Mean String Field Theory: Landau-Ginzburg theory for 1-form symmetries

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based on work with
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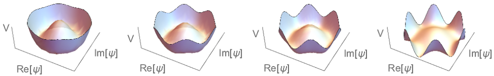
2106.12610 and work in progress.



2021-06-28

Motivation: Enlarged Landau Paradigm.

Landau paradigm: (basis of most condensed matter understanding)

1. Phases of matter are classified by how they represent their symmetries.
2. At a critical point, critical dofs are fluctuations of order parameter.

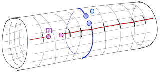
Gapless excitations or degeneracy (in a phase) are Goldstone modes for spontaneously broken symmetries.

Landau-Ginzburg theory is an implementation of this point of view for finding representative states, for understanding gross phase structure; it quantitatively describes phase transitions in large enough dimensions.

Some apparent exceptions:

- topological order [Wegner, Wen]

e.g. deconfined phase of discrete gauge theory, fractional quantum Hall states.



- other deconfined states of gauge theory (*e.g.* Coulomb phase of E&M).

As we enlarge our understanding of what constitutes a symmetry, we can also enlarge the Landau paradigm.

Goal: Our goal here will be to understand how to generalize the idea of Landau-Ginzburg theory.

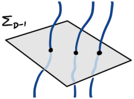
Higher-form symmetries.

[Gaiotto-Kapustin-Seiberg-Willett, Sharpe, Hofman-Iqbal, Lake...]

($D \equiv$ number of spacetime dimensions.)

0-form symmetry:

$\partial^\mu J_\mu = 0$ (*i.e.* $d \star J = 0$) $\implies Q = \int_{\Sigma_{D-1}} \star J$ is independent of time-slice Σ, *i.e.* **is topological.**



Charged particle worldlines can't end

(except on charged operators).

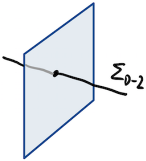
Discrete ($\mathbb{Z}_p$) version: particles can disappear in groups of p.

Charged objects are local operators

$\mathcal{O}(x) \to e^{\mathbf{i}\alpha}\mathcal{O}(x), \quad d\alpha = 0.$

1-form symmetry:

$J_{\mu\nu} = -J_{\nu\mu}$ with $\partial^\mu J_{\mu\nu} = 0$ (*i.e.* $d \star J = 0$) $\implies Q_\Sigma = \int_{\Sigma_{D-2}} \star J$ depends only on the topological class of Σ.



Charged string worldsheets can't end (except on charged operators).

Discrete ($\mathbb{Z}_p$) version: strings can disappear or end in groups of p.

Charged objects are loop operators:

$W[C] \to e^{\mathbf{i}\oint_C \Gamma} W[C], \quad d\Gamma = 0.$

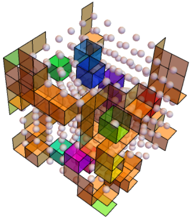
Physics examples of one-form symmetries:

- Maxwell theory with electric charges:
  $J^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} = (d\tilde{A})^{\mu\nu}$ is conserved: $\nabla_\mu J^{\mu\nu} = 0$.
  (Charged operator is the 't Hooft line, $W^E = e^{\mathbf{i}\oint_C \tilde{A}}$.)

- Pure $\mathsf{SU}(N)$ gauge theory
  or $\mathbb{Z}_N$ gauge theory
  or $\mathsf{U}(1)$ gauge theory with charge-N matter
  has a $\mathbb{Z}_N$ 1-form symmetry ('center symmetry').
  (Charged line operator is the Wilson line in the minimal irrep,
  $W[C] = \text{tr} P e^{\mathbf{i}\oint_C A}$.)

- The 3d Ising model

  has a $\mathbb{Z}_2$ 1-form symmetry
  reflecting the integrity of domain walls.
  (Charged line operator is the disorder operator.)



Higher-form symmetries can be broken spontaneously.

[Kovner-Rosenstein, Nussinov-Ortiz, Gaiotto-Kapustin-Seiberg-Willett, Hofman-Iqbal, Lake]

0-form symmetry:

Unbroken phase: correlations of charged operators are short-ranged, decay when the charged object ($S^0 =$ two points) grows.

$\langle \mathcal{O}(x)^\dagger \mathcal{O}(0) \rangle \sim e^{-m|x|}$

($|x| = \text{Area}(S^0(x))$.)

Broken phase for 0-form sym:

$\langle \mathcal{O}(x)^\dagger \mathcal{O}(0) \rangle = \langle \mathcal{O}^\dagger \rangle \langle \mathcal{O} \rangle + ...$

independent of size of S^0.

1-form symmetry:

Unbroken phase: correlations of charged operators are short-ranged, decay when the charged object grows.

$\langle W(C) \rangle \sim e^{-T_{p+1}\text{Area}(C)}$

For E&M, area law for $\langle W^E(C) \rangle$ is the superconducting phase.

Broken phase for 1-form sym:

$\langle W(C) \rangle = e^{-T_p \text{Perimeter}(C)} + ...$

(set to 1 by counterterms local to C: large loop has a vev)

Higher-form symmetries, a fruitful idea:

- Topological order as SSB [Nussinov-Ortiz 06, Gaiotto-Kapustin-Seiberg-Willett 14]
- Photon as Goldstone boson [Kovner-Rosenstein 92, Gaiotto et al, Hofman-Iqbal, Lake 18]
- A new organizing principle for magnetohydrodynamics [Grozdanov-Hofman-Iqbal 16]
- New anomaly constraints on IR behavior of QFT [Gaiotto-Kapustin-Komargodski-Seiberg 17, many others]

Robustness of higher-form symmetries.

We are used to the idea that consequences of emergent (aka accidental) symmetries are only approximate:

Explicitly breaking a 0-form symmetry gives a mass to the Goldstone boson.

Q: The existence of magnetic monopoles with $m = M_{\text{monopole}}$ explicitly breaks the 1-form symmetry of electrodynamics:

$$\partial^\mu J^E_{\mu\nu} = j^{\text{monopole}}_\nu \ .$$

If the photon is a Goldstone for this symmetry, does this mean the photon gets a mass? No!

Cheap explanation #1: By dimensional analysis (take $m_e \to \infty$).
$m_\gamma \to 0$ when $M_{\text{monopole}} \to \infty$.

Cheap explanation #2: By dimensional reduction.
$m_\gamma \overset{[\text{Polyakov}]}{\sim} e^{-M_{\text{monopole}} R} \overset{R\to\infty}{\to} 0.$

Cheap explanation #3: The operators that are charged under a 1-form symmetry are loop operators – they are not local. We can't add non-local operators to the action at all.

Mean String Field Theory

Method of the missing box.

Landau-Ginzburg mean field theory is our zeroth order tool for understanding symmetry-breaking phases and their neighbors.

0-form symmetry : mean field theory

$::$

1-form symmetry : ?

Landau-Ginzburg-Wilson reminder.

Order parameter for $\mathsf{U}(1)$ 0-form symmetry-breaking, $\phi(x) \mapsto e^{\mathbf{i}\alpha}\phi(x)$.

ϕ is a coarse-grained object, this is an effective long-wavelength description.

All local, symmetric terms, organized by derivative expansion

(what else could it be):

$$S_{\text{Landau-Ginzburg-Wilson}}[\phi] = \int d^D x \left(r|\phi|^2 + u|\phi|^4 + \cdots + |\partial\phi|^2 + \cdots \right) \ .$$

One way to make contact with a microscopic Hamiltonian H is by the variational principle:

Product-state ansatz: $|\phi\rangle \equiv \otimes_x |\phi(x)\rangle$.

$H_{LGW}[\phi] = \langle\phi|H|\phi\rangle$ determines specific coefficients.

Working by analogy.

Ingredients:

$\phi(x)$: function from space of **points** to linear rep of G

$\phi(x) \to \phi(x) e^{\mathbf{i}\alpha} \qquad d\alpha = 0$

$\frac{\partial}{\partial x^\mu}\phi(x)$

$\int d^D x$

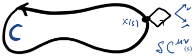
Coupling to bg field: $\mathcal{A} \to \mathcal{A} + d\alpha$

$\frac{\partial}{\partial x^\mu}\phi(x) \rightsquigarrow \left(\frac{\partial}{\partial x^\mu} - \mathbf{i}\mathcal{A}_\mu(x)\right)\phi(x)$

$\psi[C]$: functional from space of **loops** to linear rep of G

$\psi[C] \to \psi[C] e^{\mathbf{i}\oint_C \Gamma} \qquad d\Gamma = 0$

$\frac{\delta}{\delta C^{\mu\nu}}$: area derivative [Migdal, Polyakov]



$\int [dC] \equiv \int [dX] e^{-mL[C]}$

Coupling to background field: $\mathcal{B} \to \mathcal{B} + d\Gamma$

$\frac{\delta}{\delta C^{\mu\nu}(s)}\psi[C] \rightsquigarrow \left(\frac{\delta}{\delta C^{\mu\nu}(s)} - \mathbf{i}\mathcal{B}_{\mu\nu}(x(s))\right)\psi[C]$

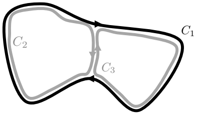
[Soo-Jong Rey 89]

Mean String Field Theory.

All terms consistent with basic principles in (area) derivative expansion:

$$S_{\rm LGW}[\psi] = \int [dC] \left(V\left(|\psi[C]|^2\right) + \frac{1}{2L[C]} \oint ds \frac{\delta\psi^\star[C]}{\delta C_{\mu\nu}(s)} \frac{\delta\psi[C]}{\delta C^{\mu\nu}(s)} + \cdots \right) + S_r[\psi],$$

$v(x) \equiv rx + ux^2 + \cdots,$ $\quad \frac{\delta}{\delta C^{\mu\nu}}$: area derivative [Migdal, Polyakov]



Topology-changing recombination terms:

$$S_r[\psi] = \int [dC_{1,2,3}] \delta[C_1 - (C_2 + C_3)] \left(\lambda\psi[C_1]\psi^\star[C_2]\psi^\star[C_3] + h.c.\right)$$

$+\cdots$ also respect 1-form symmetry.

The action $S_{\rm LGW}[\psi]$ is Wilson-natural* under the following assumptions:

- Invariance under the 1-form symmetry
- **Locality:** a single integral over the center-of-mass position.
- Ordinary rotation and translation invariance
- A certain translation invariance in loop space

Mean String Field Theory.

$$S_{\rm LGW}[\psi] = \int [dC] \left(V\left(|\psi[C]|^2\right) + \tfrac{1}{2L[C]} \oint ds \frac{\delta\psi^\star[C]}{\delta C_{\mu\nu}(s)} \frac{\delta\psi[C]}{\delta C^{\mu\nu}(s)} + \cdots \right) + S_r[\psi]$$

- Important disclaimer: Not at all UV complete, no gravity.
  We expect no connection to 'real' string field theory and are trying to do something much less difficult involving only effective strings.
- A *gauged* version of this model (without S_r) was studied [Soo-Jong Rey, 1989] as a description of 2-form Higgs mechanism, and [Franz 07, Beekman-Sadri-Zaanen 11] as a dual description of a $3+1$d superfluid.
- Still-difficult but well-posed Q: what does this model describe?
  Plausible goal: develop a crude picture of the phase diagram (and transitions) for systems with 1-form symmetries.

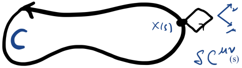
Classical mechanics of Mean String Field Theory.

Equations of motion: $0 = \frac{\delta S[\psi]}{\delta \psi^\star[C]}$

$$0 = -\frac{1}{2} e^{mL[C]} \oint ds \frac{\delta}{\delta C_{\mu\nu}(s)} \left(ds \frac{e^{-mL[C]}}{L[C]} \frac{\delta \psi[C]}{\delta C^{\mu\nu}(s)} \right) + \psi[C] V'(|\psi[C]|^2) + \frac{\delta S_r}{\delta \psi^\star[C]}$$

Requires a boundary condition at small loops.

This BC says: a small loop can shrink to nothing.

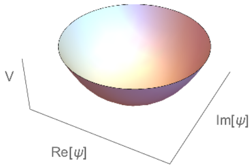


Setting ψ[small, contractible loop] $= g^{-2}$, some constant

- is consistent with the symmetries, since for a small, contractible loop, $C = \partial R$, $\psi[C] \to e^{\mathbf{i} \oint_C \Gamma} \psi[C] = e^{\mathbf{i} \int_R d\Gamma} \psi[C] = \psi[C]$ is neutral, and
- will match nicely to gauge theory in the broken phase.

Unbroken phase.

Let's ignore S_r for a moment, and take $r > 0$:



$$S_{\rm LGW}[\psi] = \int [dC] \left(r\psi[C]\psi^\star[C] + \frac{1}{2L[C]} \oint ds \frac{\delta\psi^\dagger[C]}{\delta C_{\mu\nu}(s)} \frac{\delta\psi[C]}{\delta C^{\mu\nu}(s)} + \cdots \right),$$

$r > 0 \implies \psi[C] \sim 0$. ($\psi = 0$ is not consistent with B.C.)

Ansatz: $\psi[C] = e^{-s(A[C])}$, $A[C] = \min_{\Sigma, \partial\Sigma = C} \text{Area}(\Sigma)$

For large r, A, solution is self-consistently: $(s'(A))^2 = r + \mathcal{O}\left(A^{-1/2}\right)$

$$\implies \boxed{\psi[C] \simeq e^{-\sqrt{r}A[C]}}.$$

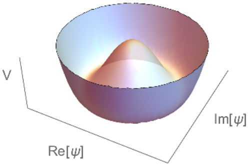
Area law. Confinement.

String tension $= \sqrt{r}$.

Broken phase.

Now consider $r < 0$:
$\psi[C] \sim v$ $(v = \sqrt{\frac{|r|}{2u}})$ "string condensed phase"

[Levin-Wen]



Fluctuations about groundstate:

$$\psi[C] = v \exp\left(\oint_C ds\,(\mathbf{i}t(x(s)) + \mathbf{i}a_\mu(x(s))\dot{x}^\mu(s) + \mathbf{i}h_{\mu\nu}(x(s))\dot{x}^\mu\dot{x}^\nu + \cdots)\right) \quad .$$

Plug back into action (worldline techniques, *e.g.* [Strassler's thesis]):

$$S[\psi] = \frac{v^2}{2}\int d^D x f_{\mu\nu}f^{\mu\nu} + \text{massive modes}, \quad (f \equiv da)$$

- Photon = Goldstone boson (slowly-varying 1-form symmetry transf).
- Gauge coupling is $g^2 = \frac{1}{2v^2}$, determined by stiffness.
- All other unprotected dofs massive.
- Perimeter-law factors $e^{-\oint_C \mathcal{L}} = e^{-mL[C]+\cdots}$ ambiguous by field redefinition of $\psi[C]$.

Topological defects in the broken phase.

Another purpose of ordinary LG theory is to provide an understanding of topological defects of the broken phase [e.g. Mermin 1979].

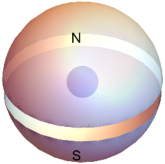
Take $X \subset$ spacetime with $\psi \neq 0$ defines a map $LX \to \mathsf{U}(1)$, where LX is the free loop space, maps $S^1 \to X$.

Defects linked with X are then labelled by homotopy classes of such maps $[LX, \mathsf{U}(1)]$.

If $\pi_1(X) = 0$, then

$$[LX, S^1] = \pi_2(X).$$

For example, take $X = S^{q-1}$ surrounding a codim q locus. This predicts that the magnetic monopole is the *only* topological defect for $G = \mathsf{U}(1)$.



Discrete 1-form symmetries.

To break the $\mathsf{U}(1)$ 1-form symmetry down to a $\mathbb{Z}_p$ subgroup, add

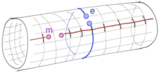
$$S_p = h \int [dC] \psi^p[C] + h.c.$$

In the broken phase, this is $S_p = 2hv^p \sum_C \cos\left(p \oint_C a\right)$.
For $h \gg 1$, minimizing S_p requires $\oint_C a = \frac{2\pi k}{p}, k = 0, \cdots p-1$ for all loops C, including nearby loops $\implies da = 0$.
Introducing a $D-2$-form Lagrange multiplier b to set $pda = 0$ gives

$$\int [d\psi] e^{-S_{\mathrm{LGW}}[\psi] - S_p[\psi]} U^k_{\mathcal{M}_{D-2}} \sim \int [dadb] e^{\mathbf{i}\frac{p}{2\pi}\int b \wedge da} e^{\mathbf{i}k \int_{\mathcal{M}_{D-2}} b}$$

where $U^k_{\mathcal{M}_{D-2}}$ is the 1-form symmetry operator.

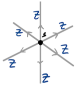
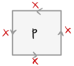
This is an EFT for $\mathbb{Z}_p$ gauge theory. ✓



Regularization on the lattice.

A simple example of a system with 1-form symmetry:

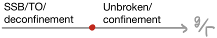
$\mathbb{Z}_p$ gauge theory aka (perturbed) toric code. Cell complex, $\mathcal{H} = \otimes_{\text{links},\,\ell}\mathcal{H}_p$.

$$H_{\text{TC}} = -\infty \sum_{\text{sites},\, s}$$  $$-\Gamma \sum_{\text{plaquettes},\, p}$$  $$-\, g \sum_{\text{links},\, \ell} Z_\ell .$$

$$g = 0:\ |\text{gs}\rangle = \sum_{\text{collections of closed loops, C}} |C\rangle$$

(where $|$  $\rangle \equiv |Z_\ell = -1\rangle$).

$g \sim$ electric string tension.



'Product-state' ansatz: $|\psi\rangle =: e^{\sum_{c,\ \text{connected}} \psi[c] W[c]} : |0\rangle$

where $W[c]|0\rangle = |c\rangle$ creates the loop c. [Related ansatze: Levin-Wen 04, Vidal et al]

$$E[\psi] \equiv \langle \Psi | H_{\text{TC}} | \Psi \rangle$$

$$= \sum_c \left(- \sum_{\partial p \cap c \neq 0} \psi^\star[c]\psi[c + \partial p] + g L[c] \psi^\star[c]\psi[c] \right) - \underbrace{\sum_p \psi[\partial p]}_{\text{small-loop BC}} + \underbrace{H_r}_{\text{recombination}}$$

$0 = \frac{\delta E}{\delta \psi^\star}$ gives a lattice version of the MSFT EoM.

Thoughts about phase transitions

Phase transitions.

If we were to ignore S_r

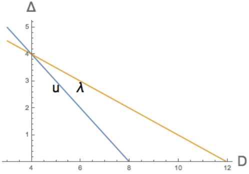
(for example if there were an additional $\psi \to -\psi$ symmetry):

- Continuous mean field transition at $r = r_c = 0$, $\psi[C] \sim e^{-\sqrt{r-r_c}A}$.
  Numerical estimate of this exponent: tension $\sim (r - r_c)^x$
  in $d = 3$ $\mathbb{Z}_2$ case is $x \simeq 1.26 \neq 0.5$ [Hasenbusch 93]
- Dimensional analysis says the upper critical dimension is 8!
  [Parisi 79] using estimate of fractal dimension of random surfaces $= 4$.

But: $S_r \sim \psi^3$.

The fact that the U(1) 1-form symmetry admits a cubic term strongly suggests that the generic transition is first order.

This provides an appealing explanation for the anecdotal evidence from many numerical simulations. *e.g.* [Creutz-Jacobs-Rebbi, ..., Kawai-Nio-Okamoto, Allais, Florio-Lopes-Matos-Penedoñes]



$D = 4$ is special.

Phase transitions.

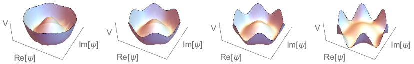
[Important input from Diego Hofman]

Two notions of lower critical dimension: ($D_U^1 = D_U^2 = 2$ for 0-form symms.)

- where Hohenberg-Mermin-Wagner-Coleman forbids symmetry breaking ($D_U^1 = 3$ [Gaiotto et al, Lake])
- where linearly-transforming fields are dimensionless ($D_U^2 = 4$).

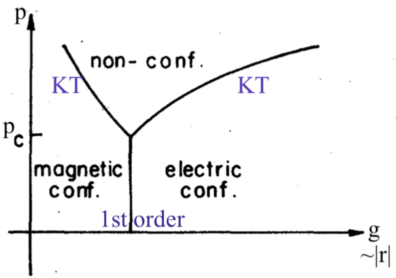
In $D = 4$ there can be a KT-like transition.

The dimension of h in
$S_p = -h \int [dC] \psi^p[C] + h.c.$
is $\Delta_p(g) = \frac{g^2 p^2}{32\pi^2}$ [Kapustin 05].
For large-enough p, $\Delta_p(g)$ passes through 4 at some $g_c < \sqrt{4\pi}$.



Seen in $3+1$d $\mathbb{Z}_p$ lattice gauge theory simulations for large-enough p.

[Elitzur et al, Horn et al, Windey et al, Svetitsky-Yaffe, Creutz-Jacobs-Rebbi]



Final thoughts.

- There is much more to understand about this theory.
  It is not quite under control yet, but likely can be understood.
- Can we find new RG fixed points this way?
- By adding topological and WZW terms, we can describe 1-form SPTs, and realize more general gauge theories as the broken phase.
- What is a gauge theory?
  Contrast with the work of Polyakov, Migdal, Makeenko and others reformulating a particular gauge theory as a field theory in loop space:
  Here, by writing a field theory in loop space, we arrive at some universal properties of gauge theory.

Final thought.

Q: Does the enlarged Landau paradigm
(including all generalizations of symmetries, and their anomalies
– see Shu-Heng Shao's talk on Thursday)
incorporate all phases of matter
(and transitions between them)
as consequences of symmetry?

Landau was even more right than we thought.
This seems to be a fruitful principle.

The end.

Thanks for listening.

Thanks to the organizers of the Strings meeting.