = \int d^3x \left(\frac{1}{2} \Omega_{ij} \partial_\mu \chi^i \partial^\mu \chi^j + \frac{\lambda}{4} (\Omega_{ij} \chi^i \chi^j)^2 \right)$$

- In d>4 this quartic theory has a UV fixed point at large N.
- Consider instead the cubic Sp(N) invariant theory, which is weakly coupled in 6-ε dimensions.

$$S = \int d^d x \left(\frac{1}{2} \Omega_{ij} \partial_\mu \chi^i \partial^\mu \chi^j + \frac{1}{2} (\partial_\mu \sigma)^2 + \frac{1}{2} g_1 \Omega_{ij} \chi^i \chi^j \sigma + \frac{1}{6} g_2 \sigma^3 \right)$$

- The beta functions are related to those of the O(N) theory via N-> -N
- For Sp(N) there are IR stable fixed points at imaginary couplings for all positive even N.

Symmetry Enhancement for N=2

- The N=2 model may be written as

$$S = \int d^d x \left(\partial_\mu \theta \partial^\mu \bar{\theta} + \frac{1}{2} (\partial_\mu \sigma)^2 + g_1 \sigma \theta \bar{\theta} + \frac{1}{6} g_2 \sigma^3 \right)$$

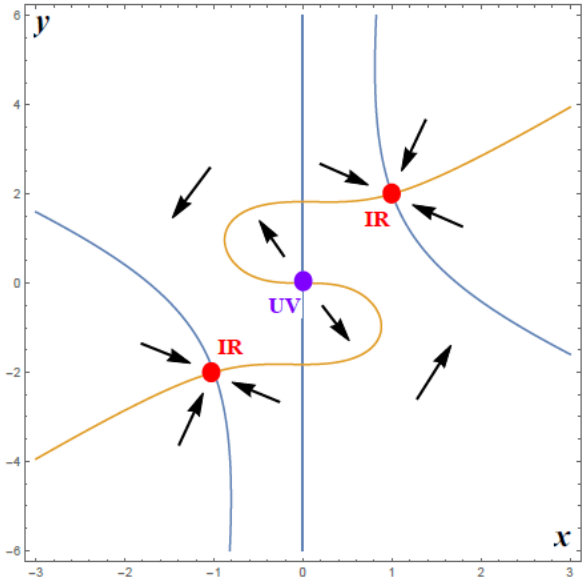
- At the fixed point

$$g_2^* = 2 g_1^* \,, \qquad g_1^* = i \sqrt{\frac{(4\pi)^3 \epsilon}{5}} \left(1 + \frac{67}{180} \epsilon + O(\epsilon^2) \right)$$

- There is symmetry enhancement from Sp(2) to the supergroup Osp(1|2)

$$\delta \theta = \sigma \alpha \ , \quad \delta \bar{\theta} = \sigma \bar{\alpha} \ , \quad \delta \sigma = -\alpha \bar{\theta} + \bar{\alpha} \theta$$

- Defining $g_1 = i\sqrt{\frac{(4\pi)^3\epsilon}{5}}x,\ g_2 = i\sqrt{\frac{(4\pi)^3\epsilon}{5}}y$



- The scaling dimensions of commuting and anti-commuting scalars are equal

$$\Delta_\sigma = \Delta_\theta = 2 - \frac{8}{15}\epsilon - \frac{7}{450}\epsilon^2 - \frac{269 - 702\zeta(3)}{33750}\epsilon^3$$

Connection with the Potts Model

- (n+1) state Potts model can be described in 6-ε dimensions by a cubic field theory of n scalar fields Zia, Wallace

$$S = \int d^d x \left(\partial_\mu \phi^i \partial^\mu \phi^i + \frac{1}{6} g d_{ijk} \phi^i \phi^j \phi^k \right)$$

$$d^{ijk} = \sum_{\alpha=1}^{n+1} e_\alpha^i e_\alpha^j e_\alpha^k$$

- The vectors e_α^i describe the vertices of the n-dimensional generalization of tetrahedron.

- The 6-ε expansions have been developed for any q-state Potts model.
- We find that, in the formal limit q-> 0, they are the same as at the fixed point with the emergent Osp(1|2) symmetry.
- The zero-state Potts model can be defined on a lattice using the spanning forest model, and Monte Carlo results for scaling exponents are available in d=3,4,5 where the model has second order phase transitions. Deng, Garoni, Sokal

Conclusions

- The ε expansions in the O(N), Gross-Neveu, Nambu-Jona-Lasinio, and other vectorial CFTs, are useful for applications to condensed matter and statistical physics.
- They provide “checks and balances” for the new numerical results using the conformal bootstrap.
- They serve as nice playgrounds for the RG inequalities (C-theorem, a-theorem, F-theorem) and for the higher spin AdS/CFT and dS/CFT correspondence.

- Some small values of N are special cases where there are enhanced IR symmetries.
- Cubic QFT in d<6 can exhibit the enhanced OSp(1|2) supergroup symmetry. It describes a known statistical system, the spanning forests (equivalent to the zero-state Potts model).
- Yukawa CFTs in d<4 can exhibit emergent supersymmetry with 2 or 4 supercharges. Perhaps this can be realized in condensed matter systems and observed experimentally.