Tensor Network and Holography

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Outline

Some background on Rényi entropy and Entanglement entropy

Tensor networks

Correlation functions

Lattice symmetry - Coxeter group

Summary/Future directions

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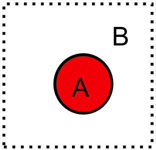
Reduced density matrix and Entanglement Entropy

- Reduced density Matrix

$$\rho = |\Psi\rangle\langle\Psi|, \qquad \rho_A = \mathrm{tr}_B \rho$$



- The entanglement entropy is a limiting case of the Rényi entropy by analytically continuing n.

$$S_{EE} = -\mathrm{tr}_A(\rho_A \ln \rho_A) = \lim_{n\to 1} S_n = -\partial_n \ln \mathrm{tr} \rho_A^n|_{n=1}$$

Holographic EE

Given that it is a difficult task generally to compute EE/Rényi entropy in field theory, how about using holography?

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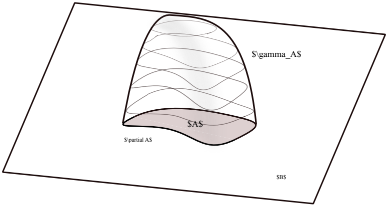
- Ryu-Takayanagi Formula

$$S_1 = \frac{\mathrm{Min}[\mathcal{A}_{d-1}]|_{\partial\mathcal{A}=\partial A}}{4G_N}$$

Ryu & Takayanagi (proof by Casini, Huerta, Myers; Aitor Lewkowycz, Juan Maldacena)



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Tensor network and wavefunctions

Tensor network representation of wavefunction:

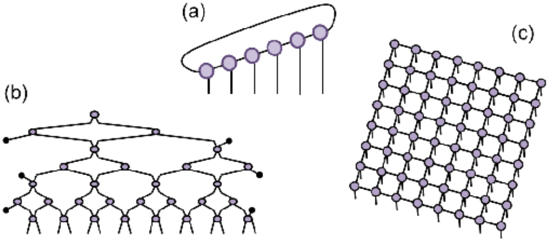

figure courtesy Singh and Vidal, Global symmetries in tensor network states: symmetric tensors versus minimal bond dimension

Mera and RT formula

It was first pointed out by Brian Swingle :

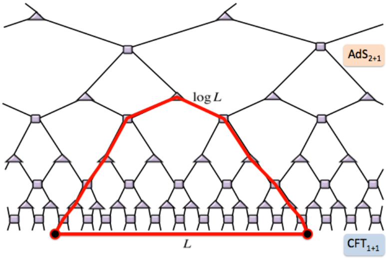

Figure 13: (color online) In scale-invariant $1d$ MERA, two sites separated by distance L in physical space are separated by distance $O(\log L)$ in holographic space. If the MERA reproduces the properties of a $1d$ quantum critical lattice system with a CFT limit, then one can understand the open indices in the TN as a discrete version of a $(1+1)d$-CFT, living at the boundary of a bulk that is a discrete version of a $(2+1)d$-AdS gravitational dual.

figure courtesy Orus, Advances on Tensor Network Theory: Symmetries, Fermions, Entanglement, and Holography

Holographic code – a proposal

Patawski, Yoshida, Harlow, Preskill

- Use Perfect tensors as a building block
- Perfect tensors $T_{a_1 a_2 \cdots a_{2n}}$ – unitary transformation no matter how you split the indices into 2 groups of the same size

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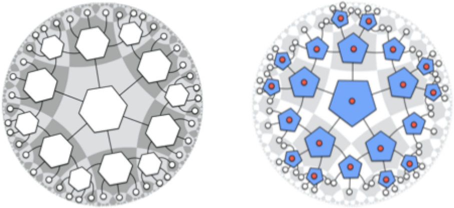
- Use Perfect tensors as a building block
- Perfect tensors $T_{a_1 a_2 \cdots a_{2n}}$ – unitary transformation no matter how you split the indices into 2 groups of the same size
- Products of perfect tensors easily form unitaries and isometries (norm-preserving maps)

$$|I\rangle = T_{Ib}|b\rangle, \qquad T_{Ia}T^{\dagger}_{aJ} = \delta_{IJ}, \qquad D(I) \leq D(a) \tag{1}$$

Holographic code– a proposal

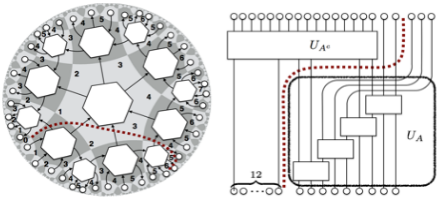


(a) Holographic hexagon state (b) Holographic pentagon code

Figure 4. White dots represent physical legs on the boundary. Red dots represent logical input legs associated to each perfect tensor.

courtesy Pastawski, Yoshida, Harlow,Preskill, Holographic quantum error-correcting codes: Toy models for the bulk/boundary correspondence

Holographic code– RT formula



(a) Circuit interpretation construction (b) Bulk operator reconstruction circuit

Figure 19. Here we illustrate how to construct a unitary circuit interpretation of a holographic state which witnesses the RT entanglement for a simply connected boundary region. (a) The following steps for the construction are illustrated: i) Label the node located at one end of the boundary as 0. ii) Label all other nodes according to the distance from this node. iii) Direct all tensor indices such that the larger label lies to the right. (b) In the example, the circuit interpretation for the network has depth 12. For this reason we provide a full sequential presentation of the circuit interpretation along one side of the geodesic, condensing the remaining 7 gates into U_{A^c}. Note that there are outputs that are produced directly by U_A, without going through U_{A^c} as well as inputs that are fed directly to U_{A^c} without going through U_A.

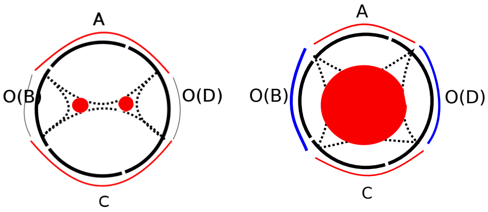
$$\rho_A = U^{\beta_1\cdots\beta_N}_{a_1\cdots a_{|A|}} U^{\dagger\ \beta_1\cdots\beta_N}_{a'_1\cdots a'_{|A|}} |a_1, \cdots, a_{|A|}\rangle\langle a'_1 \cdots, a'_{|A|}| \tag{2}$$

courtesy Pastawski, Yoshida, Harlow,Preskill, Holographic quantum error-correcting codes: Toy models for the bulk/boundary correspondence

Outline

Obvious issues with this code

- Correlation functions? NO connected " few-point" correlation functions between local operators



Correlation functions

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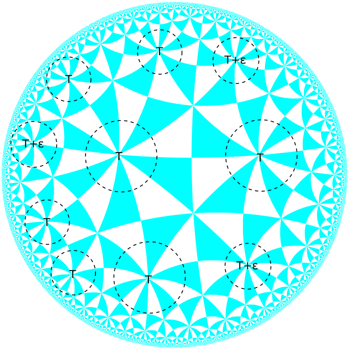
- assume that the perfect tensors are the leading order contribution to this wave-function.
- $$T \to T + \epsilon t, \tag{3}$$

  for some other tensor t, and that ϵ is assumed to be very small.

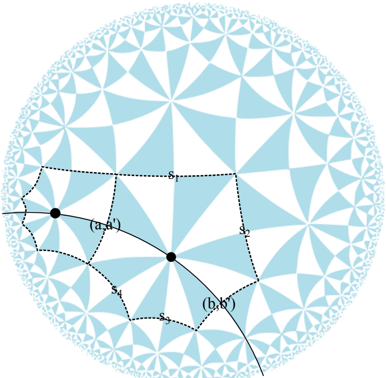
2-point functions

Leading order contribution linear in ϵ?
They vanish.



2-point functions

- The leading contributions come from a geodesic-full of nodes replaced by t!



2-point functions

Natural to define ”Transfer Matrix”, if we assume homogeneity and isometry:

$$(\mathcal{T}_{o_1})^{ab}_{a'b'} = [(T.t^\dagger + t.T^\dagger)_v]^{ab}_{a'b'} = T_{as_1 s_2 bs_3 s_4} t^\dagger_{a' s_1 s_2 b' s_3 s_4} + \text{h.c.}. \quad (4)$$

This gives

$$\langle O_1 O_2 \rangle = O_{1\,aa'} \mathcal{T}^{aa'}_{bb'} \mathcal{T}^{bb'}_{cc'} \cdots \mathcal{T}^{ss'}_{tt'} O_{2\,tt'} \quad (5)$$

Diagonalizing $\mathcal{T}$, gives

$$(U.\mathcal{T}.U^\dagger)^{aa'}_{bb'} = \lambda_{aa'} \delta^{aa'}_{bb'} \quad (6)$$

conformal dimension

The conformal dimension of operators are thus given by

$$\langle O^{aa'}(x_1)O^{aa'}(x_2)\rangle = \epsilon^{\gamma}(\lambda_{aa'})^{\gamma}, \ \Delta_{aa'} = -\log(\lambda\epsilon), \ \gamma \sim \log|x_2 - x_1| \tag{7}$$

- Some extra conditions: e.g. for the 6 leg tensor code $\mathcal{T}_{o_i}$ and $\mathcal{T}_{o_j}$ has the same spectra.

e.g.

$$(\mathcal{T}_{o_2})^{ab}_{a'b'} = T_{s_1\,a\,s_2 s_3\,b\,s_4} t^{\dagger}_{s_1\,a'\,s_2 s_3 b'\,s_4} + \mathrm{h.c.} \tag{8}$$

For an [L,M] lattice, geodesics are most naturally running across the L-gon in $L/2$ different ways.

3-pt function

The same game can be played for three point function. Applying the same argument, the leading contribution would be given by three geodesics connected at a point.

- if the bulk tensor network is a discretization of the AdS space, one could find an optimal point that minimizes

$$S(3) = \sum_{i=1}^{3} \Delta_i \gamma_i(z), \tag{9}$$

  where Δ_i is the conformal dimension of the three operators.

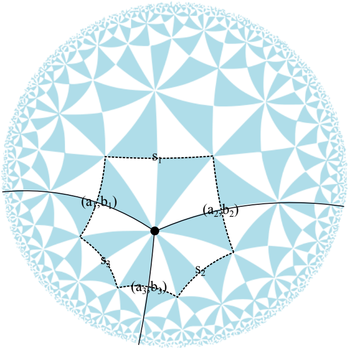
- In ads space, this is basically

$$\gamma = \log \frac{z\epsilon}{z^2 + (x - x_i)^2}, \tag{10}$$

  which is the geodesic distance from (x_i, ϵ) to some point (x, z) in the bulk.

3-pt function

- In ads-space, the 3-point correlation function would automatically satisfy the universal form enforced by conformal symmetry if one finds the appropriate point (x, z). Janik, Surowka, Wereszczynski



Fusion matrix

This shows that one can define a fusion matrix

$$\mathcal{F}^{(a_1,b_1),(a_2,b_2),(a_3,b_3)} =$$

$$(T_{a_1 s_1 a_2 s_2 a_3 s_3} t^{\dagger}_{b_1 s_1 b_2 s_2 b_3 s_3} + t_{a_1 s_1 a_2 s_2 a_3 s_3} T^{\dagger}_{b_1 s_1 b_2 s_2 b_3 s_3}). \quad (11)$$

where the position of the contracted index s depends on the orientation of node where the three geodesics meet.

Outline

Coxeter Group

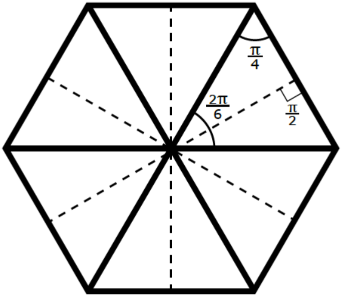
- To describe these lattices systematically, it is natural to use the classification of these lattices based on the Coxeter group.

Coxeter Group

- One basic feature of the coxeter group is to specify the angles in the triangulation. [p,q,r]
- Hyperbolic space

$$\frac{\pi}{p} + \frac{\pi}{q} + \frac{\pi}{r} < \pi \tag{12}$$

- the lattice preserves all the symmetries generated by reflection across the planes defining the triangulation



Lattice symmetry - Coxeter group

- identify polytopes with nodes, and edges (codimension one surfaces) with legs
- e.g. the HAPPY hexagonal code can be identified with the [4,6,2] lattice
- allows more detailed analysis of the lattice structure : e.g. computing code distance for a general lattice
  1. $\frac{N_{bulk}}{N_{boundary}}([6,8]) \to 0.2508$
  2. $\frac{N_{bulk}}{N_{boundary}}([6,10]) \to 0.2504$
  3. $\frac{N_{bulk}}{N_{boundary}}([6,16]) \to 0.25$

BTZ

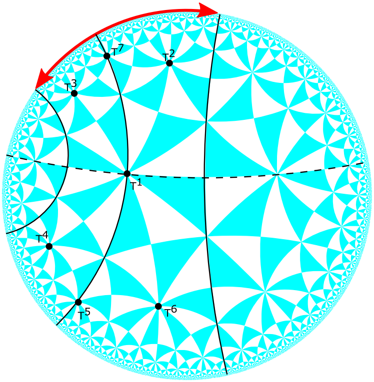
Symmetries allow us to build a BTZ black hole, which is an orbifold:

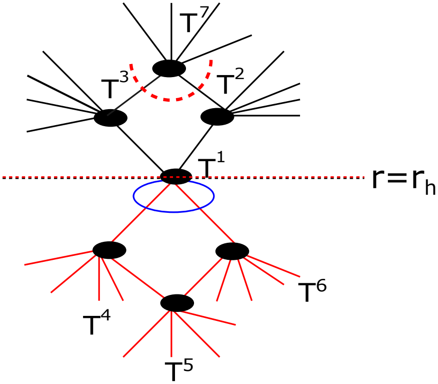
$$ds^2 = \frac{dz^2 + dx^2 + dt^2}{z^2},$$

where

$$z = \frac{r_h}{r} \exp(2\pi r_+ \phi), \qquad x + it = \frac{r}{r_h} \exp(2\pi r_+ (\phi + i\tau))$$

BTZ bh





Recover two RT surfaces, one wrapping the horizon, the other not. – same result recovered using Random tensors

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- Is this capturing ”quantum” gravity?
- higher point functions???
- Rényi entropy S_n —- no n dependence! – a consequence of the unitaries
- isometries and the stress tensor using the Coxeter group???
- time dynamics??? e.g. recover time-like geodesics??? sending shock waves??? black holes and scrambling????

Thank you!!