Discussion on Integrability

Vladimir Kazakov   & Gregory Korchemsky

"It does not matter if the solution is exact,
it matters if it is correct"

Alexey Zamolodchikov

Disclamer: **no citations! No complaints accepted**   😁

$\mathcal{N}$=4 SYM and ABJM model are integrable in planar limit

- What does it mean?

- Origins of integrability?

 - What is achieved due integrability?

 - What else might be achieved?

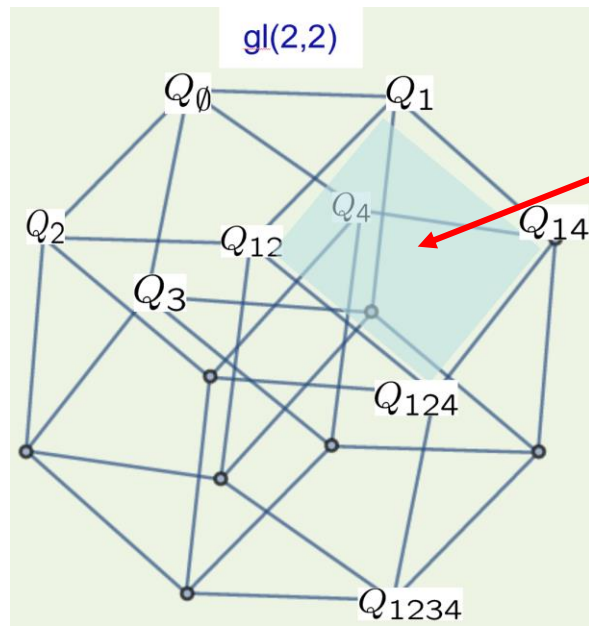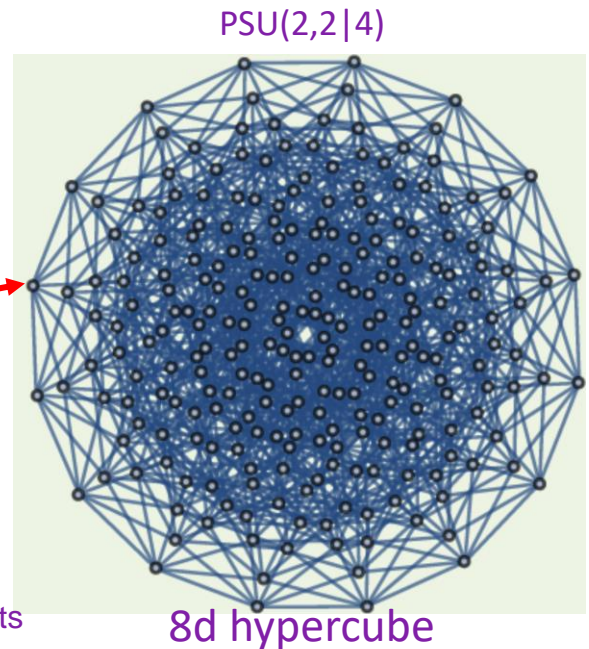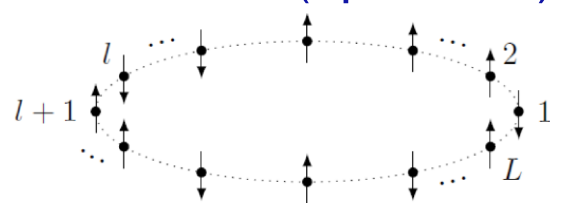# The origins of this integrability still mysterious

- Some observations :

    - Explicit integrability in lowest orders of PT
    - (Quasi)-classical integrability of the worldsheet sigma-model
    - Mysterious cancellations in planar PT (highest transcendentality)
    - Fishnet CFT limit of Υ-twisted $\mathcal{N}$=4 SYM - a chance
      for understanding its integrability

- Problem of spectrum of anomalous dimensions is solved
  (any local, and some non-local operators, any coupling…)

- Ultimate formalism: Quantum Spectral Curve (QSC) :
  a Riemann-Hilbert problem on a few Baxter functions $Q_{ijk...}(u)$
  with the known algebraic (grassmanian) and analytic structure.

- It is a generalization of Baxter functions (operators) $Q(u) = \prod\limits_{j=1}^{\text{Length}} (u - u_j)$
  in Heisenberg spin chain

Bethe roots

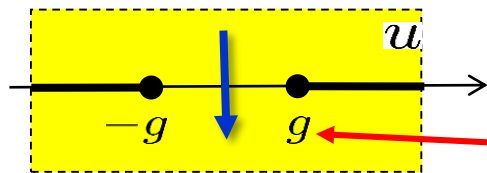8d hypercube

$$Q_1(u)Q_{124}(u) = \begin{vmatrix} Q_{12}(u + \frac{i}{2}) & Q_{14}(u + \frac{i}{2}) \\ Q_{12}(u - \frac{i}{2}) & Q_{14}(u - \frac{i}{2}) \end{vmatrix}$$

Plücker relations

gl(2,2)

$Q_\emptyset$ Q_1

Q_2 Q_4 Q_{14}

Q_{12}

Q_3

Q_{124}

Q_{1234}

large u asymptotics $Q_{123} \sim u^{(\pm\Delta\pm S_1\pm S_2)/2}$

cut on physical sheet

u

$-g$ g

't Hooft coupling

gluing rules (monodromy) $Q_{123} \propto \overline{Q_{25678}}, \cdots$

Hasse diagram (8d hypercube)

- QSC provides for anomalous dimensions and qq-potentials:

  - regular weak coupling expansion (~20 orders...)
  - strong coupling (a few orders, no regular expansion yet)
  - other approximations for QSC: large spins, next-to-next-to-LO for BFKL spectrum ...
 - quasi-exact numerics (easily >50 digits)

$\mathrm{Tr}\,\mathcal{D}^S Z^L$

$L = 3,\ S = 2$

$L = 2,\ S = 2\ (Konishi)$

$$\Delta_{Konishi} \simeq 2\lambda^{1/4} + \frac{2}{\lambda^{1/4}} + \frac{-3\zeta_3 + \frac{1}{2}}{\lambda^{3/4}} + \frac{\frac{15}{2}\zeta_5 + 6\zeta_3 - \frac{1}{2}}{\lambda^{5/4}} + \cdots$$

$\lambda = 16\pi^2 g^2$

$\mathrm{Tr}\,(\Phi\,\nabla^S\,\Phi)$

$g = 0.3$

$\Delta(S, g)$
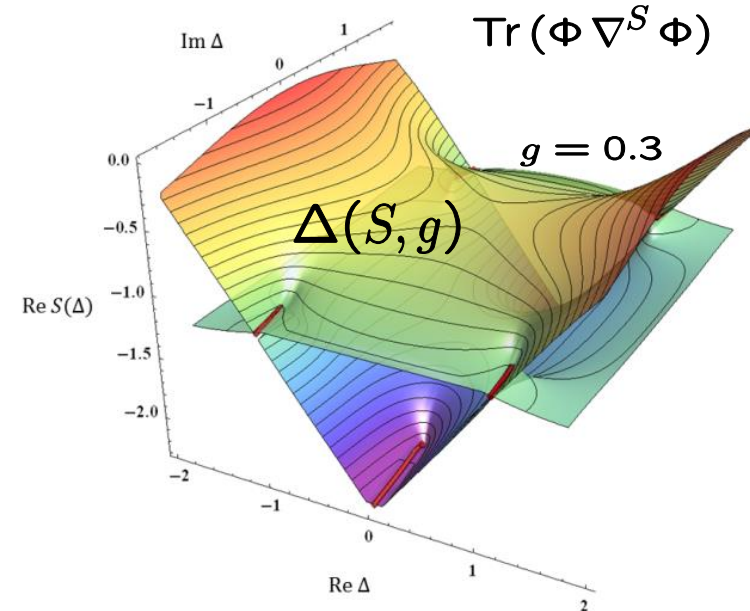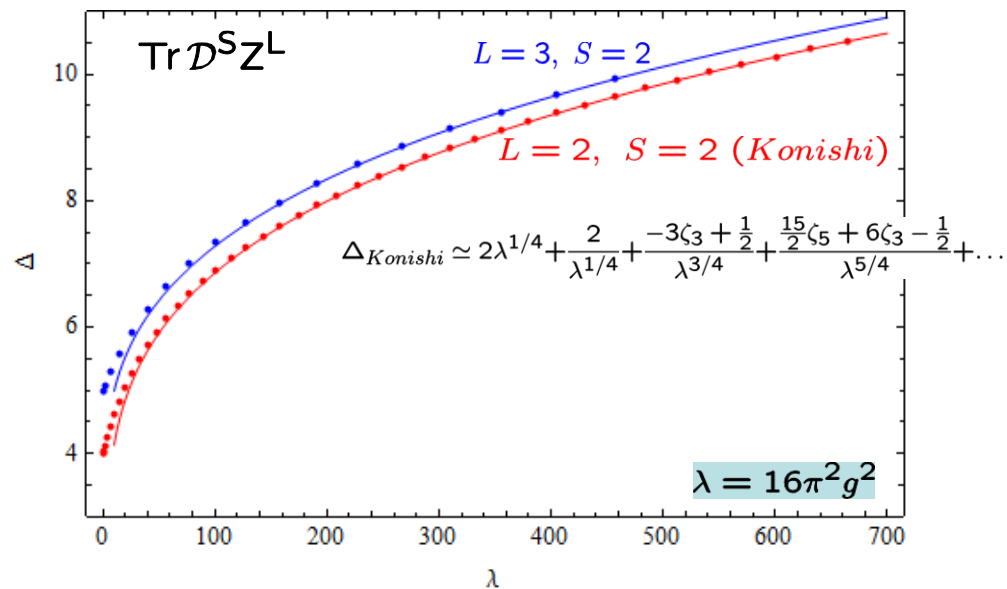
- Perfect fit with other data: perturbation theory, quasiclassical computations in string sigma model,… AdS/CFT seen in full glory!

- Fishnet CFT limit of ϒ-twisted 𝒩=4 SYM:
  large complex twist & weak coupling

$$\mathcal{L} = \mathrm{tr}\left[\partial\bar{X}\partial X + \partial\bar{Z}\partial Z + \xi^2\, \bar{X}\bar{Z}XZ \right]$$

- explicitly integrable in all loops (non-compact SU(2,2) spin chain). QSC in form of TBA…
- Feynman graphs have a regular lattice, "fishnet" structure
- In most general case (with 3 couplings) one gets more complicated "dynamical fishnet", still regular planar structure
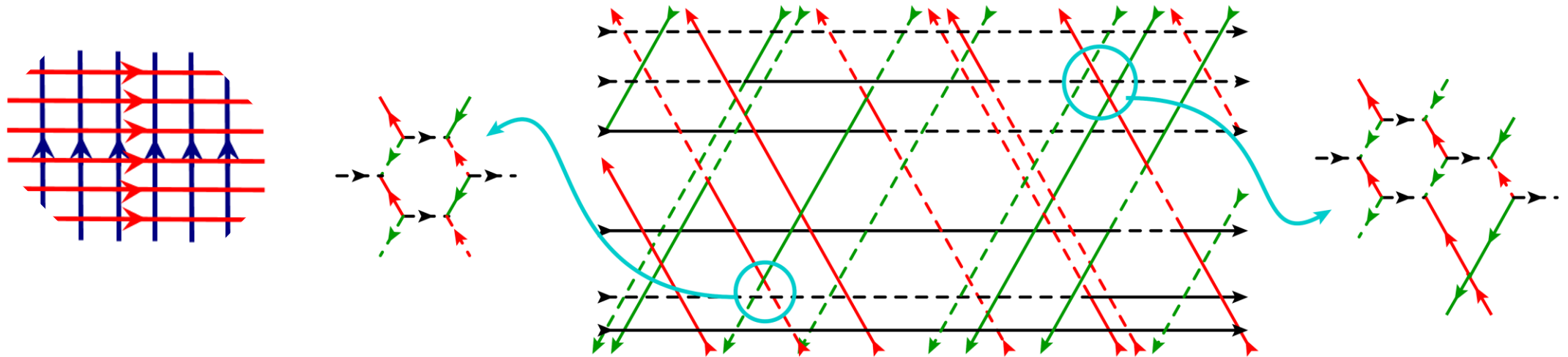
- Hidden regular dynamical lattice structure of planar graphs of full 𝒩=4 SYM ?

- Explicit AdS/CFT picture : "Fish-chain" (chain of particles on AdS)

- Integrable O(1,5) sigma model for dense fishnet graphs

- Integrable Fichnet CFT exists in any dimension

$$R_{AdS} = \frac{\sqrt{3}}{m\xi}$$
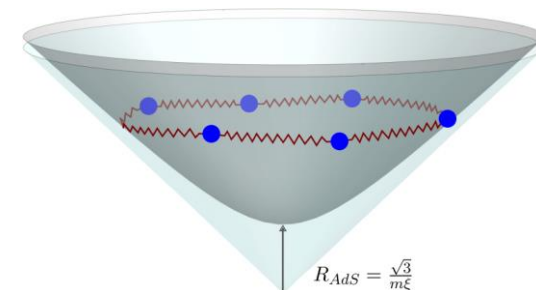
Planar $\mathcal{N} = 4$ super Yang-Mills is integrable

Use integrability to compute various quantities for arbitrary `t Hooft coupling

A lot of progress in solving the theory:

- ✔ Spectrum of the dilatation operator
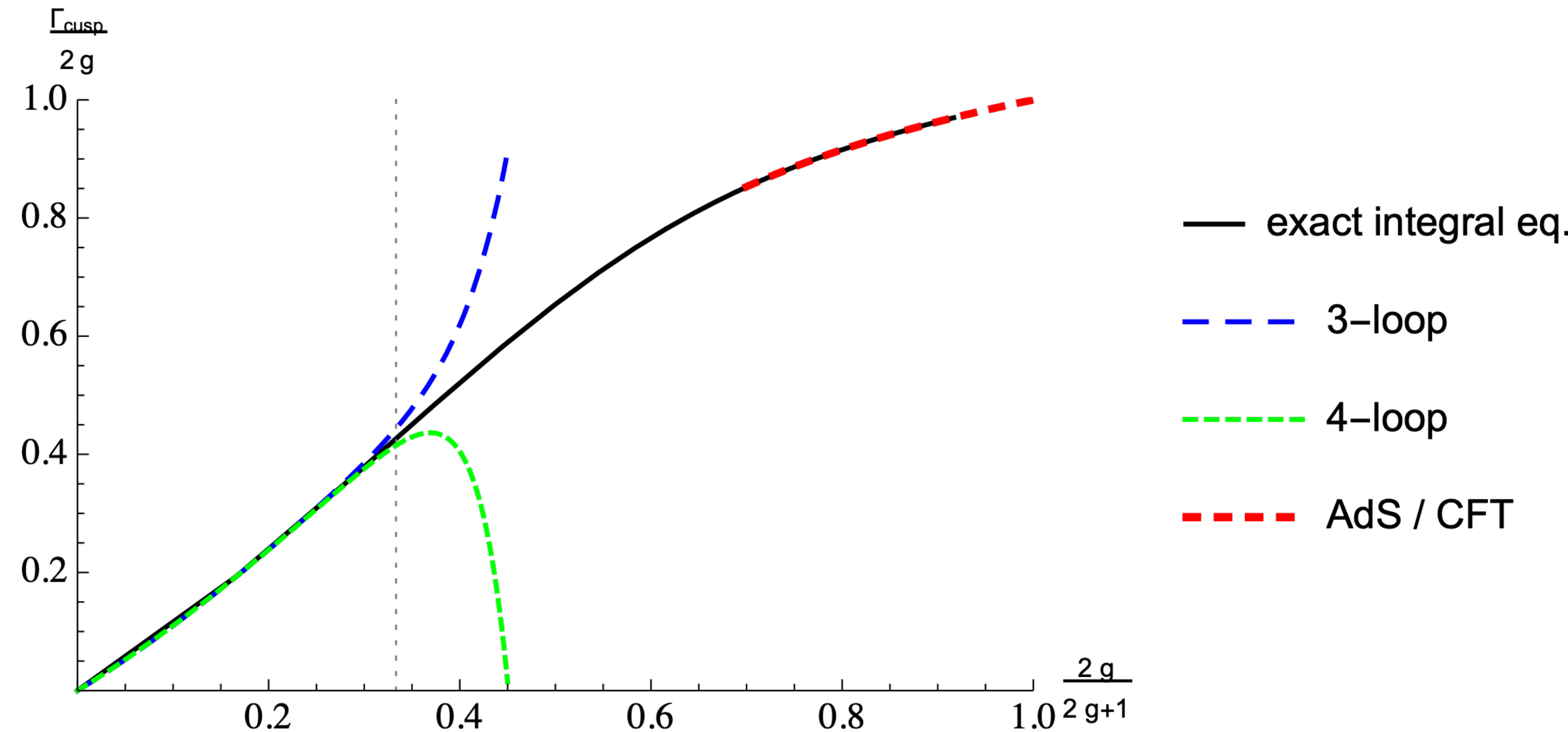- ✔ Scattering amplitudes
- ✔ Correlation functions

What do we expect to find?

Success story: cusp anomalous dimension

Weak coupling: Feynman diagrams

Strong coupling: GKP string

Finite coupling: integrability

State of art: on-shell amplitudes

Duality between amplitudes $A_n(k)$ and light-like Wilson loops $\mathscr{W}(x), (k_i = x_i - x_{i+1})$

Dual superconformal + Yangian symmetry

— Weak coupling:

   Beautiful mathematical structures of integrands (positive geometry, cluster algebras)

   Amplitudes can be bootstrapped from their analytic properties

—Strong coupling:

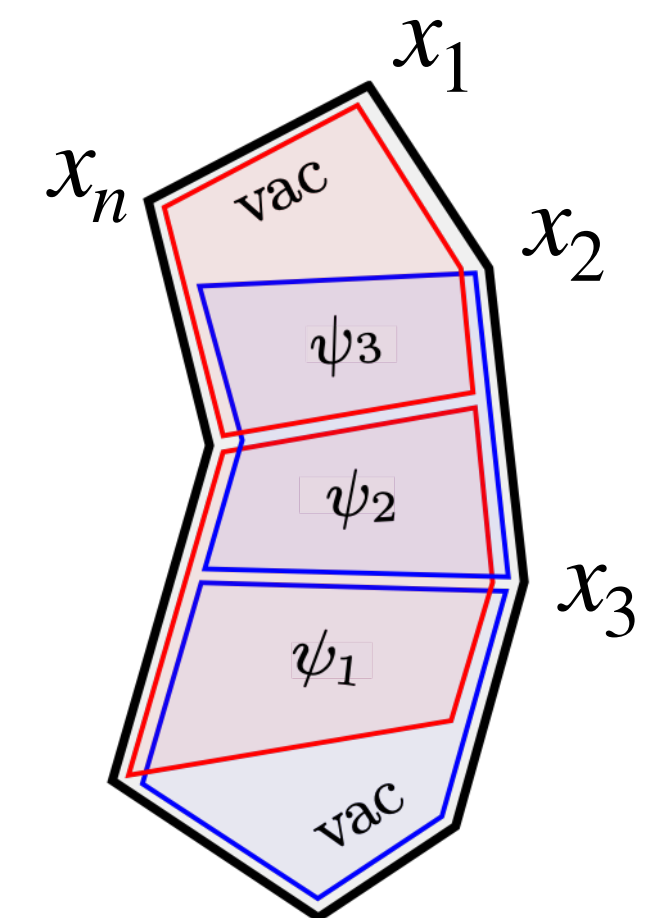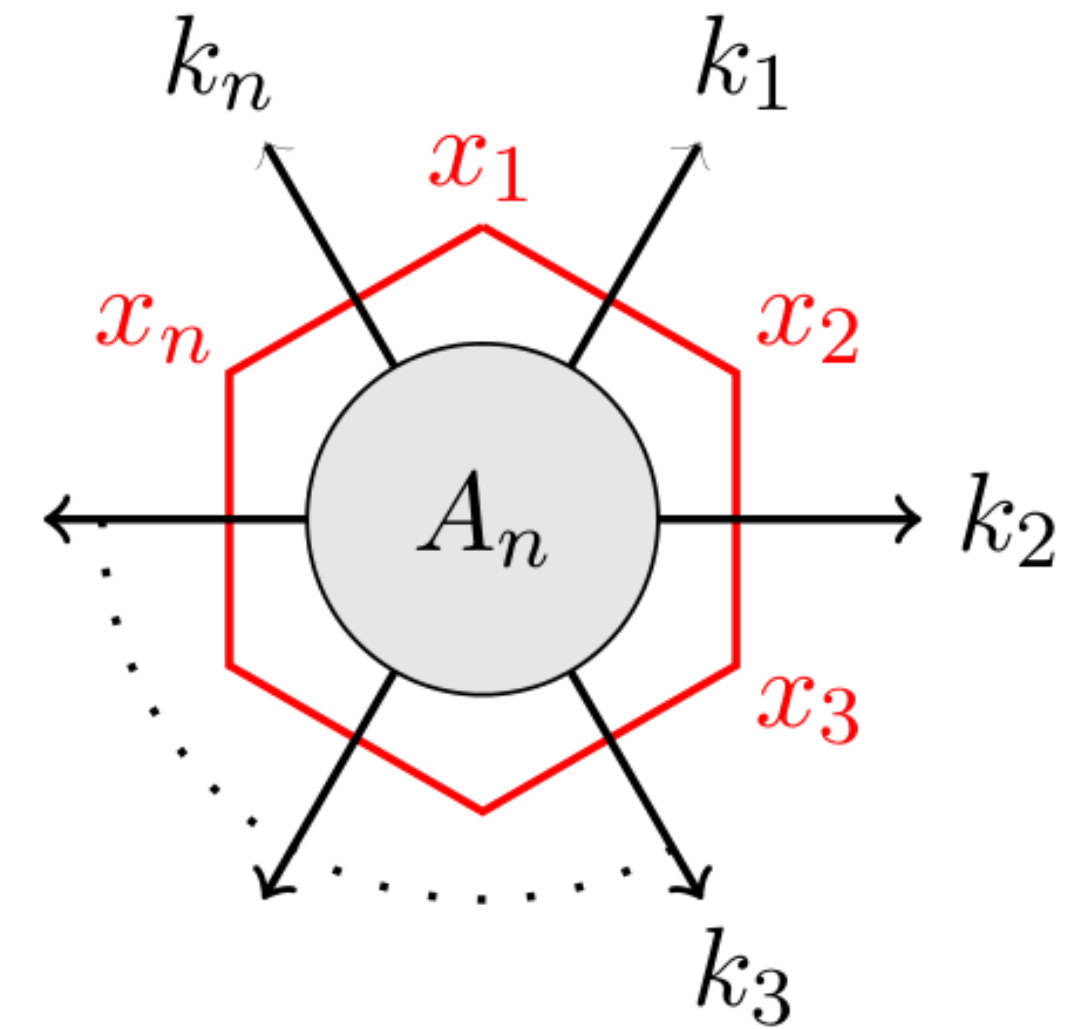   Relation to the minimal area in AdS attached to the polygon null Wilson loop

—Finite coupling:

   Generalization of the Operator Product Expansion to the null Wilson loop

$$\mathscr{W} = \sum_{\psi_i} e^{-E_i\tau + ip_i\sigma + im_i\phi} P(0|\psi_1)P(\psi_1|\psi_2)\ldots P(\psi_n|0)$$

Pentagon transitions $P(\psi|\psi')$ fixed by integrability

Sum over excitation of the GKP string; hard to evaluate for finite coupling

State of art: 4pt correlation functions

$$\left\langle O_{R_1}(x_1) O_{R_2}(x_2) O_{R_3}(x_3) O_{R_4}(x_4) \right\rangle \qquad O_R(x) = \mathrm{tr}[(y \cdot \phi(x))^R] \quad \text{1/2 BPS operator with R charge}$$

—Weak coupling : Correlators can be bootstrapped using $\mathcal{N} = 4$ symmetry + OPE conditions

—Strong coupling: Scattering amplitudes in AdS (supergravity + strings)

—Finite coupling: Hexagonalization approach
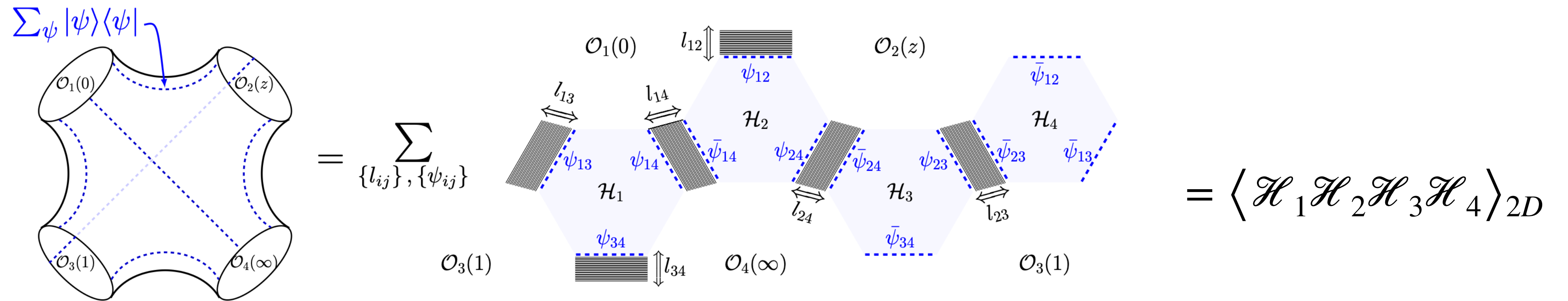
Glue together hexagons and sum up over excitations propagating through their egdes

Hexagon form factors are fixed by integrability

It is very difficult to evaluate the sums for generic R charges

Simplification in the large R charge limit: $\left\langle \mathcal{H}_1 \mathcal{H}_2 \right\rangle \left\langle \mathcal{H}_3 \mathcal{H}_4 \right\rangle = \text{Octagon}^2$

Analytical results for the Octagon; surprising relation to 2d Ising model

Open questions

- ***What is the origin of integrability?***

—Do we have a "good" definition of integrability?

   How to construct conserved currents/quantize sigma model on AdS5 x S5 beyond the classical limit?

—Explicit "spin-chain" formalism for spectral integrability: Baxter Q-operators, Sklyanin's separated variables

—What did we learn about string theory from hexagonalization? Could it hold for other string theories? in flat space?

- ***Is $\mathcal{N} = 4$ SYM a fishnet CFT in disguise?***

—Does the fishnet limit of $\mathcal{N} = 4$ SYM (strong imaginary γ-twist + weak coupling) reveal essential features of integrability?

—Can we expand around $\mathcal{N} = 4$ SYM or around the fishnet to study less symmetric (nonintegrable) theories?

—Can integrability help in the first principle derivation of the holographic principle for interacting CFTs in the planar limit?

   Holographic dual of fishnet CFT — fishchain model (discretised string on AdS)

Open questions

- **Did we solve planar $\mathcal{N}=4$ SYM?**

  —How to derive a regular strong coupling expansion of anomalous dimensions from QSC?

  —How to evaluate infinite sums for amplitudes/correlations at finite coupling?

  —How simple could the final solution be given amount of information carried by amplitudes/correlations?

  —What interesting physics could we find with an exact solution which we could not find otherwise?

    (new kinematical limits in scattering amplitudes / light-cone limits of correlators)

- **What did we learn about integrability in high-energy QCD?**

  — At weak coupling, high-energy QCD shares some of integrability properties of $\mathcal{N}=4$ SYM:

    special sectors of the dilatation operator, amplitudes in the high-energy (Regge) limit, dual conformal symmetry

  —We know from numerous examples (anomalous dimensions, form factors, multi-Regge limit) that $\mathcal{N}=4$ SYM

    captures "the most complicated" part of the answer (principle of maximal transcendentality), why?

  —Novel methods for computing Feynman integrals in $\mathcal{N}=4$ SYM have become a standard tool for QCD calculations

  —Can we view QCD as a perturbation around $\mathcal{N}=4$ SYM?

Open questions

- **Beyond the planar limit: If and how integrability is reflected in $1/N_c$ corrections ?**

— Integrability of the dilatation operator is broken by nonplanar corrections … but

— Circular Wilson loop can be found exactly to any order in $1/N_c$ from localisation

— Localization predicts that 4-point functions of *short* operators integrated with certain weight can be expressed

   in terms of matrix integrals for any coupling and any $1/N_c$

— Nonplanar corrections to amplitudes in $\mathcal{N} = 4$ SYM have "directional" dual conformal symmetry

Open questions

- **What is the origin of integrability?**

- **Did we solve planar $\mathcal{N} = 4$ SYM?**

- **Is $\mathcal{N} = 4$ SYM a fishnet CFT in disguise?**

- **Beyond the planar limit: If and how integrability is reflected in $1/N_c$ corrections?**

- **What did we learn about integrability in high-energy QCD?**

Further interesting questions to be found on Slack workspace