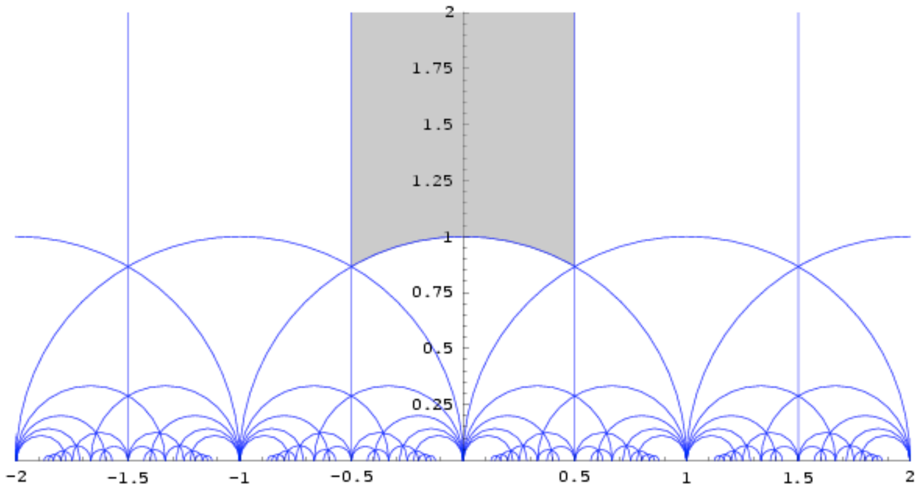
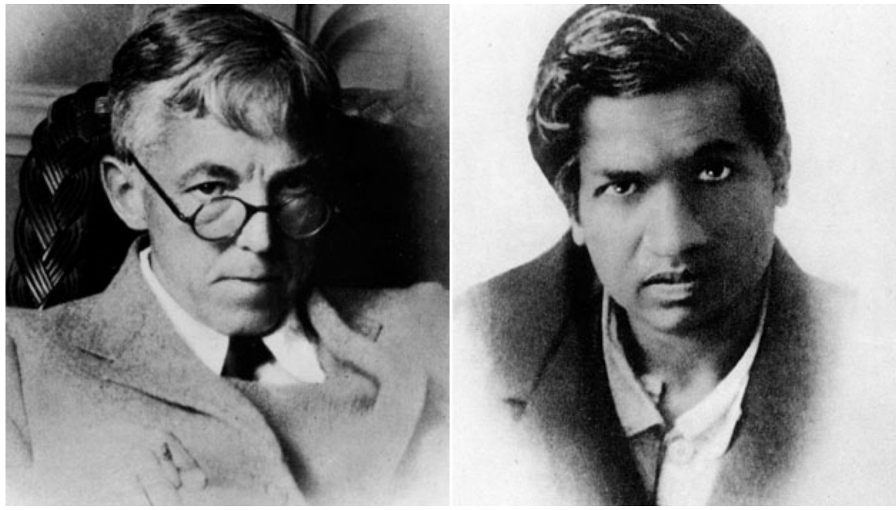
Mock Modular Mathieu Moonshine





Shamit Kachru
Strings 2014

Any new material in this talk is based on:

MOCK MODULAR MATHIEU MOONSHINE MODULES

Miranda C. N. Cheng[1], Xi Dong[2], John F. R. Duncan[3], Sarah Harrison[2], Shamit Kachru[2], and Timm Wrase[2]

which just appeared on the arXiv, and work in progress.

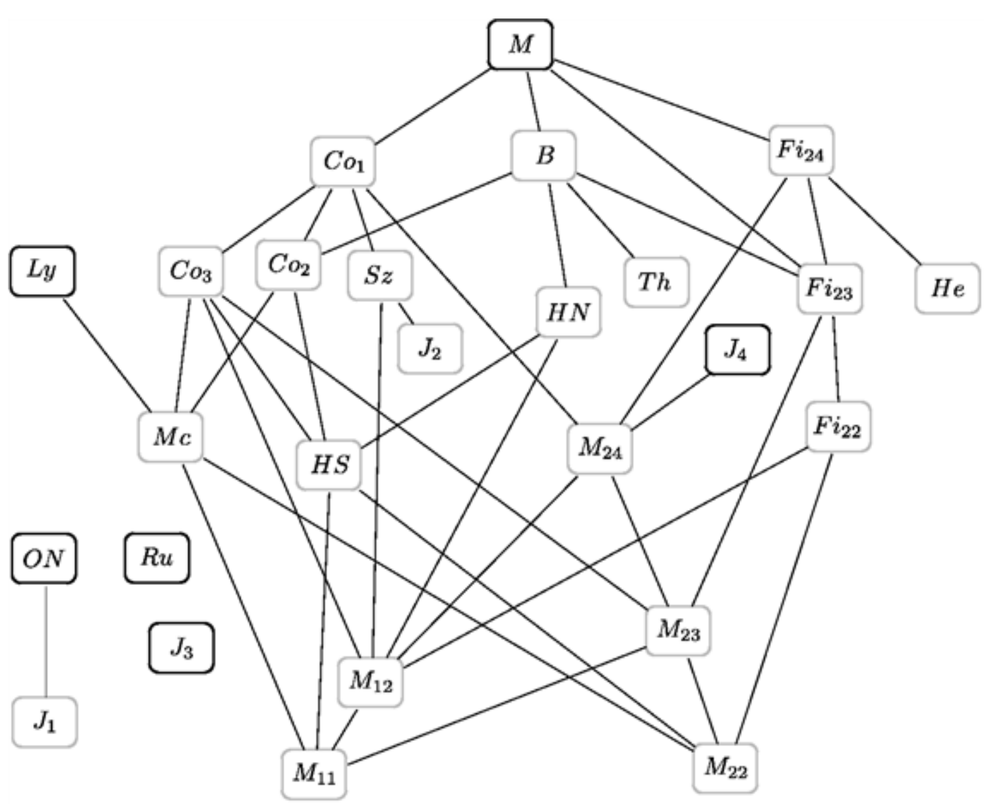
Outline of talk:

I. Introduction
II. Geometric motivation
III. M22/M23 moonshine

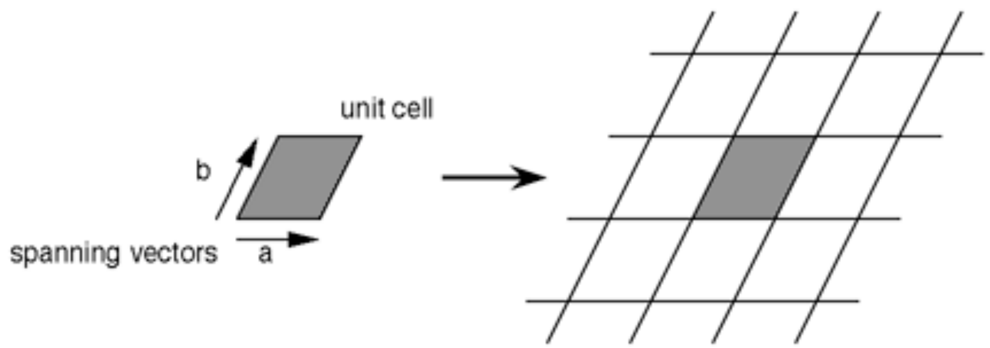
1. Introduction

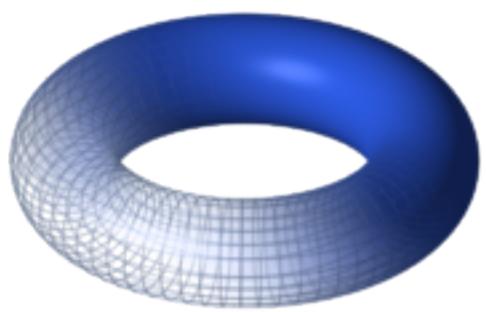
A talk about moonshine needs to begin with some recap of the story to date, and the objects involved.

At the heart of the story are two classes of beautiful and enigmatic objects in mathematics:

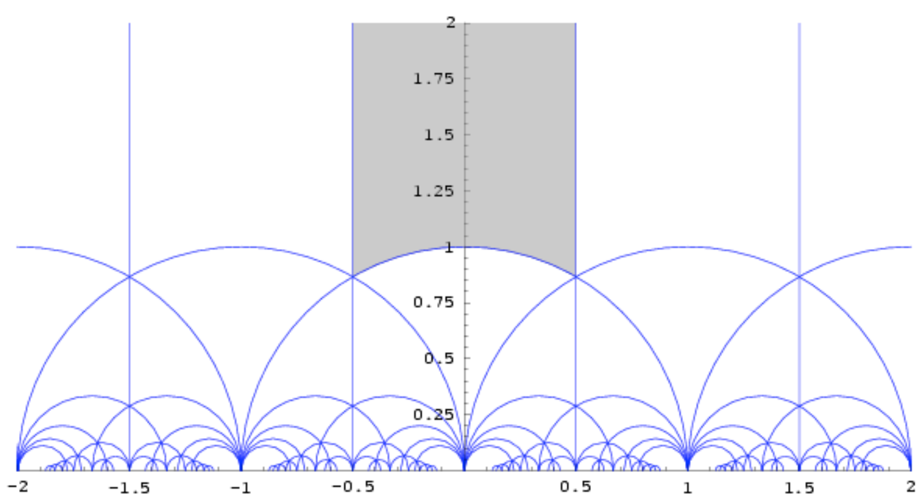


First off, we have the sporadic finite groups -- the 26 simple finite groups that do not come in infinite families.





And secondly, we have the modular functions and forms: objects which play well with the modular group (c.f. worldsheet partition functions, S-duality invt. space-time actions, ...).



A common example: consider SL(2,Z) acting on the UHP via fractional linear transformations:

$$\tau \to \frac{a\tau+b}{c\tau+d} \equiv A\cdot\tau$$

Then a modular function is a meromorphic function which satisfies:

$$f(A\cdot\tau) = f(\tau)$$

while a modular form of weight k satisfies instead:

$$f(A\cdot\tau) = (c\tau+d)^k f(\tau)$$

Monstrous moonshine originated in the observation that:

Dims of irreps of M	
d_1	1
d_2	196,883
d_3	21,296,876
d_4	842,609,326

while

$$j(\tau) = \frac{1}{q} + 744 + 196,884\ q + 21,493,760\ q^2 + \cdots$$

$$q = e^{2\pi i\tau}$$

One notices immediately the "coincidence":

$$196,884 = 196,883 + 1$$
$$21,493,760 = 21,296,876 + 196,883 + 1$$

The "McKay-Thompson series" greatly strengthen evidence for some real relationship:

Suppose there is a physical theory whose partition function is j, and which has Monster symmetry. Then:

$$V = V_{-1} \oplus V_1 \oplus V_2 \oplus V_3 \oplus \dots$$

$$V_{-1} = \rho_0, \; V_1 = \rho_1 \oplus \rho_0, V_2 = \rho_2 \oplus \rho_1 \oplus \rho_0, \dots$$

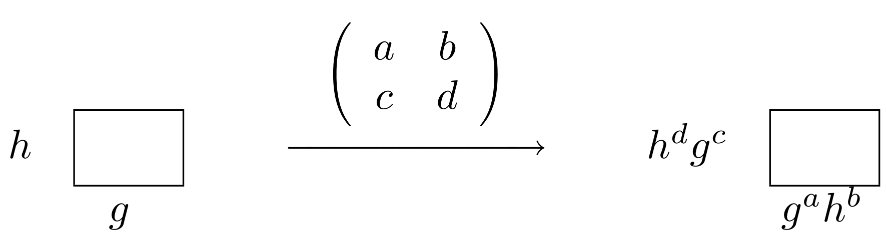
This suggests to also study the MT series:

$$ch_\rho(g) = Tr(\rho(g)), \quad g \in M$$

$$T_g(\tau) = ch_{V_{-1}}(g)\ q^{-1} + \sum_{i=1}^{\infty} ch_{V_i}(g)\ q^i$$

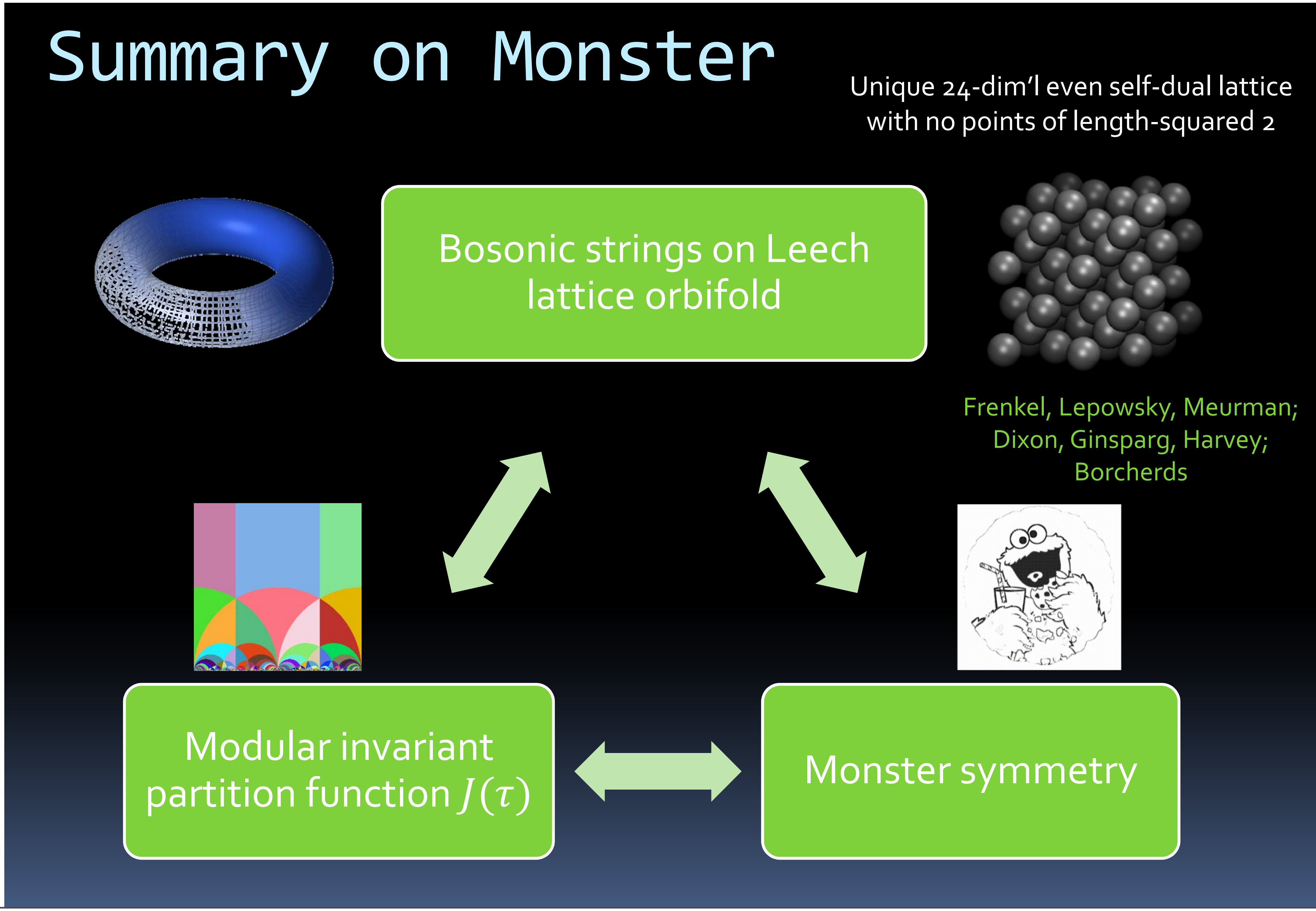
For each conjugacy class of M, we get such a series. Now while the partition function is modular under SL(2,Z), in general the MT series are not:

$$Z_{[g]}(\tau) = \mathrm{Tr}(g q^{L_0}) = T_{[g]}(\tau)$$



...but we should still get modular functions under a subgroup of SL(2,Z) that preserves the B.C.

This gave many further non-trivial checks. Eventually, a beautiful and complete (?) story was worked out, by Frenkel-Lepowsky-Meurman, Borcherds, ...



II. Geometric Motivation

This is where matters stood until the work of EOT (2010) brought K3 into the story:

* For any (2,2) SCFT, one can compute the elliptic genus:

$$Z_{\mathrm{ell}}(q, \gamma_L) = \mathrm{Tr}_{RR}(-1)^{F_L} q^{L_0} e^{i\gamma J_L} (-1)^{F_R} \bar{q}^{\bar{L}_0}$$

It is a generalization of a modular form, known as a Jacobi form.

Math object definition slide

A Jacobi form of level 0 and index m behaves as

$$\phi\left(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\right)=e\left[\frac{mcz^2}{c\tau+d}\right]\phi(\tau,z),\qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL_2(\mathbf{Z})$$

$$\phi(\tau,z+\lambda\tau+\mu)=(-1)^{2m(\lambda+\mu)}e[-m(\lambda^2\tau+2\lambda z)]\phi(\tau,z),\qquad \lambda,\mu\in\mathbf{Z}$$

under modular transformations and elliptic transformations. The latter is encoding the behavior under the "spectral flow" of N=2 SCFTs.

It was computed for K3 in 1989 by EOTY; expanded in N=4 characters in Ooguri's thesis; and interpreted in terms of moonshine in 2010:

$$\phi(\tau,\gamma) = 8 \sum_{i=2}^{4} \frac{\theta_i(\tau,\gamma)^2}{\theta_i(\tau,0)^2}$$

$$= 20\ ch^{\text{short}}_{1/4,0}(\tau,\gamma) - 2\ ch^{\text{short}}_{1/4,1/2}(\tau,\gamma) + \textstyle\sum_{n=1}^{\infty} A_n\ ch^{\text{long}}_{1/4+n,1/2}(\tau,\gamma)$$

The values of the As are given by:

$$A_1 = 90 = \mathbf{45} + \overline{\mathbf{45}}$$

$$A_2 = 462 = \mathbf{231} + \overline{\mathbf{231}} \longleftarrow$$ dims of irreps of M24!

$$A_3 = 1540 = \mathbf{770} + \overline{\mathbf{770}}$$

From: Hirosi Ooguri <h.ooguri@gmail.com>
Subject: My PhD thesis
Date: May 15, 2014 2:32:56 PM PDT
To: Shamit Kachru <shamit.kachru@gmail.com>
Cc: Ooguri Hirosi <h.ooguri@gmail.com>

Dear Shamit,

Here is a copy of my PhD thesis. Please see (3.16). I did not know how to divide these by two.

Regards, Hirosi

Math object definition slide

M24 is a sporadic group of order

$$|\mathrm{M24}| = 2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23 \;=\; 244{,}823{,}040$$

"Automorphism group of the unique doubly even self-dual binary code of length 24 with no words of length 4 (extended binary Golay code)."

* Consider a sequence of 0s and 1s

* Any length 24 word in G has even overlap with all codewords in G iff it is in G

* The number of 1s in each element is divisible by 4 but not equal to 4

* The subgroup of S24 that preserves G is M24

M22 and M23 are the subgroups of permutations in M24 that stabilize one or two points

This M24 moonshine, and a list of generalizations associated with each of the Niemeier lattices called "umbral moonshine," have been investigated intensely in the past few years. The full analogue of the story of Monstrous moonshine is not yet clear.

Cheng, Duncan, Harvey

In particular, no known K3 conformal field theory (or auxiliary object associated with it) gives an M24 module with the desired properties.

Gaberdiel, Hohenegger, Volpato

The starting point for the work I'll report was the desire to extend these kinds of results to Calabi-Yau manifolds of higher dimension.

For Calabi-Yau threefolds, there is a story involving the heterotic/type II duality:

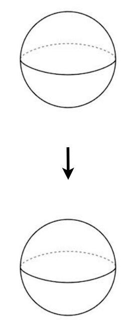
Heterotic strings on K3xT2

$$\int_{K3} c_2(V_1) + c_2(V_2)$$

$$\equiv n_1 + n_2 = 24$$

dilaton S

Type IIA on Calabi-Yau threefolds



elliptic fibration over

$F_n, \; n_1 = 12 + n, n_2 = 12 - n$

size of base P^1

A related M24 structure appears in the heterotic string on K3 as a (dualizable) quantity governing space-time threshold corrections. So it appears in GW invariants of the dual Calabi-Yau spaces.

Cheng, Dong, Duncan, Harvey, Kachru, Wrase

We were then led to think about Calabi-Yau fourfolds.

The elliptic genus of a Calabi-Yau fourfold has an interesting structure.

It is a linear combination of two Jacobi forms of weight 0 and index 2:

C.D.D. Neumann, 1996

$$\mathcal{Z}_{\rm ell}(\tau, z) = \chi_0(X)\ E_4(q)\phi_{-2,1}(q,y)^2 + \frac{\chi(X)}{144}\ \left(\phi_{0,1}(q,y)^2 - E_4(q)\phi_{-2,1}(q,y)^2\right)\ .$$

$$E_4(q) = 1 + 240\sum_{k=1}^{\infty}\sigma_3(k)q^{2k} = 1 + 240q^2 + 2160q^4 + 6720q^6 + \cdots$$

$$\phi_{-2,1}(q,y) = \frac{\phi_{10,1}(q,y)}{\eta(q)^{24}} = \left(\frac{1}{y} - 2 + y\right) - \left(\frac{2}{y^2} - \frac{8}{y} + 12 - 8y + 2y^2\right) q + \ldots\ ,$$

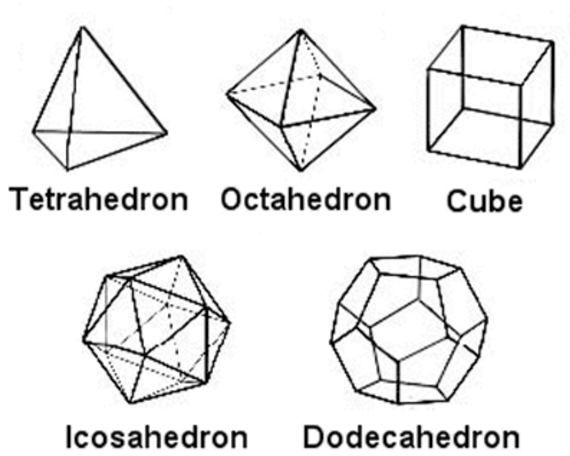
$$\phi_{0,1}(q,y) = \frac{\phi_{12,1}(q,y)}{\eta(q)^{24}} = \left(\frac{1}{y} + 10 + y\right) + \left(\frac{10}{y^2} - \frac{64}{y} + 108 - 64y + 10y^2\right) q + \ldots$$

Standard generators for ring of weak Jacobi forms

The piece that is present universally has a nice expression:

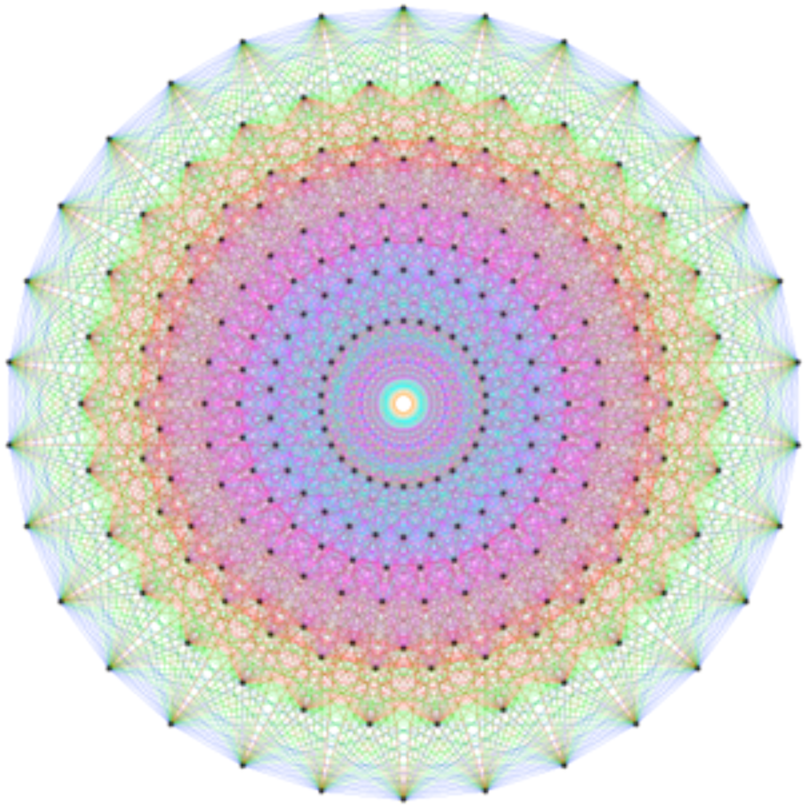
$$Z_{univ} \sim \frac{1}{\eta(\tau)^{12}} \sum_{i=1}^{4} \theta_i(\tau, 2z)\theta_i(\tau, 0)^{11}$$

As we'll see momentarily, this has a very suggestive q-expansion and hints at many interesting things. But first, we switch to a setting where all statements can be made precise, without randomly selecting a fourfold.



We can consider this a move to Platonic ideals instead of real-world grubby fourfolds...

III. The Platonic realm of M22/M23 moonshine



We begin with the chiral SCFT on the E8 root lattice.

It can be formulated in terms of 8 bosons and their Fermi superpartners.

Next, we orbifold: $$(X^i, \psi^i) \mapsto -(X^i, \psi^i)$$

The partition function of this theory can be computed by elementary means. It is given by:

$$Z_{\rm orb}^{NS}(q) = \frac{1}{2}\left[\frac{E_4(q)\theta_3(q,1)^4}{\eta(q)^{12}} + 16\frac{\theta_4(q,1)^4}{\theta_2(q,1)^4} + 16\frac{\theta_2(q,1)^4}{\theta_4(q,1)^4}\right]$$
$$= \frac{1}{\sqrt{q}} + 0 + 276\sqrt{q} + 2048q + 11202q^{3/2} + \cdots$$

The coefficients are interesting:

$$2048 = 1 + 276 + 1771$$
$$11202 = 1 + 276 + 299 + 1771 + 8855$$
$$\ldots$$

Natural decomposition into Co1 reps!

In fact, it was realized some years ago that this model has a (not so manifest) Conway symmetry, which commutes with the N=1 supersymmetry.

FLM; Duncan '05

$$
\begin{aligned}
Z^{NS}_{\mathrm{orb}}(q) &= \frac{1}{\sqrt{q}} + 276\sqrt{q} + (1+276+1771)q \\
&\qquad + (1+276+299+1771+8855)q^{3/2} + \cdots \\
&= \mathrm{ch}_{h=0} + 276\mathrm{ch}_{h=1} + 1771\mathrm{ch}_{h=\frac{3}{2}} \\
&\qquad + (299+8855)\mathrm{ch}_{h=2} + \cdots
\end{aligned}
$$

This CFT played a significant role in attempts to find a holographic dual of pure supergravity in AdS3.

Witten '07

We have realized several new things about this model; explaining them will occupy the rest of this talk.

1. It admits an N=4 description with more or less manifest M22 moonshine.

2. The elliptic genus and twining functions that arise match expectations for every conjugacy class. They have a beautiful interpretation as Rademacher sums.

3. The natural objects appearing in the genera are mock modular forms. This gives a completely explicit example of mock moonshine with a full construction of the module.

4. An analogous story holds for an N=2 description with M23 mock moonshine.

One can construct a full string theory simply related to this module by considering the asymmetric orbifold generated by this $\mathbb{Z}_2$ acting separately on left/right.

Its elliptic genus is:

$$Z_{\text{ell}}(q,y) = Z(q,y) Z^{\text{Witten}}_{\text{right}} \quad , \quad Z^{\text{Witten}}_{\text{right}} = 24$$

$$Z(q,y) = \frac{1}{2}\left[\frac{E_4(q)\theta_1(q,y)^4}{\eta(q)^{12}} + 16\sum_{i=2}^{4}\frac{\theta_i(q,y)^4}{\theta_i(q,1)^4}\right]$$

Now, expand this thing in N=4 characters:

$$Z(q,y) = 21\ \mathrm{ch}^{\mathrm{BPS}}_{h=1/2,l=0}(q,y) + \mathrm{ch}^{\mathrm{BPS}}_{h=1/2,l=1}(q,y)$$

$$+\ 560\ \mathrm{ch}_{h=3/2,l=1/2}(q,y) + 8470\ \mathrm{ch}_{h=5/2,l=1/2}(q,y) + \cdots$$

$$+\ 210\ \mathrm{ch}_{h=3/2,l=1}(q,y) + 4444\ \mathrm{ch}_{h=5/2,l=1}(q,y) + \cdots$$

these two lines are governed by mock modular forms!

The coefficients are dimensions of M22 representations:

$$21 = 21$$

$$560 = 280 + \overline{280}$$

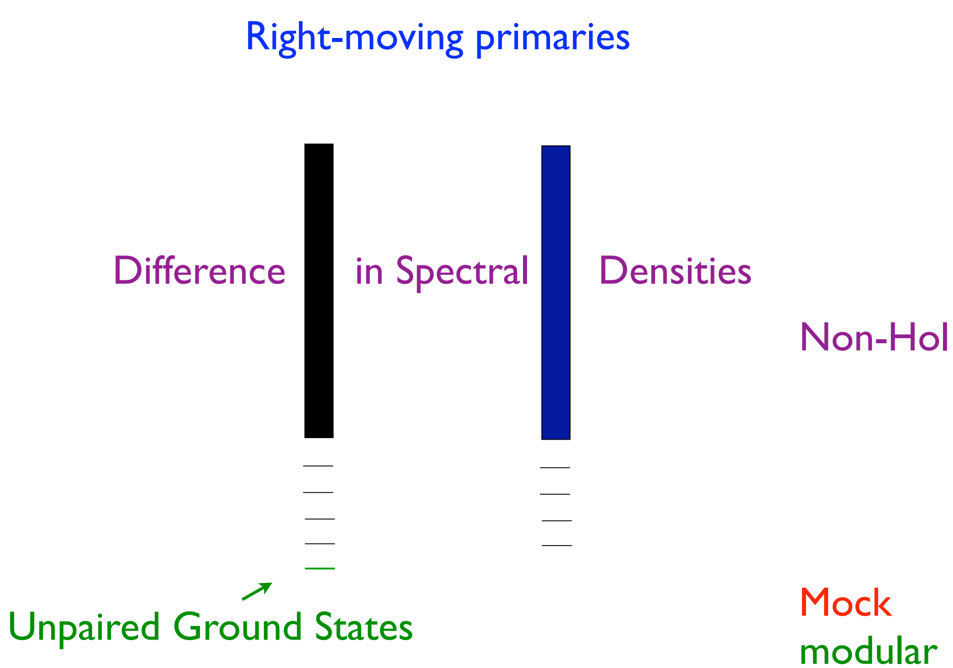
$$210 = 210$$

And, no virtual representations appear.

Math object definition slide

A "mock modular form" arises when a theory has to make a choice between modularity and holomorphy.

Typical example: supersymmetric index in a theory with



$$Z_{\rm ell}(q,\gamma_L) = {\rm Tr}_{RR}(-1)^{F_L} q^{L_0} e^{i\gamma J_L} (-1)^{F_R} \bar q^{\bar L_0}$$

may not quite manage to localize on right-moving ground states, due to mismatch in densities of fermions and bosons at finite energy.

There is an equivalent description of the E8 orbifold theory as a theory of 24 free fermions orbifolded by the action of $(-1)^F$.

* The orbifold has a manifest Spin(24) symmetry.

* Choosing an N=1 superalgebra actually reduces this symmetry to Co_0 .

* Now, to construct an N=4 superalgebra, choose three of the fermions.

The currents of the SU(2) R-symmetry of N=4 are then:

$$J_i = -i\epsilon_{ijk}\lambda_j\lambda_k\,, \qquad i,j,k \in \{1,2,3\}\,.$$

One can check quickly that

$$J_i(z)J_j(0) \sim \frac{1}{z^2}\delta_{ij} + \frac{i}{z}\epsilon_{ijk}J_k(0)\,.$$

By bosonizing the fermions, writing the currents in bosonic language, and writing the N=1 supercurrent in the same way, one can show that one obtains from OPEs

$$J_i(z)W(0) \sim -\frac{i}{2z}W_i(0)$$

The algebra of the supercurrents is then calculated to be:

$$
\begin{aligned}
W_i(z)W_j(0) &\sim \delta_{ij}\left[\frac{8}{z^3}+\frac{8}{z}T(0)\right]+2i\epsilon_{ijk}\left[\frac{2}{z^2}J_k(0)+\frac{1}{z}\partial J_k(0)\right]\\
W(z)W_i(0) &\sim -2i\left(\frac{2}{z^2}+\frac{\partial}{z}\right)J_i(0)\,,\\
J_i(z)W_j(0) &\sim \frac{i}{2z}(\delta_{ij}W+\epsilon_{ijk}W_k)\,.
\end{aligned}
$$

and together with the stress-energy tensor

$$T=-\frac{1}{2}\lambda_\alpha\partial\lambda_\alpha=-\frac{1}{2}\partial H_a\partial H_a$$

these fill out an N=4, c=12 superconformal algebra.

The subgroup of Conway that commutes with the choice of the three-plane is M22.

In this language, the elliptic genus is more naturally rewritten as:

$$Z(q,y) = \frac{1}{2}\sum_{i=2}^{4} \frac{\theta_i(q,y^2)\theta_i(q,1)^{11}}{\eta(q)^{12}}$$

This is equivalent to our earlier expression (from the "E8 viewpoint") by nontrivial identities on Jacobi forms.

The 21 "non N=4" fermions give the 21 of M22. All higher states in the module have transformation laws that can then be derived from first principles.

One can now compute twining genera by an explicit prescription. But also, excitingly:

* One can use the magic of Rademacher sums!

In the original Monstrous Moonshine, all twining functions were given as Hauptmoduln of genus zero subgroups of the modular group.

In Mathieu moonshine, this was not true. However, the genus zero property is equivalent to arising as a Rademacher sum in the Monster case, and this property holds for the twining functions of Mathieu moonshine.

Duncan, Frenkel;
Cheng, Duncan

Here, the twinings also all arise as Rademacher sums.

Math object definition slide

A "Rademacher sum" is a close relative of a Poincare series. You can obtain a modular form by starting with an object invariant under the stabilizer of infinity, and then summing over representatives of right cosets:

$$\sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \backslash \Gamma} \mathrm{e}\left(m \frac{a\tau + b}{c\tau + d} \right) \frac{1}{(c\tau + d)^w},$$

Poincare series for modular form of weight w=2k

Rademacher sums are modified versions of this that improve the convergence properties.

* A similar story holds if instead of enlarging the N=1 supersymmetry to N=4, one enlarges it to N=2.

* The commutant of a choice of U(1) R-current (and consequent N=2 algebra) is M23.

* Again there is a manifest symmetry in the fermionic construction, twinings are computable by a simple prescription, and everything holds together nicely.

This gives examples of mock modular moonshine for M22 and M23 at c=12. The Mathieu case of M24 at c=6 remains mysterious.

In the big picture, this means:

* We now have several examples of mock modular moonshine with explicit modules.

* They arise naturally in the quantum/stringy geometry of certain asymmetric orbifolds.

* Connections to geometric models (CY fourfolds, spin(7) manifolds) require more work, but there are promising avenues...