ENTANGLEMENT, CAUSALITY, HOLOGRAPHY

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Based on: VH, H. Maxfield, M. Rangamani, & E. Tonni: 1306.4004 & 1306.4324,
VH: 1406.4611,
and W.I.P. w/ M. Headrick, VH, A. Lawrence, & M. Rangamani

Motivation

- AdS/CFT correspondence:
  - Can provide invaluable insight into strongly coupled QFT & QG
  - To realize its full potential, need to further develop the dictionary…
- Natural expectation:
  - Physically important / natural constructs one side will have correspondingly important / natural duals on the other side…
- Recent progress in QI vs. QG
  - Fundamental quantum information constructs (e.g. entanglement) seem to be intimately related to geometry!
- Hence study natural geometrical / causal constructs in bulk.
- Useful tool in defining new quantities: general covariance…

OUTLINE

- Entanglement wedge & Causal wedge

[Headrick, VH, Lawrence & Rangamani, to appear '14] ↘

[VH&Rangamani '12; VH,MR,Tonni, '13]

- Strip wedge, Rim wedge

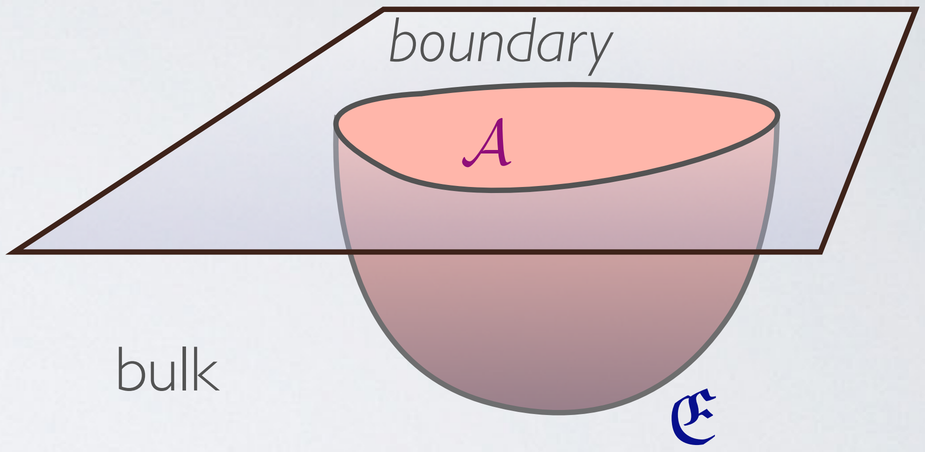
[VH, '14]

- Poincare wedge

Holographic Entanglement Entropy

Proposal [RT=Ryu & Takayanagi, '06] for *static* configurations:

In the bulk EE $S_{\mathcal{A}}$ is captured by the area of minimal co-dimension-2 bulk surface $\mathfrak{E}$ *at constant t* anchored on $\partial\mathcal{A}$ & homol. to $\mathcal{A}$.



$$S_{\mathcal{A}} = \min_{\partial\mathfrak{E}=\partial\mathcal{A}} \frac{\text{Area}(\mathfrak{E})}{4\,G_N}$$

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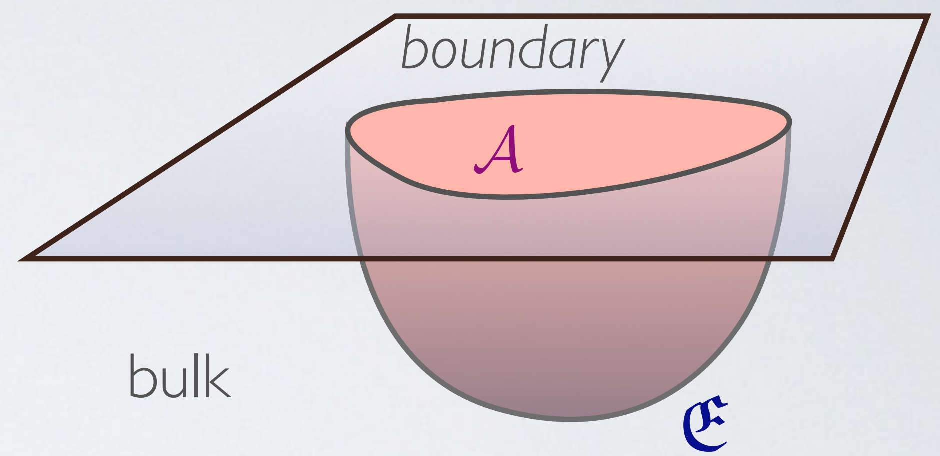

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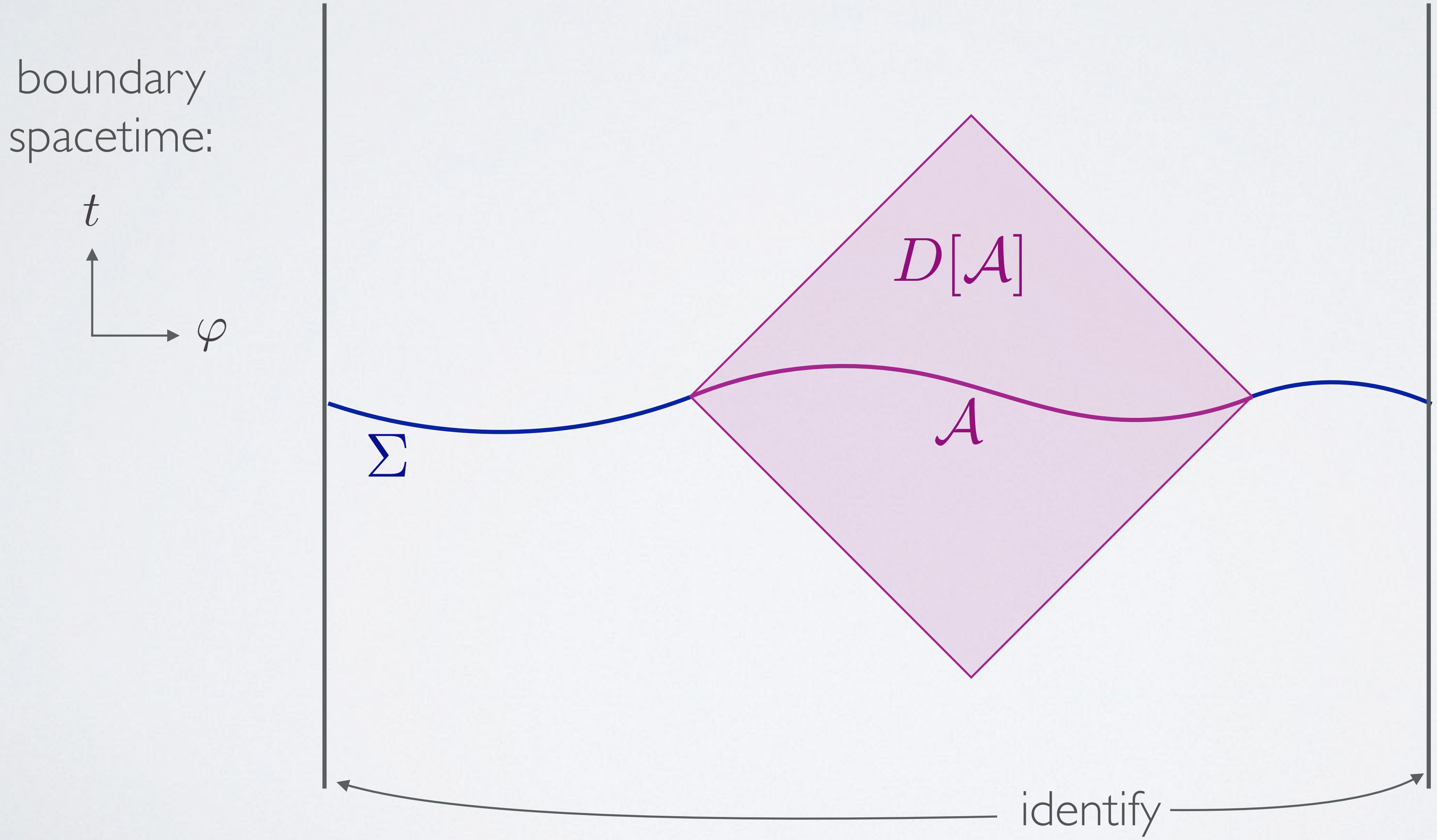
In *time-dependent* situations, covariantize: [HRT=VH, Rangamani, Takayanagi '07]

- ✶ minimal surface → extremal surface
- ✶ equivalently, $\mathfrak{E}$ is the surface with zero null expansions; (cf. light sheet construction [Bousso '02])
- ✶ equivalently, maximin construction: maximize over minimal-area surface on a spacelike slice [Wall '12]

CFT causal restriction

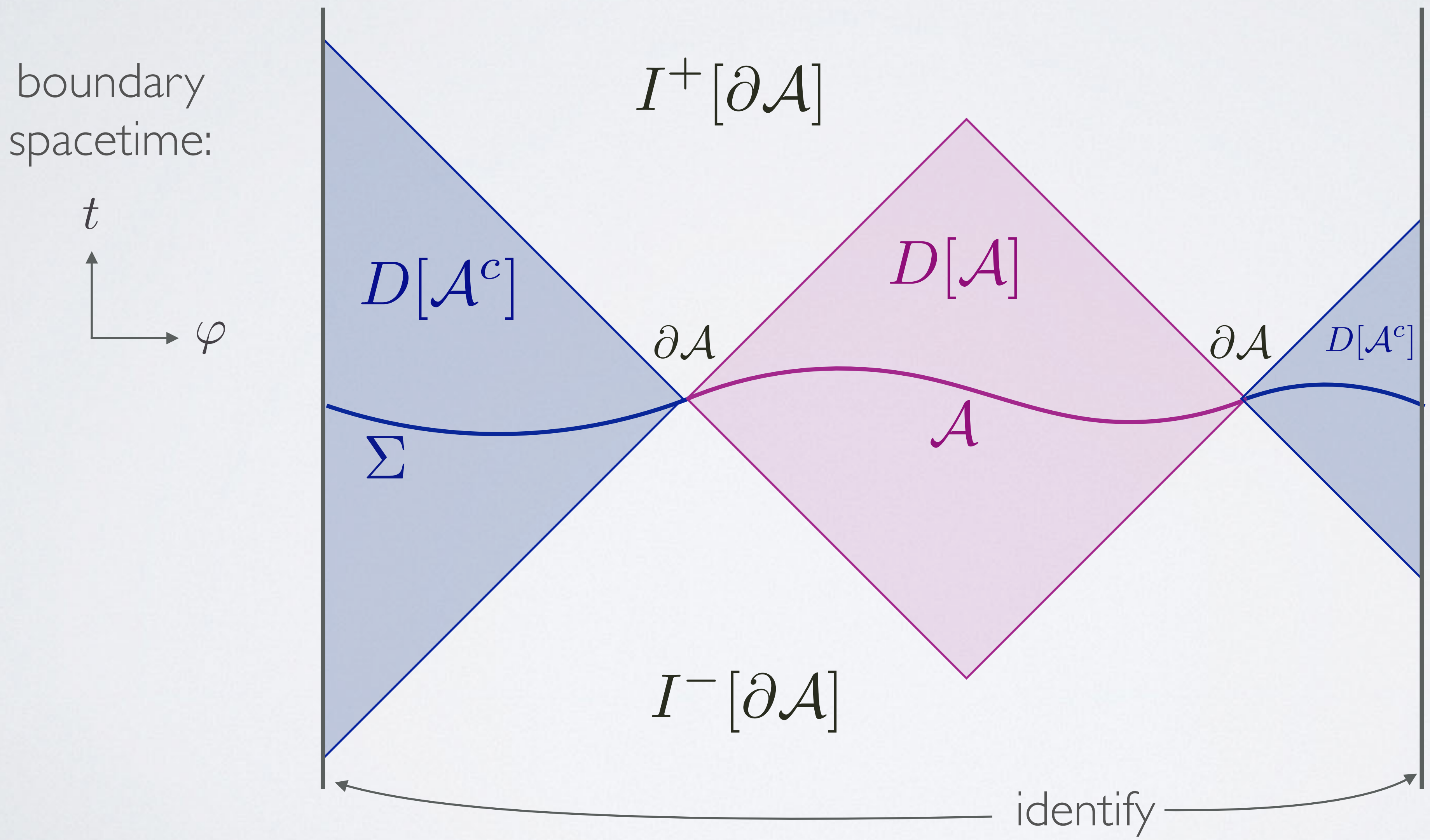
- Entanglement entropy $S_{\mathcal{A}}$ only depends on $D[\mathcal{A}]$ and not on Σ.



CFT causal restriction

- Entanglement entropy $S_{\mathcal{A}}$ only depends on $D[\mathcal{A}]$ and not on Σ.
- Natural separation of boundary spacetime into 4 regions:

$$\partial\mathcal{M} = D[\mathcal{A}] \cup D[\mathcal{A}^c] \cup I^-[\partial\mathcal{A}] \cup I^+[\partial\mathcal{A}]$$



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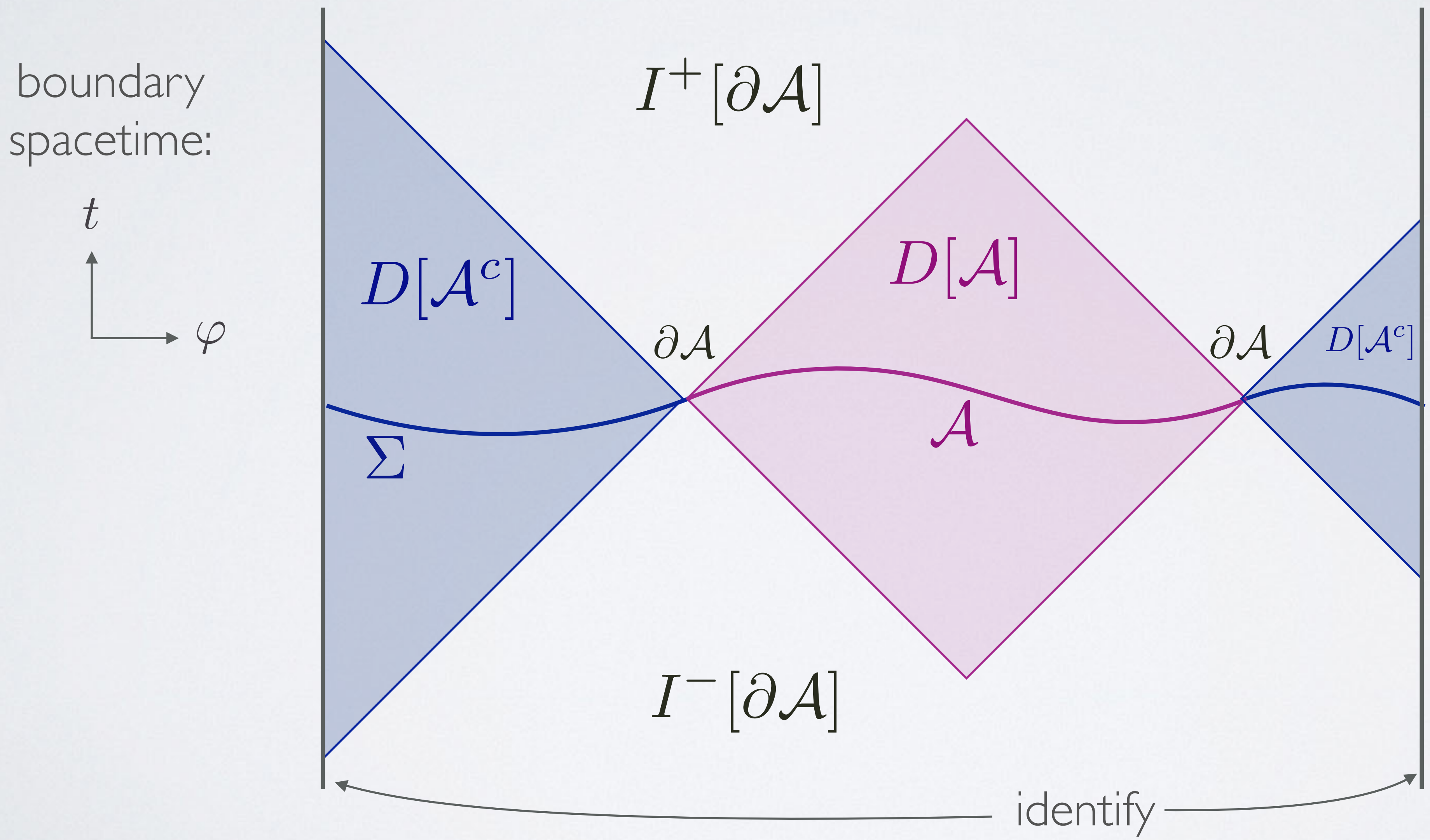


- EE should not be influenced by any change to state within $D[\mathcal{A}]$ or $D[\mathcal{A}^c]$.

Causal Wedge construction

Bulk causal region corresponding to $D[\mathcal{A}]$:

- Bulk causal wedge $\blacklozenge_{\mathcal{A}}$

$$\blacklozenge_{\mathcal{A}} \equiv J^{-}[D[\mathcal{A}]] \cap J^{+}[D[\mathcal{A}]]$$

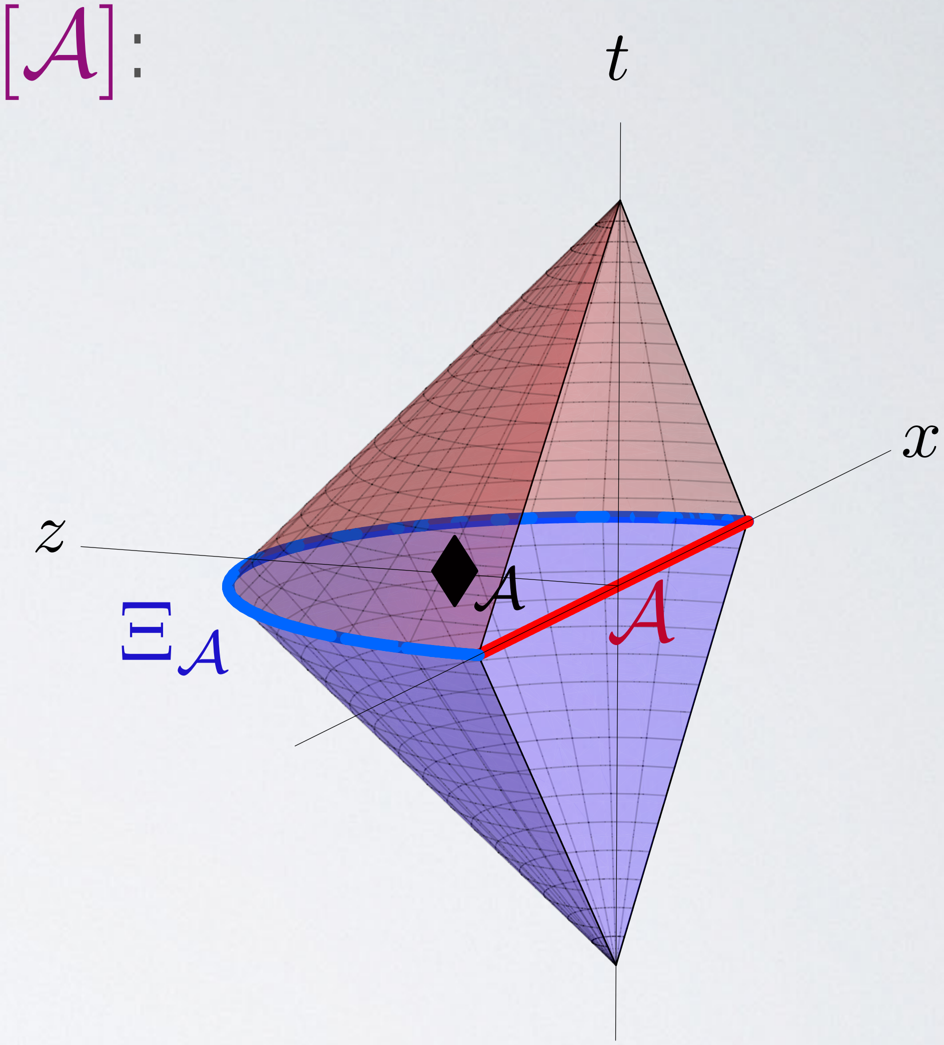
= { bulk causal curves which begin and end on $D[\mathcal{A}]$ }

- Causal information surface $\Xi_{\mathcal{A}}$

$$\Xi_{\mathcal{A}} \equiv \partial J^{-}[D[\mathcal{A}]] \cap \partial J^{+}[D[\mathcal{A}]]$$

- Causal holographic information $\chi_{\mathcal{A}}$

$$\chi_{\mathcal{A}} \equiv \frac{\text{Area}(\Xi_{\mathcal{A}})}{4\, G_N}$$



[VH&Rangamani '12]

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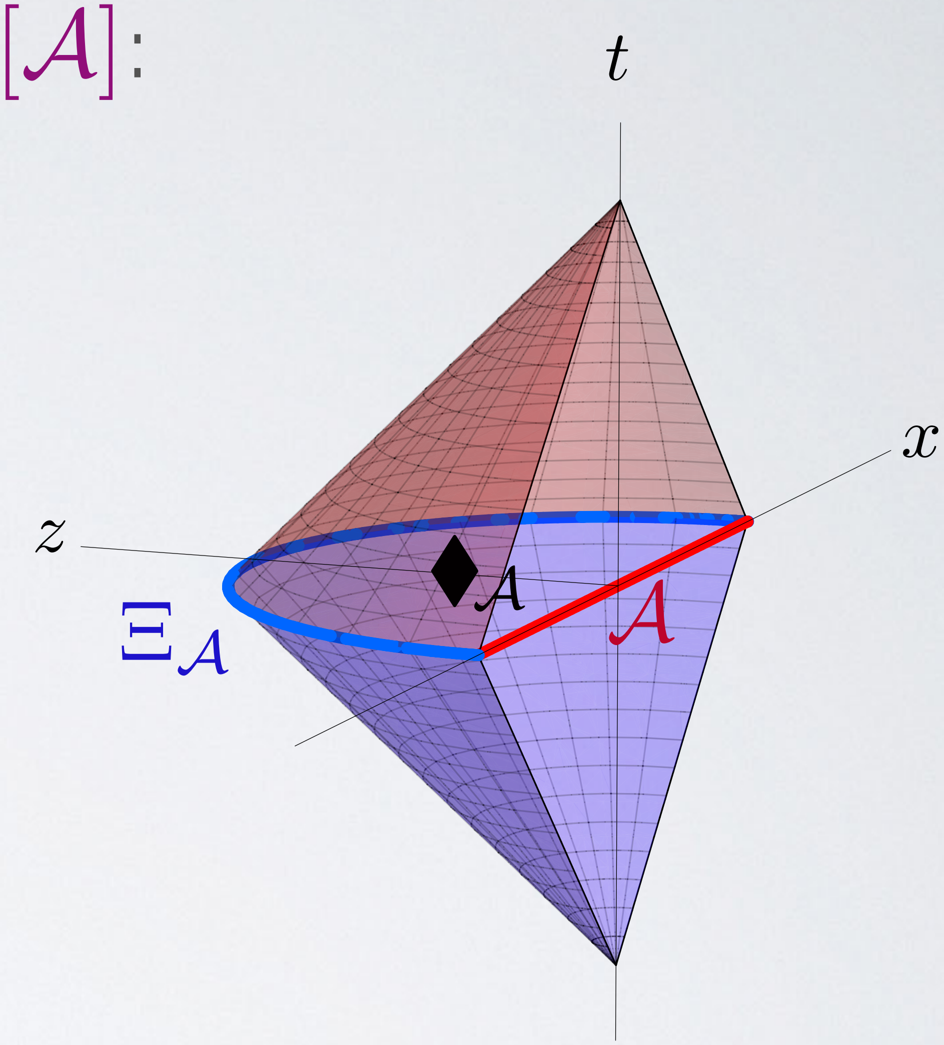


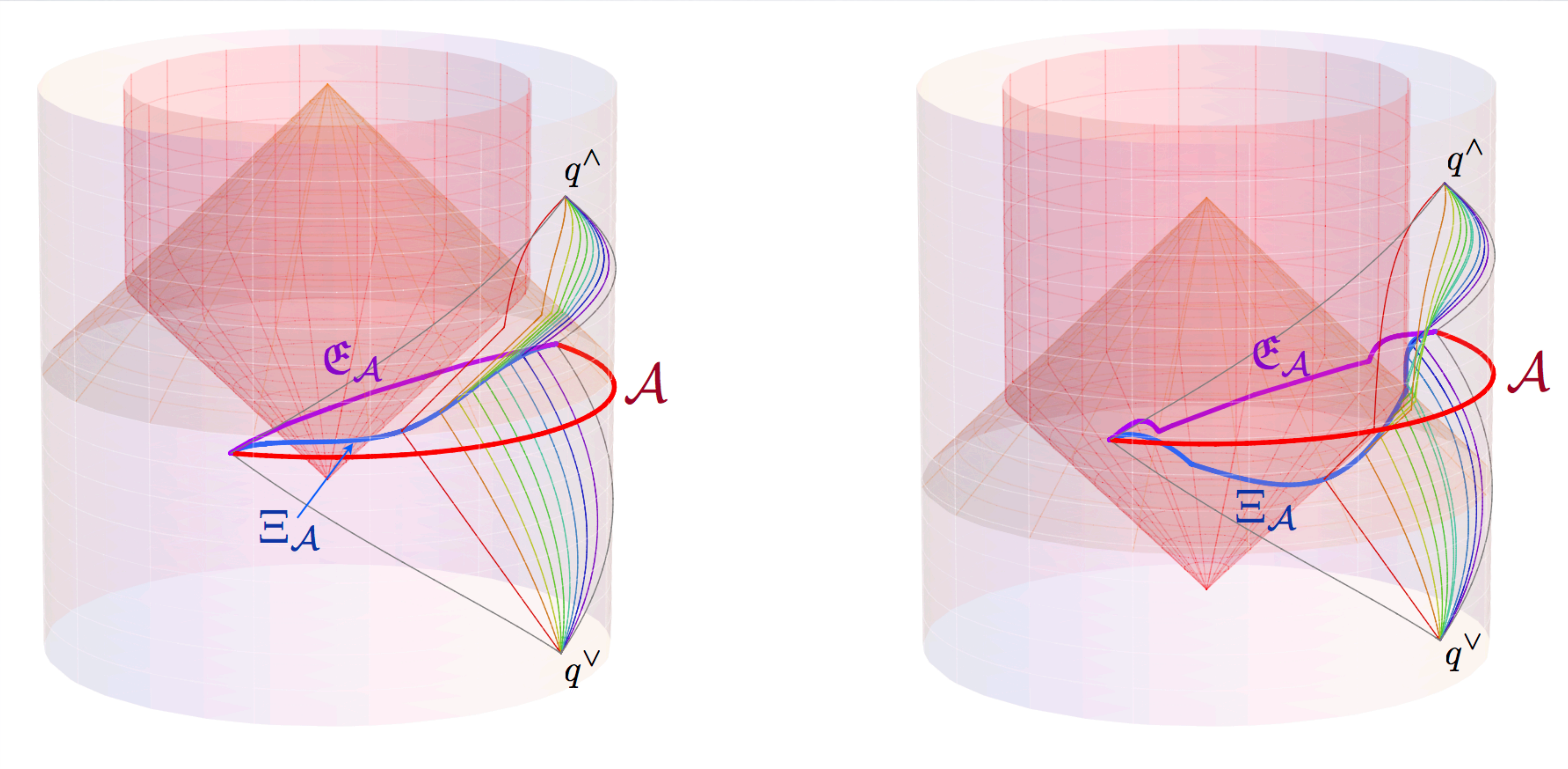
[VH&Rangamani '12]

- In special cases, $\Xi_{\mathcal{A}} = \mathfrak{E}_{\mathcal{A}} \quad \Rightarrow \quad \chi = S_{\mathcal{A}}$, but in general they differ.

- Important Q: what is their interpretation within the dual CFT ?

Causal wedge profile in Vaidya-AdS

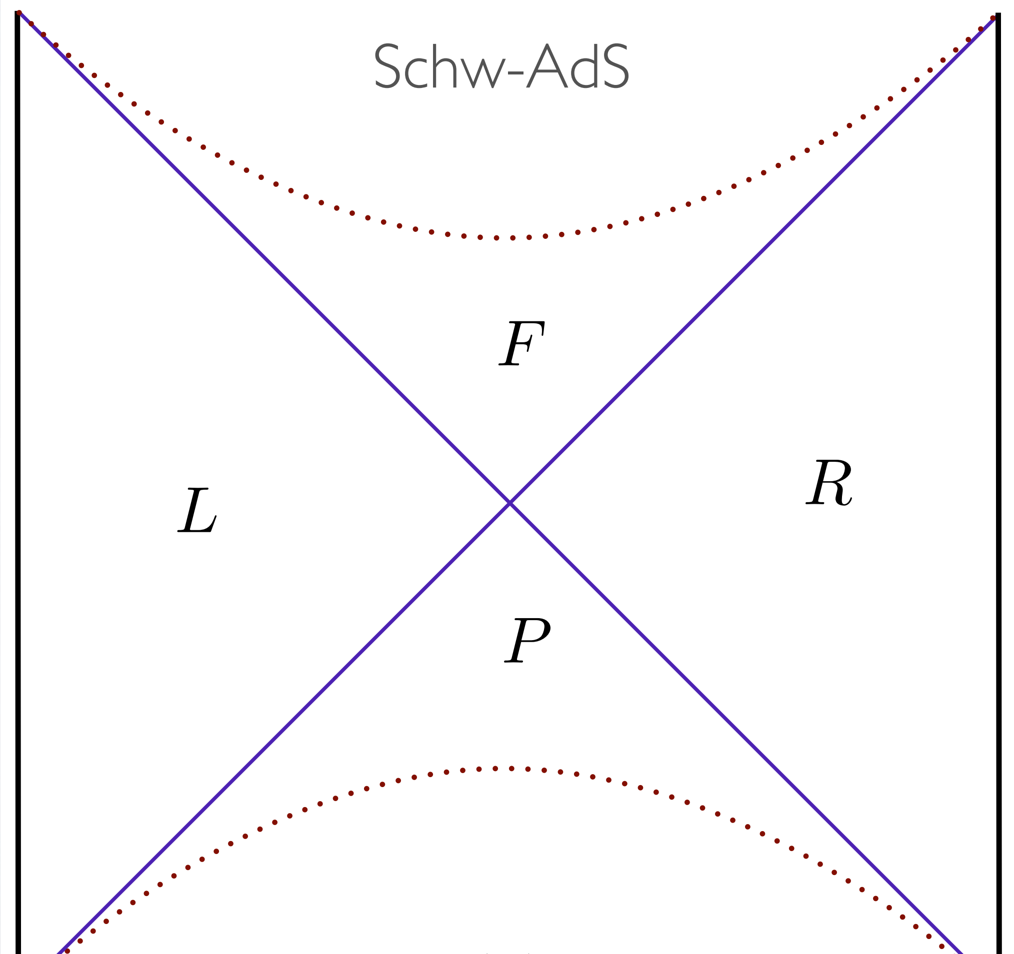
- Extremal surface cannot lie inside the causal wedge [VH&MR; Wall]
  - But in special cases $\mathfrak{E}_{\mathcal{A}}$ can be null related to $\Xi_{\mathcal{A}}$, e.g.:



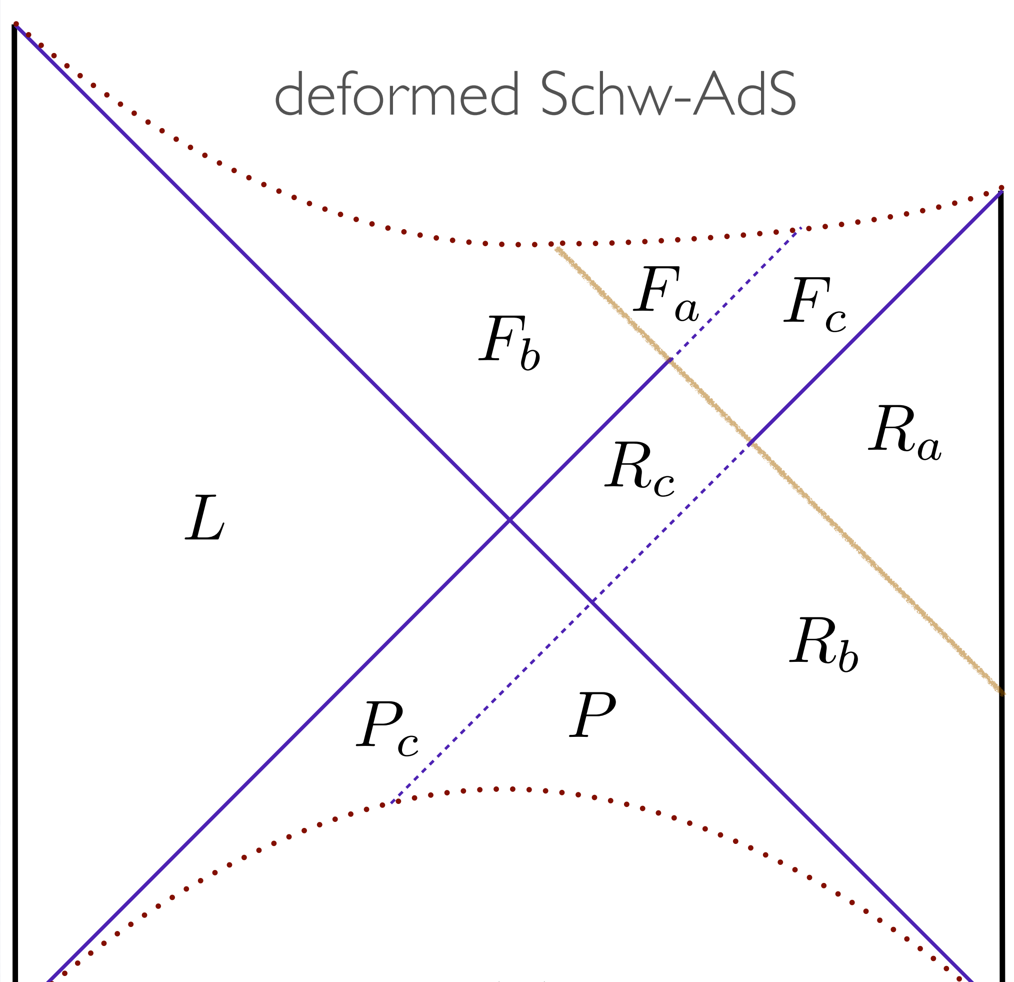
- Danger: is it possible to deform $\mathfrak{E}_{\mathcal{A}}$ s.t. timelike-separated from $\Xi_{\mathcal{A}}$?

Dynamical eternal BH geometry

- Extremal surfaces cannot penetrate static BH event horizon [VH, '12]
- But they can penetrate dynamical BH event horizon [cf. Vaidya-AdS]
- Danger: can surface from on R bdy reach to causal communication w/ L bdy?



(a)



(b)

Bulk causal restriction

- A-priori, boundary causality of EE is not manifest in the bulk:
  - Need: extremal surface to lie outside the causal wedge…
  - In eternal BH geometry, w/ 2 boundaries, need extremal surface anchored on R bdy to not reach into causal contact w/ L bdy…

Bulk causal restriction

- A-priori, boundary causality of EE is not manifest in the bulk:
  - Need: extremal surface to lie outside the causal wedge… ✔
  - In eternal BH geometry, w/ 2 boundaries, need extremal surface anchored on R bdy to not reach into causal contact w/ L bdy… ✔

- We can show that both are satisfied robustly.

  [Headrick, VH, Lawrence, & Rangamani, WIP]

  - Generically, $\mathfrak{E}_{\mathcal{A}}$ is spacelike-separated from $\Xi_{\mathcal{A}}$
  - (otherwise violates Raychaudhuri equation)

- This leads us to the notion of Entanglement Wedge:

Entanglement wedge

- Boundary spacetime separation:

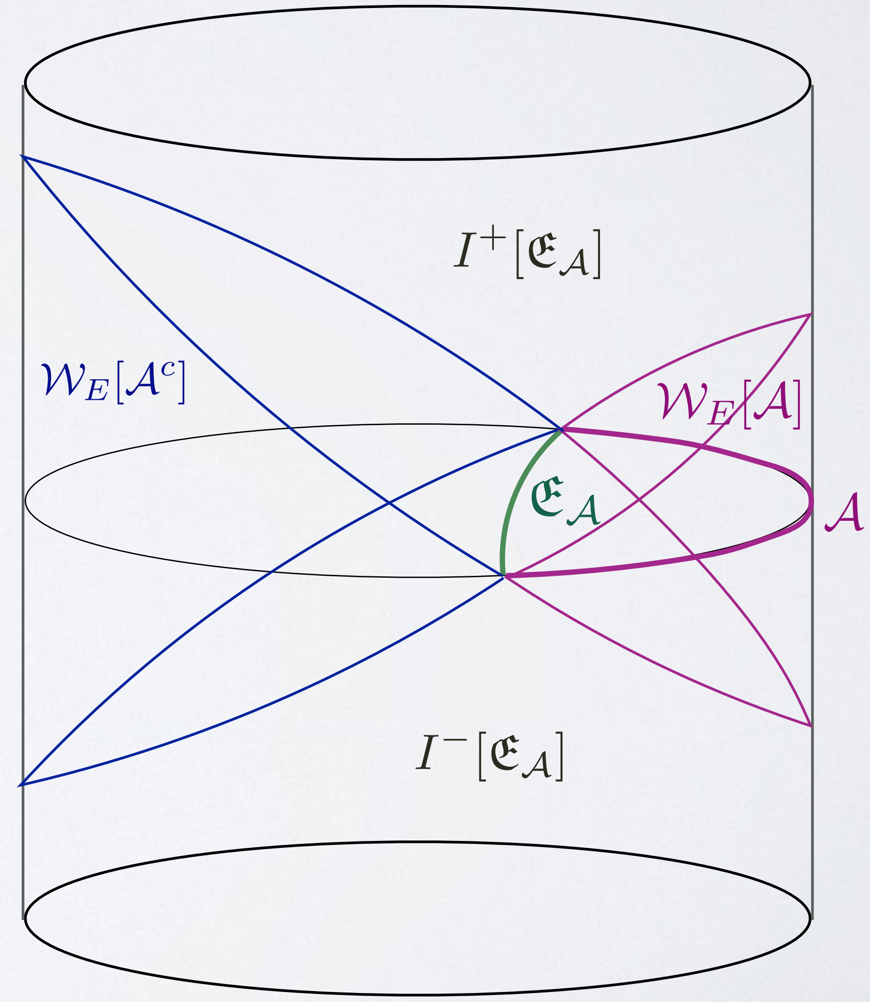
$$\partial\mathcal{M} = D[\mathcal{A}] \cup D[\mathcal{A}^c] \cup I^-[\partial\mathcal{A}] \cup I^+[\partial\mathcal{A}]$$

- This naturally induces a corresponding separation into 4 bulk regions:

$$\mathcal{M} = \mathcal{W}_E[\mathcal{A}] \cup \mathcal{W}_E[\mathcal{A}^c] \cup I^-[\mathfrak{E}_\mathcal{A}] \cup I^+[\mathfrak{E}_\mathcal{A}]$$

entanglement wedge of $\mathcal{A}$

- $\mathcal{W}_E[\mathcal{A}]$ ends on $D[\mathcal{A}]$
- contains the causal wedge $\blacklozenge_\mathcal{A}$
- generated by null geodesics normal to $\mathfrak{E}_\mathcal{A}$



Bulk dual of reduced density matrix?

?: What bulk region is reconstructable from $\rho_{\mathcal{A}}$?

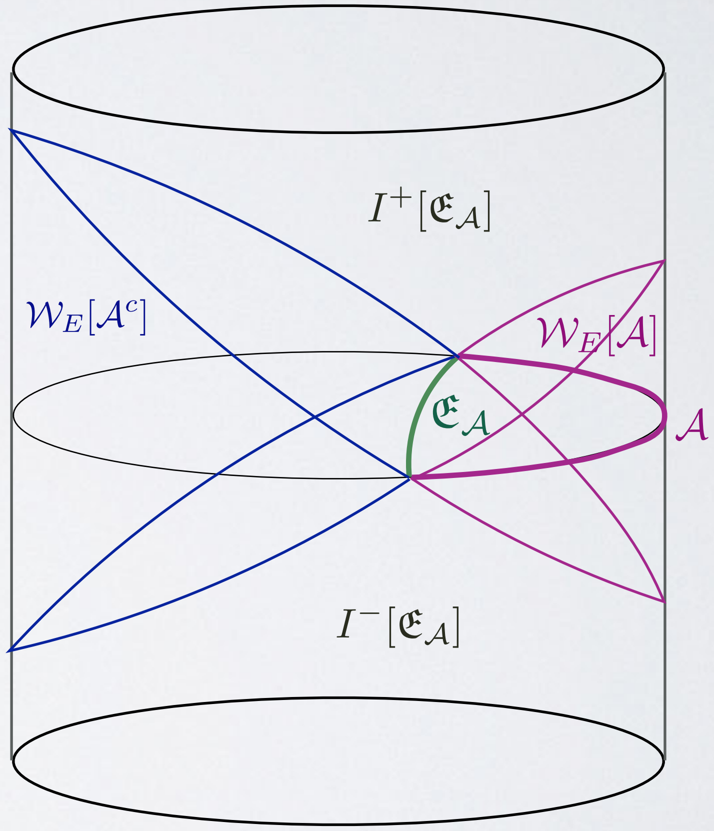
- Causal wedge $\blacklozenge_{\mathcal{A}}$?

  [Bousso, Leichenauer, & Rosenhaus, '12]

- Entanglement wedge $\mathcal{W}_E[\mathcal{A}]$?

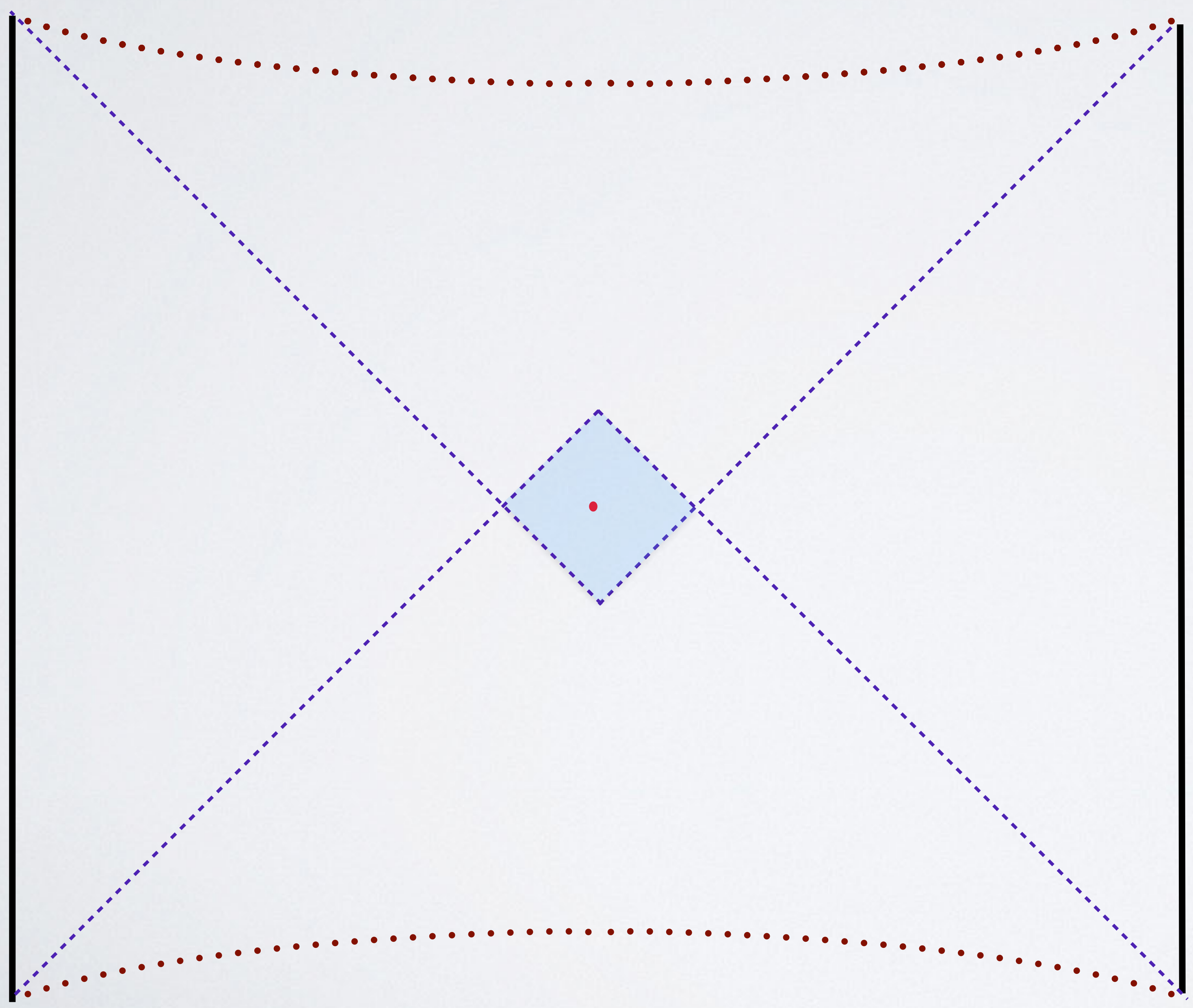
  our conjecture [HHLR]. cf. also:

  [Czech, Karczmarek, Nogueira, Van Raamsdonk, '12; Wall, '12]



Entanglement wedge in deformed SAdS

In deformed eternal Schw-AdS, (compact) extremal surface corresponding to $\mathcal{A} = \Sigma_L$ or $\mathcal{A} = \Sigma_R$ must lie in the 'shadow region'



i.e. causally disconnected from both boundaries…

(for static Schw-AdS, shadow region = bifurcation surface)

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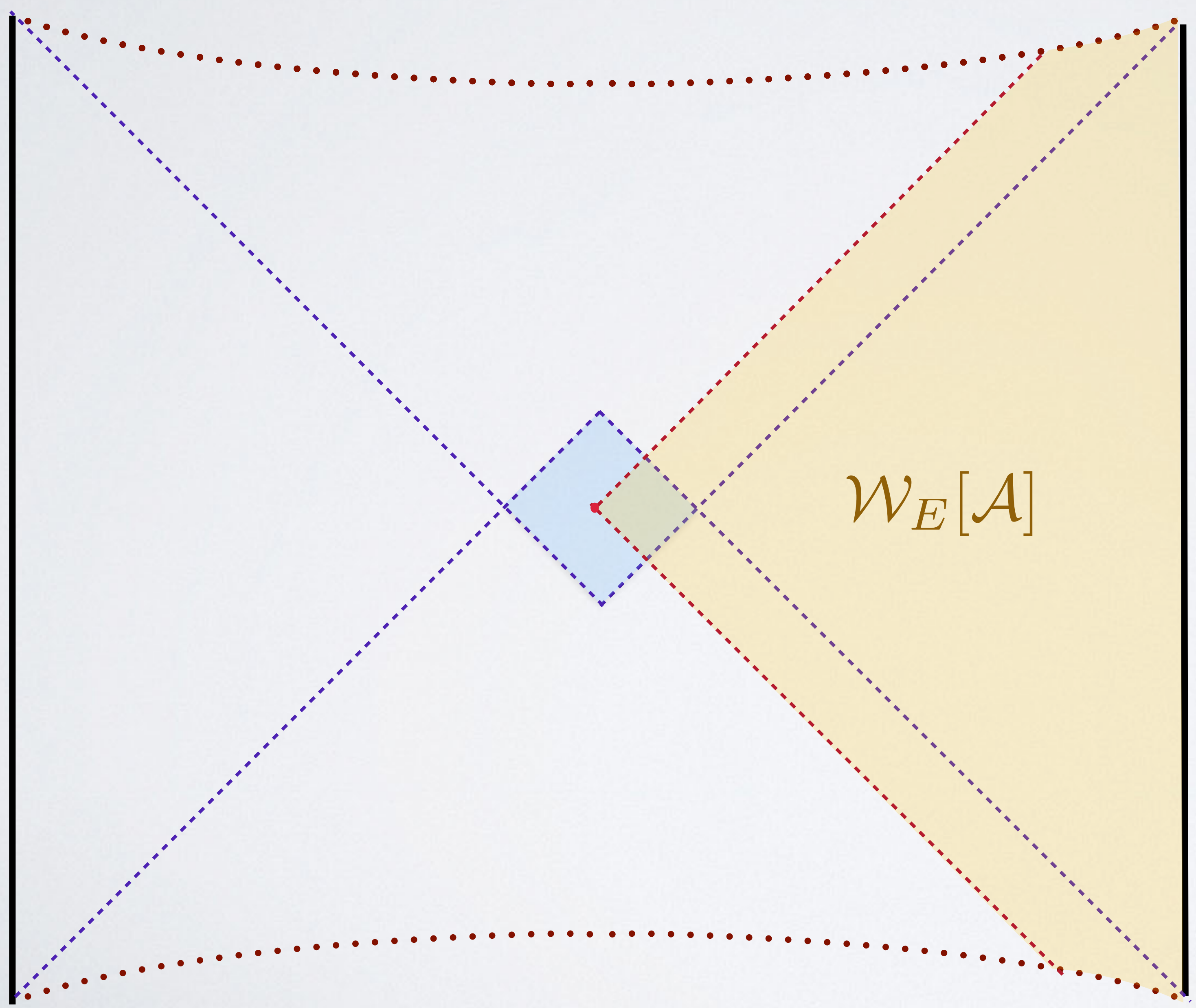
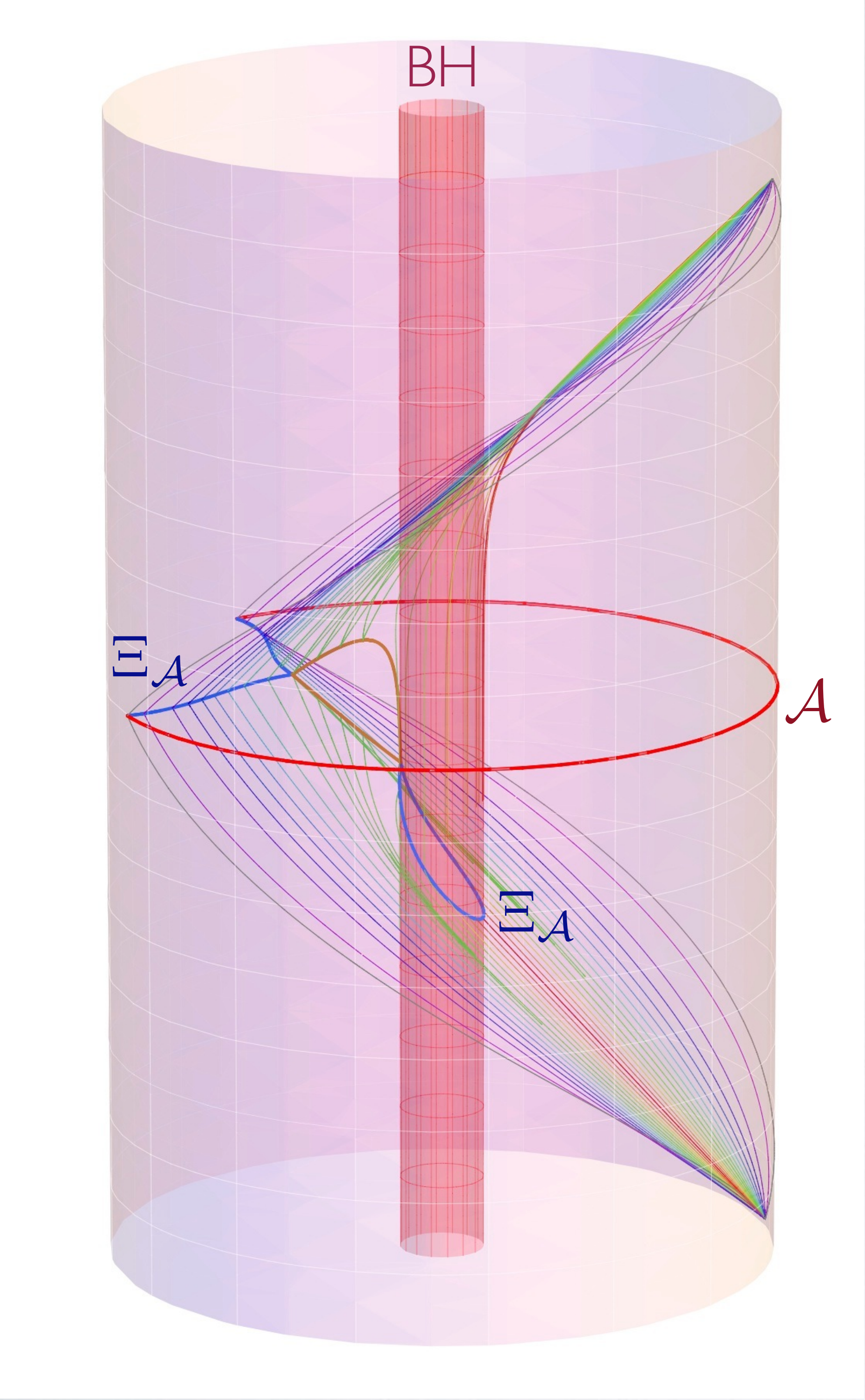


i.e. causally disconnected from both boundaries…

(for static Schw-AdS, shadow region = bifurcation surface)

$\Rightarrow$ Entanglement wedge extends past event horizon

Causal wedge can have holes



- Important implication for entanglement:
  - whenever $\mathcal{A}$ is large enough for $\Xi_{\mathcal{A}}$ to have two disconnected pieces, there cannot exist a single connected extremal (minimal) surface $\mathfrak{E}_{\mathcal{A}}$ homologous to $\mathcal{A}$!
  - in such cases, $\Rightarrow \; S_{\mathcal{A}} = S_{\mathcal{A}^c} + S_{\rm BH}$

    (saturates Araki-Lieb inequality)

    → *entanglement plateau*

    [VH, Maxfield, Rangamani, Tonni, '13]

    → two components to entanglement
  - Causal wedge argument guarantees this even for generic time-dependent BHs.

OUTLINE

- Entanglement wedge & Causal wedge
- Strip wedge, Rim wedge [VH,'14]
- Poincare wedge

Hole-ography

[Balasubramanian, Chowdhury, Czech, de Boer, & Heller, `13]

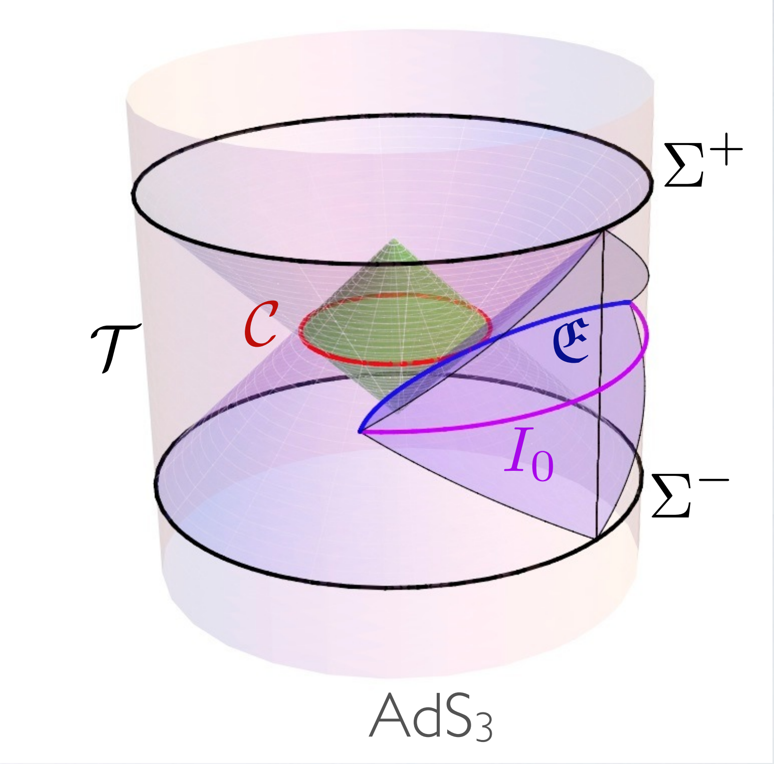
- Characterize `collective ignorance' of a family of observers:

Bulk observers:

restrict to exterior of a hole (w/ rim $\mathcal{C}$)

Boundary observers:

restrict to interior of a time strip $\mathcal{T}$



- [BCCdBH] conflated the two notions; but in general they are distinct, the construction is not reversible…
- Initially called this "residual entropy" (=E), later renamed to "differential entropy" [cf. Myers, Rao, & Sugishita, `14]
- [BCCdBH] present a formula for E:

$$E = \sum_k [S(I_k) - S(I_k \cap I_{k+1})] \quad \to \frac{1}{2}\int_0^{2\pi} d\varphi \, \frac{dS(\alpha)}{d\alpha} \quad = \frac{\text{Area}(\mathcal{C})}{4\,G_N}$$

Hole-ography

- However, the [BCCdBH] construction has severe limitations:
  - valid only in 3 dimensional bulk
  - valid only for pure AdS
  - valid only for $\mathcal{C}$ at constant t (or time-symmetric $\mathcal{T}$)
  - valid only for sufficiently 'tame' setup
- Upshot: differential entropy given by $E \sim \int d\varphi \, \frac{dS(\alpha)}{d\alpha} \, |_{\alpha(\varphi)}$ does NOT capture residual entropy.
- Q: is there a more robust notion of residual entropy, applicable for any asymp. AdS geometry in any dimension, & for any region specification?

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- Q: is there a more robust notion of residual entropy, applicable for any asymp. AdS geometry in any dimension, & for any region specification?

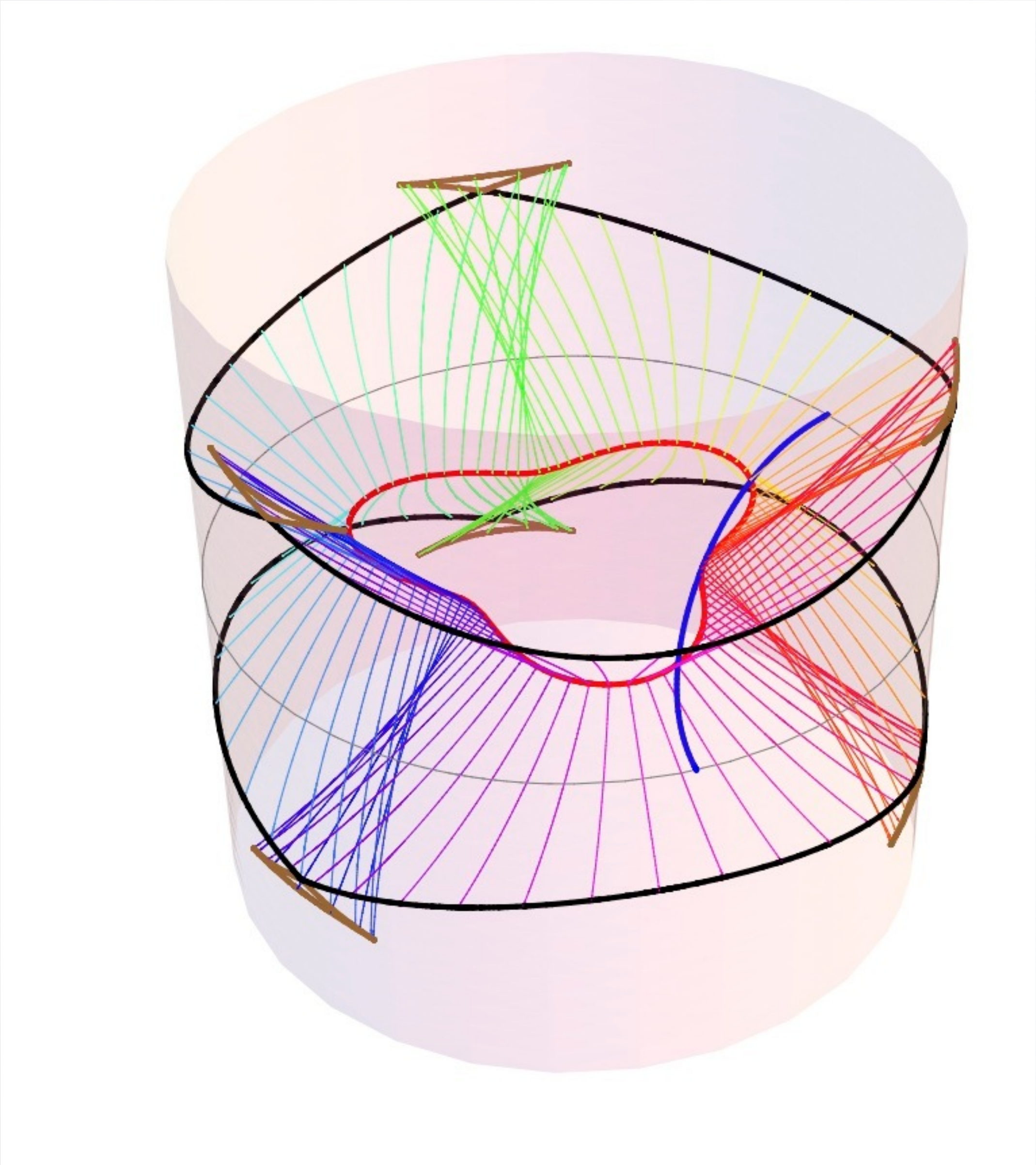
Yes!

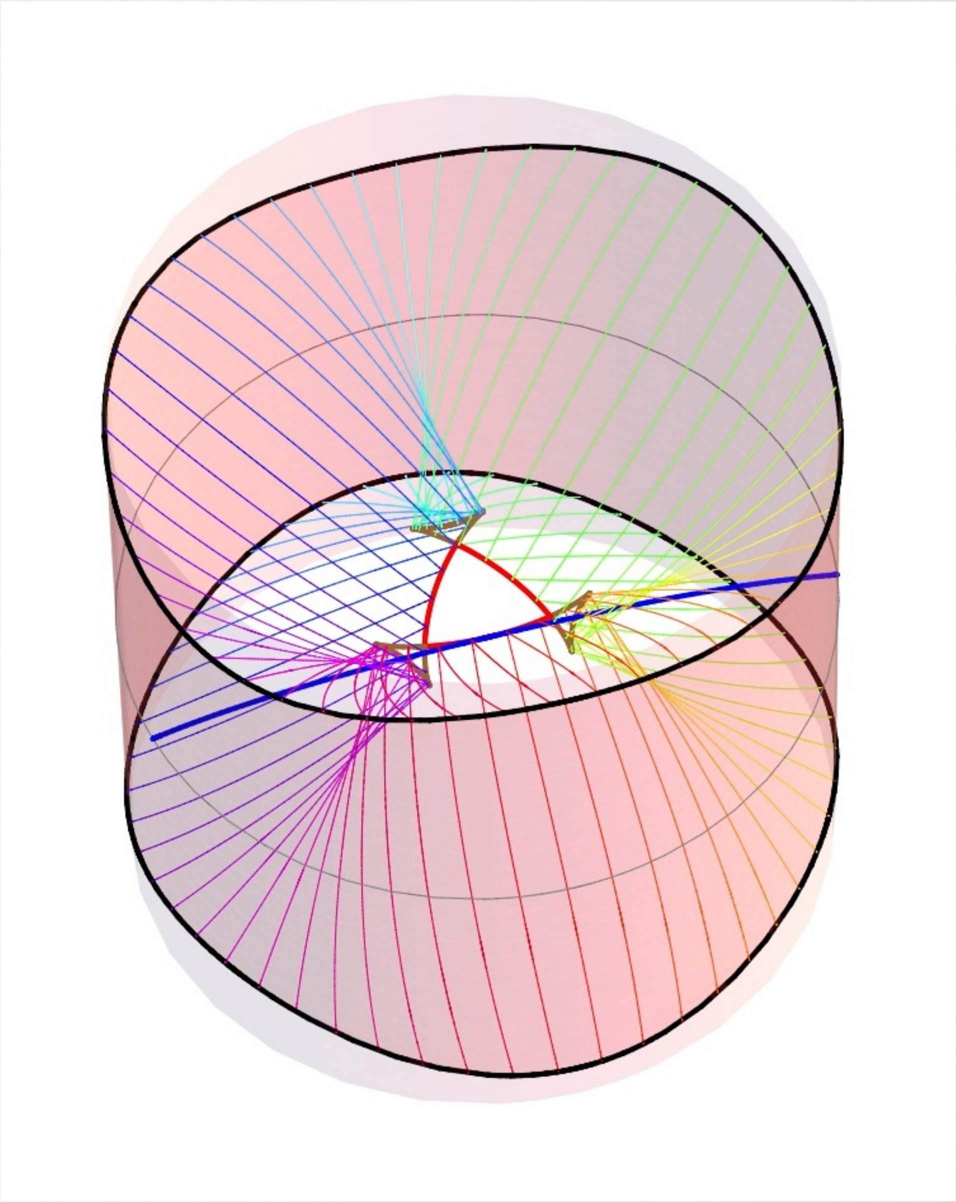
[VH,'14]

Null generators can cross

Smooth bulk curve
⇝ kinky time strip

Smooth time strip
⇝ kinky bulk curve

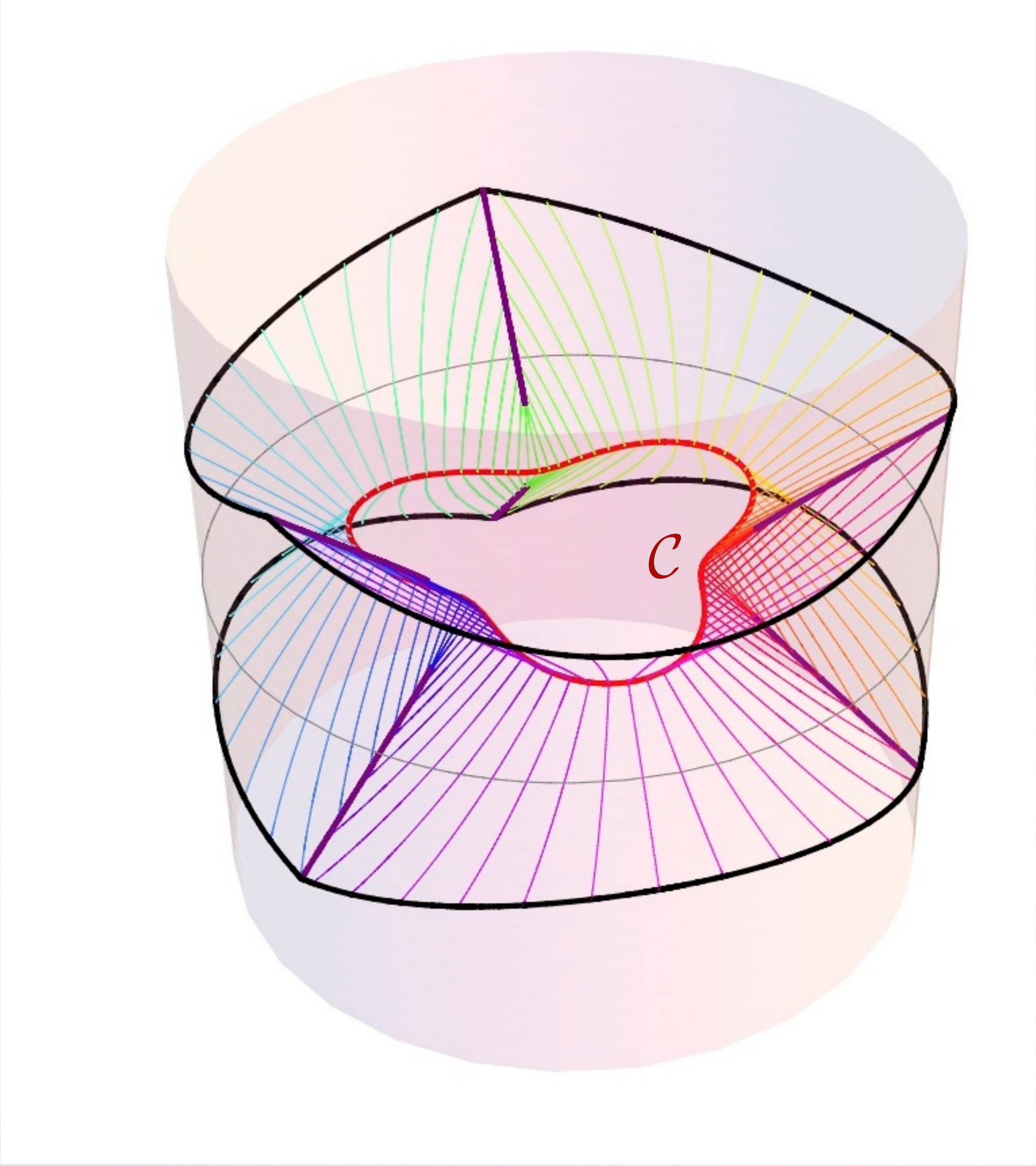




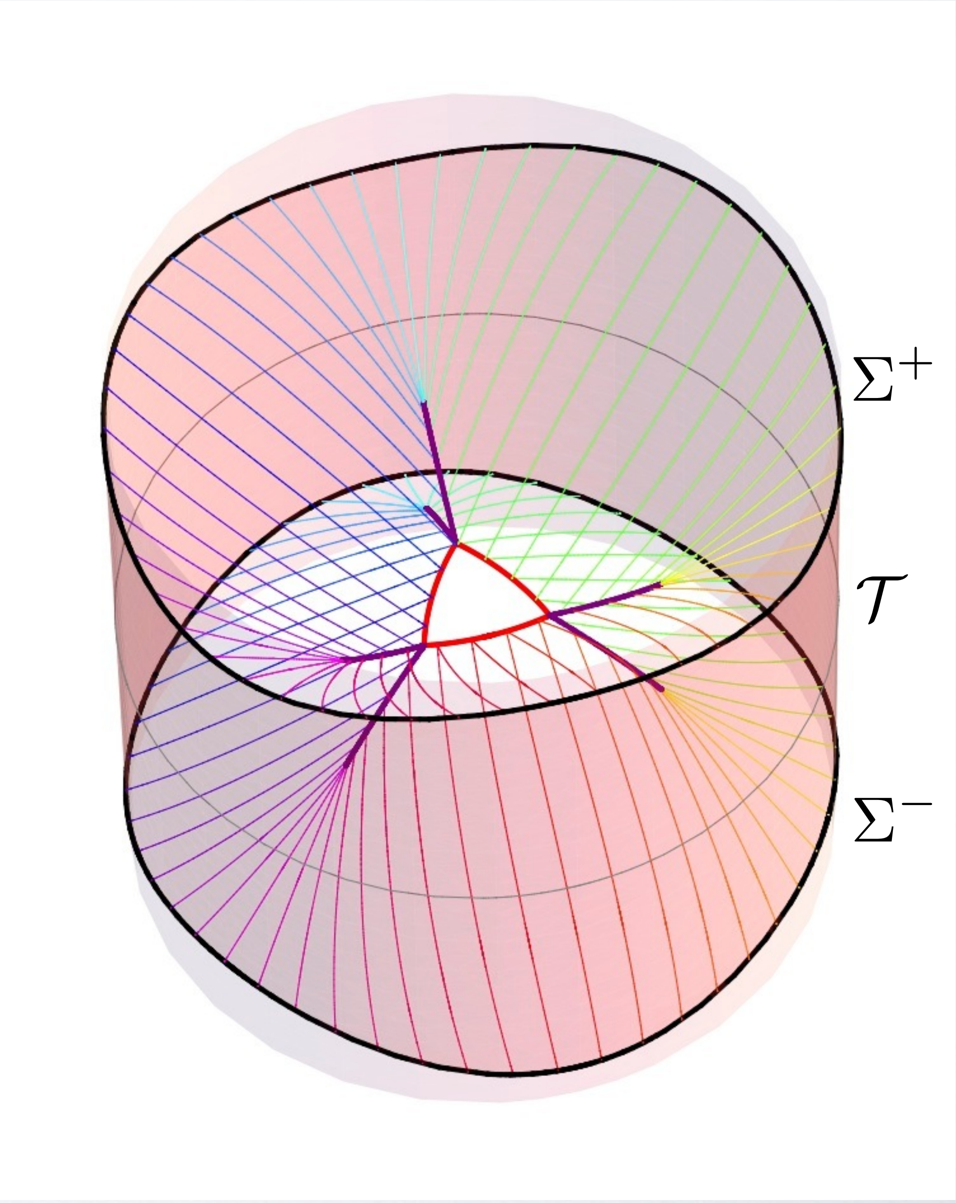
Covariant Residual Entropy

$\exists$ 2 well-defined proposals based on starting point:

bulk $\mathcal{C} \rightsquigarrow$ Rim Wedge:

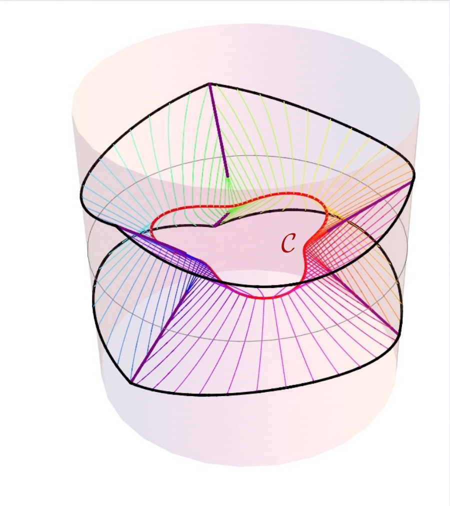


boundary $\mathcal{T} \rightsquigarrow$ Strip Wedge:



Covariant Residual Entropy

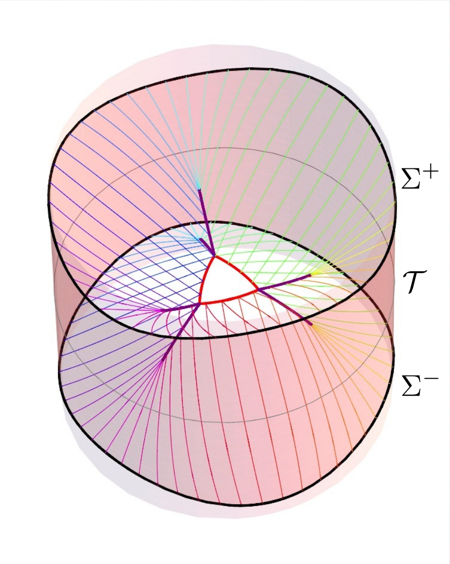
- Two covariant proposals (for bulk vs. bdy starting point)
  - bulk $\mathcal{C}$ defining bulk hole $\rightsquigarrow$ Rim Wedge:



$$\mathcal{W}_{\mathcal{C}} = \left[I^+[\mathcal{C}] \cup I^-[\mathcal{C}] \right]^c \backslash \text{hole}$$

= (closure of) spacelike-separated points from the bulk hole

  - boundary $\Sigma^{\pm}$ defining the time strip $\rightsquigarrow$ Strip Wedge:



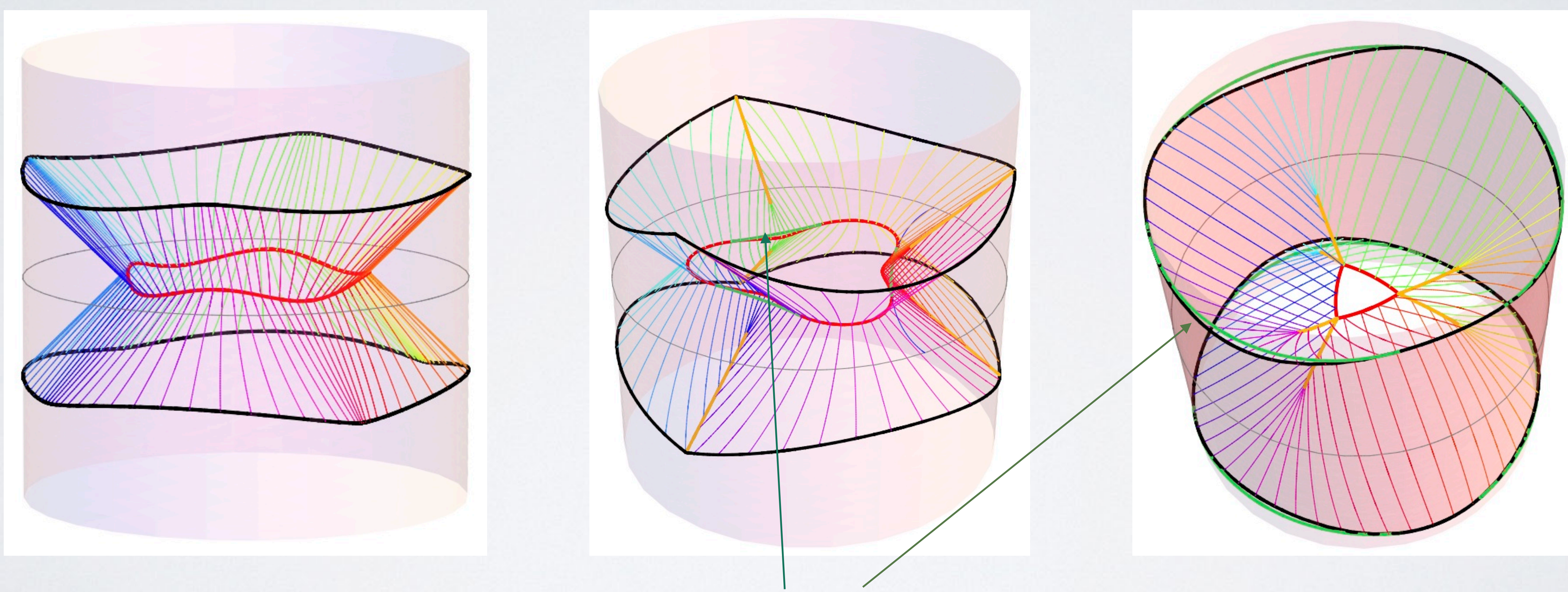
$$\mathcal{W}_{\Sigma} = J^+[\Sigma^-] \cap J^-[\Sigma^+]$$

= causally-connected (both in future and past direction) points to the boundary time strip

Covariant Residual Entropy

- These coincide only if the generators don't cross — cf. (a)
- Generally neither procedure is reversible — cf. (b) & (c)

(a) (b) (c)



Green curves = reverse-constructed wedge

- However, it is always true that $\mathcal{W}_{\Sigma} \subset \mathcal{W}_{\mathcal{C}}$

Covariant Residual Entropy - a puzzle:

- Natural expectations for residual entropy (RE):
  - Bdy RE = area of strip wedge rim

    — cf. expectation of [BCCdBH] and CHI hints [VH&Rangamani, Kelly&Wall]
  - Bulk RE = area of bulk hole rim

    — cf. bulk entanglement entropy [Bianchi&Myers, '12]
- BUT: irreversibility has important implications:
  - Distinct boundary time strips → same hole rim (i.e. same bdy RE)
  - Distinct bulk hole rims (i.e. different bulk RE) → same boundary time strip.
- Hence collective ignorance more global than composite of individual observers' ignorance…
- Apparently local boundary observers can't recover bulk RE.

OUTLINE

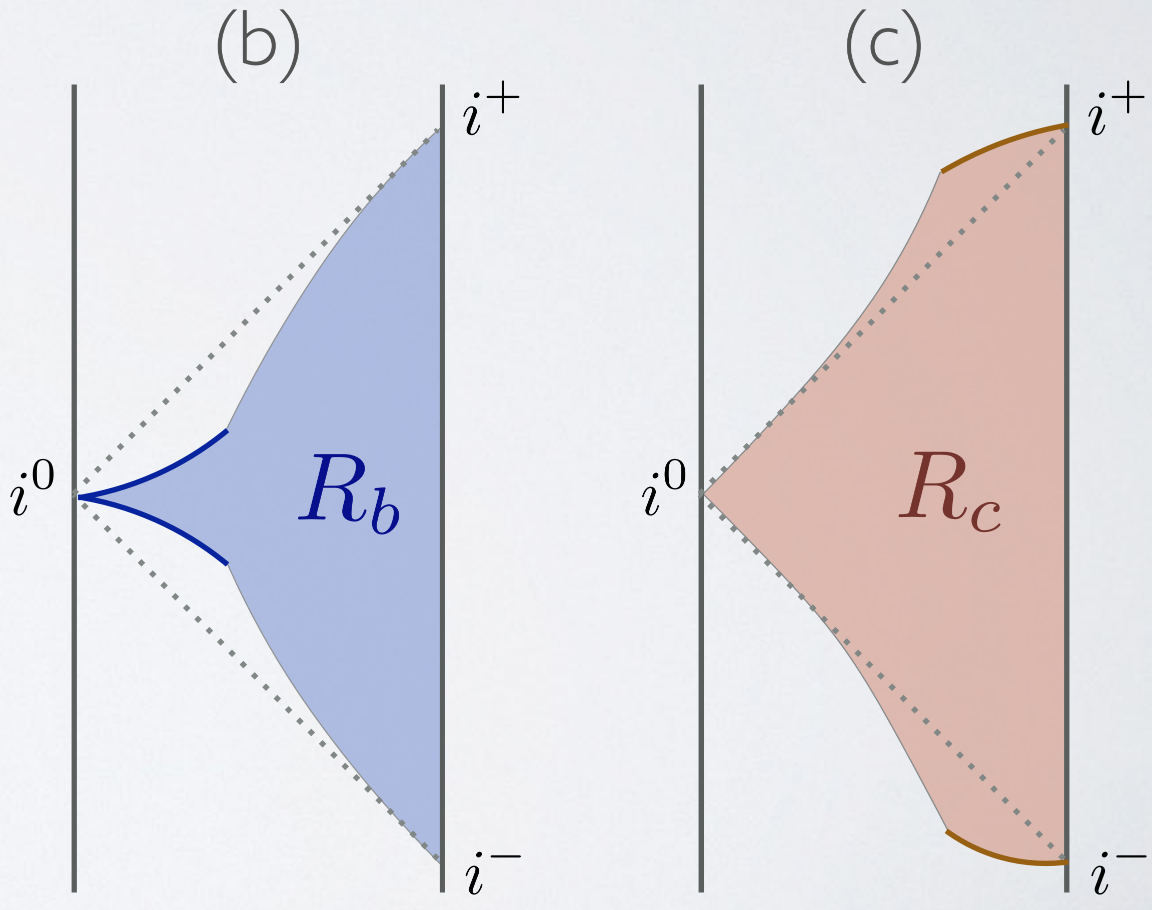
- Entanglement wedge & Causal wedge
- Strip wedge, Rim wedge
- Poincare wedge

Poincare patch for pure AdS

- = dual of CFT (in vacuum state)
- ?: what is the bulk dual for a given excited state?
- Note: asymp. AdS ⇒ same restriction to Mink. ST on bdy…
- Possible options:
  - (a) Coordinate patch inherited from Poincare patch of pure AdS
    ✗ — not covariant
  - (b) Causal wedge of bdy Mink. ST

    $$R_b = J^-[i^+] \cap J^+[i^-]$$
  - (c) Spacelike-separated points from (cf. Entanglement wedge)

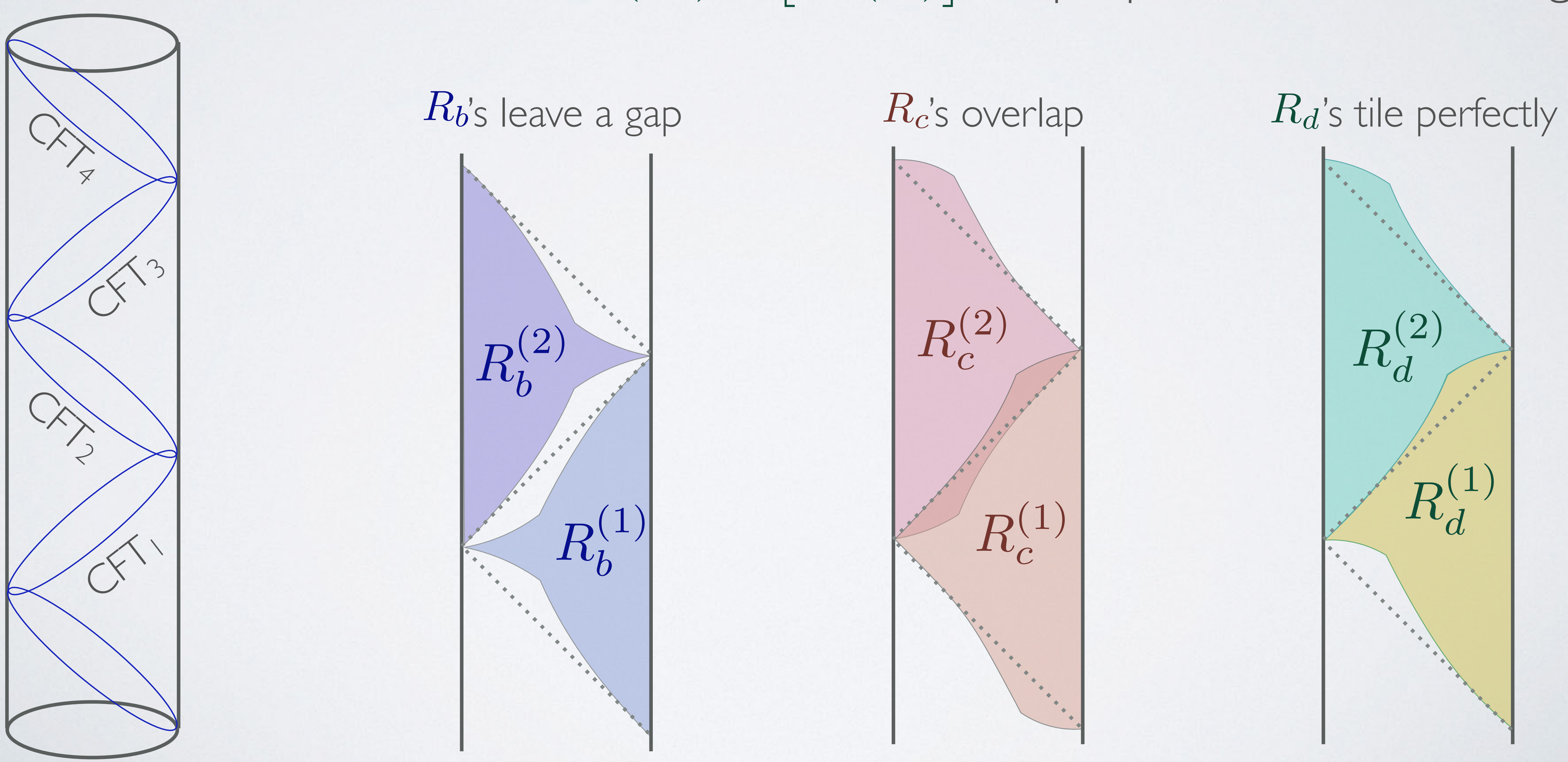
    $$R_c = \left[I^+(i^0) \cup I^-(i^0)\right]^c$$
  - (d) Some hybrid?



Poincare patch for pure AdS

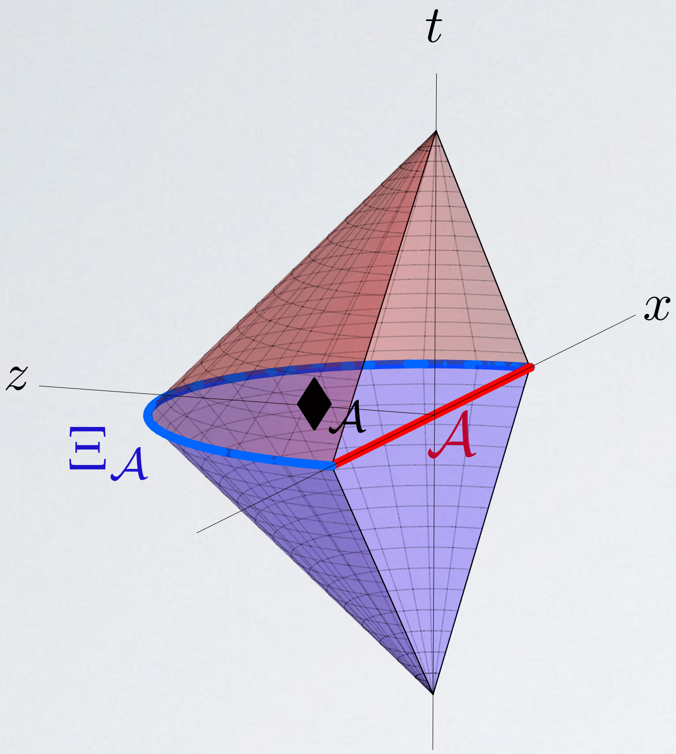
- As a hint consider tiling property in pure global AdS:
  - Global AdS boundary is tiled perfectly by Minkowski regions
  - Neither R_b nor R_c have this property, but a hybrid R_d does ∀ bulk,

    where $R_d = J^+(i^-) \cap \left[I^+(i^0)\right]^c$ = proposed Poincare wedge.

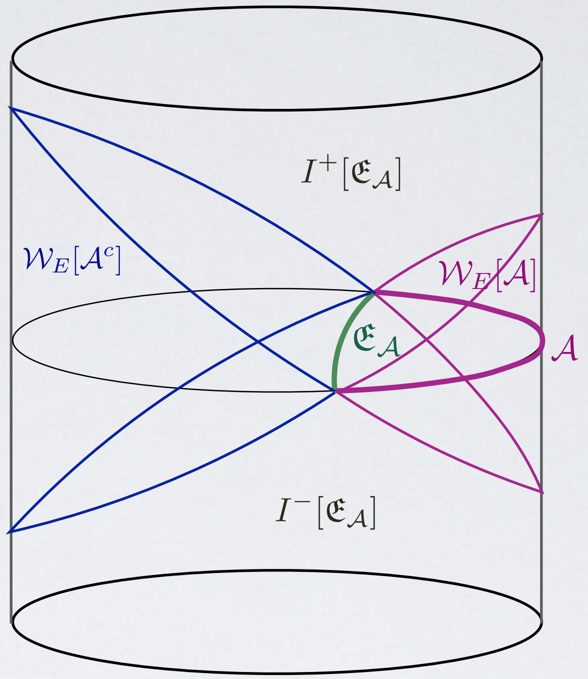


Summary

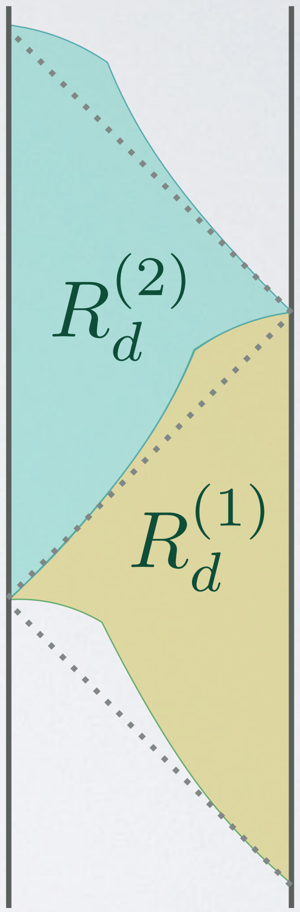
- Main lesson: general covariance is a powerful guiding principle for constructing physically interesting quantities.
- We have seen several distinct causal sets:
  - Causal wedge, Strip wedge
  - Entanglement wedge, Rim wedge
  - Poincare wedge
- Typically, their boundaries (generated by null geodesics) admit crossover seams, which has important implications.
  - Local boundary observers may not capture bulk residual entropy, there is a more nonlocal aspect to collective ignorance than {obs}…
  - Requirement of tiling bulk by Poincare wedges suggests a prescription
- HRT is consistent with causality;
- Entanglement wedge is most natural bulk dual of $\rho_{\mathcal{A}}$



t
x
z
$\Xi_{\mathcal{A}}$
$\blacklozenge_{\mathcal{A}}$
$\mathcal{A}$

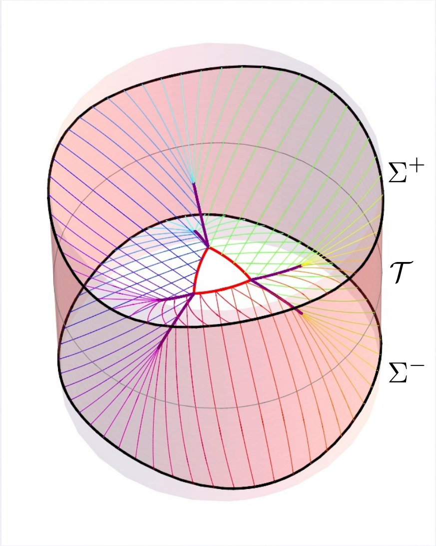


$I^+[\mathfrak{E}_{\mathcal{A}}]$
$\mathcal{W}_E[\mathcal{A}^c]$
$\mathcal{W}_E[\mathcal{A}]$
$\mathfrak{E}_{\mathcal{A}}$
$\mathcal{A}$
$I^-[\mathfrak{E}_{\mathcal{A}}]$

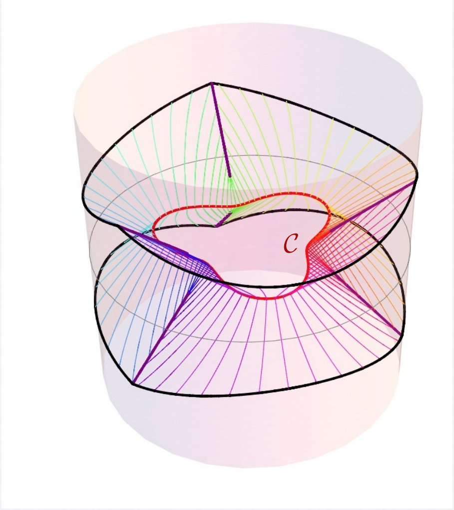


$R_d^{(2)}$
$R_d^{(1)}$

Thank you



Σ^+
$\mathcal{T}$
Σ^-



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Appendices

Summary of HEE proposals:

In all cases, EE is given by Area/4G of a certain surface which is:

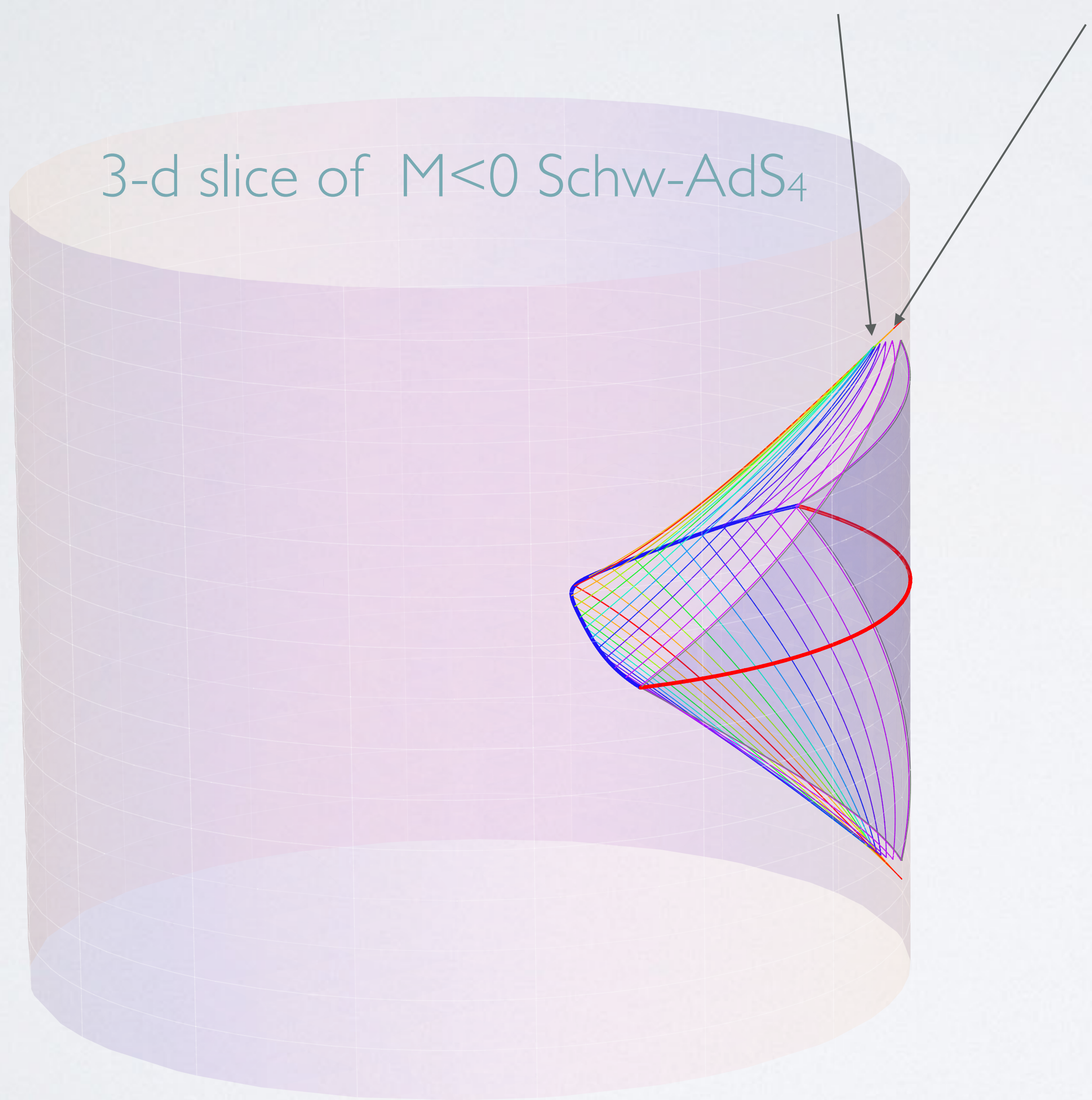
- bulk co-dimension 2 surface
- anchored on the boundary on entangling surface $\partial\mathcal{A}$
- homologous to $\mathcal{A}$ [Headrick, Takayanagi, et.al.]
- in case of multiple surfaces, $S_{\mathcal{A}}$ is given by the one with smallest area.

But the HEE proposals differ in the specification of the surfaces:

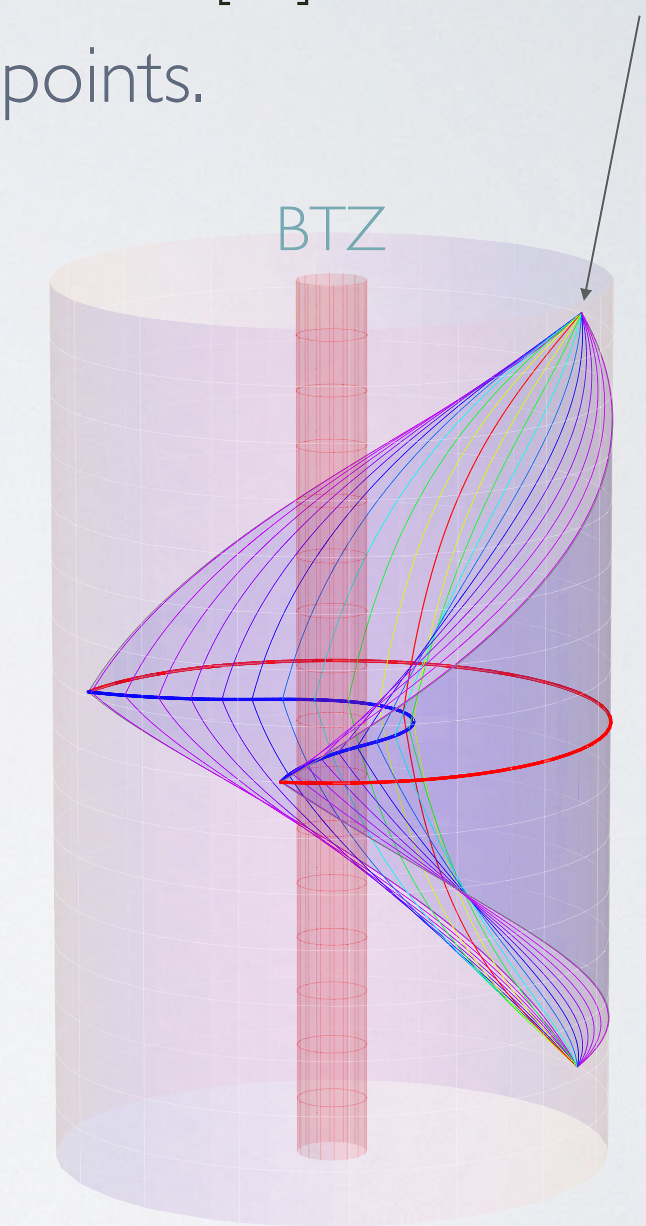
- RT [Ryu & Takayanagi] (static ST only): minimal surface on const. t slice
- HRT [Hubeny, Rangamani, & Takayanagi]: extremal surface in full ST
- maximin [Wall]: minimal surface on bulk achronal slice $\tilde{\Sigma}$, maximized over all $\tilde{\Sigma}$ containing $\mathcal{A}$ (equivalent to extremal surface)

Entanglement wedge example 1

- Only for special cases such as BTZ do generators of $\partial\mathcal{W}_E[\mathcal{A}]$ reach boundary.
- In general, the generators end at caustic / crossover points.



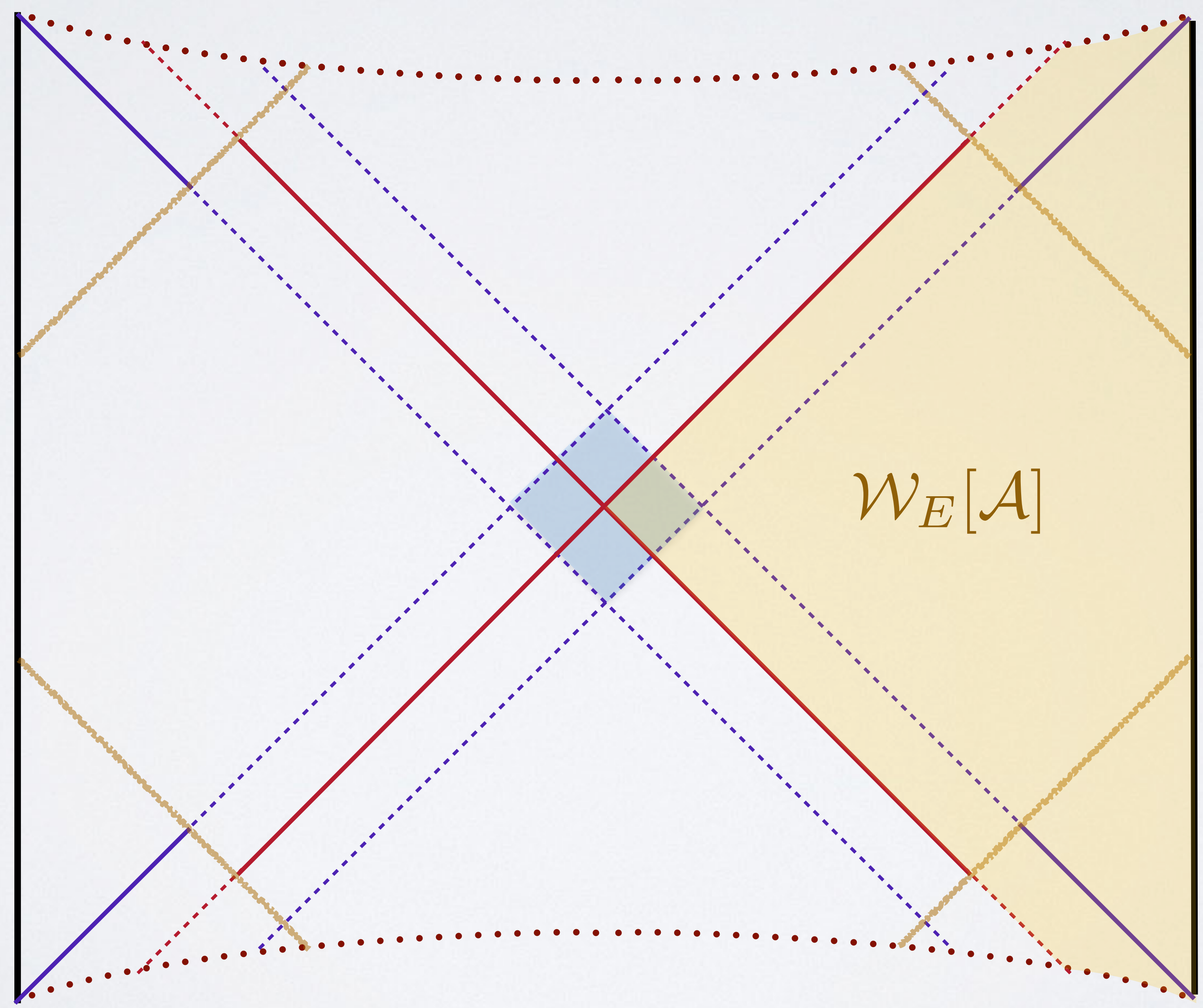
entanglement wedge ⊃ causal wedge



entanglement wedge = causal wedge

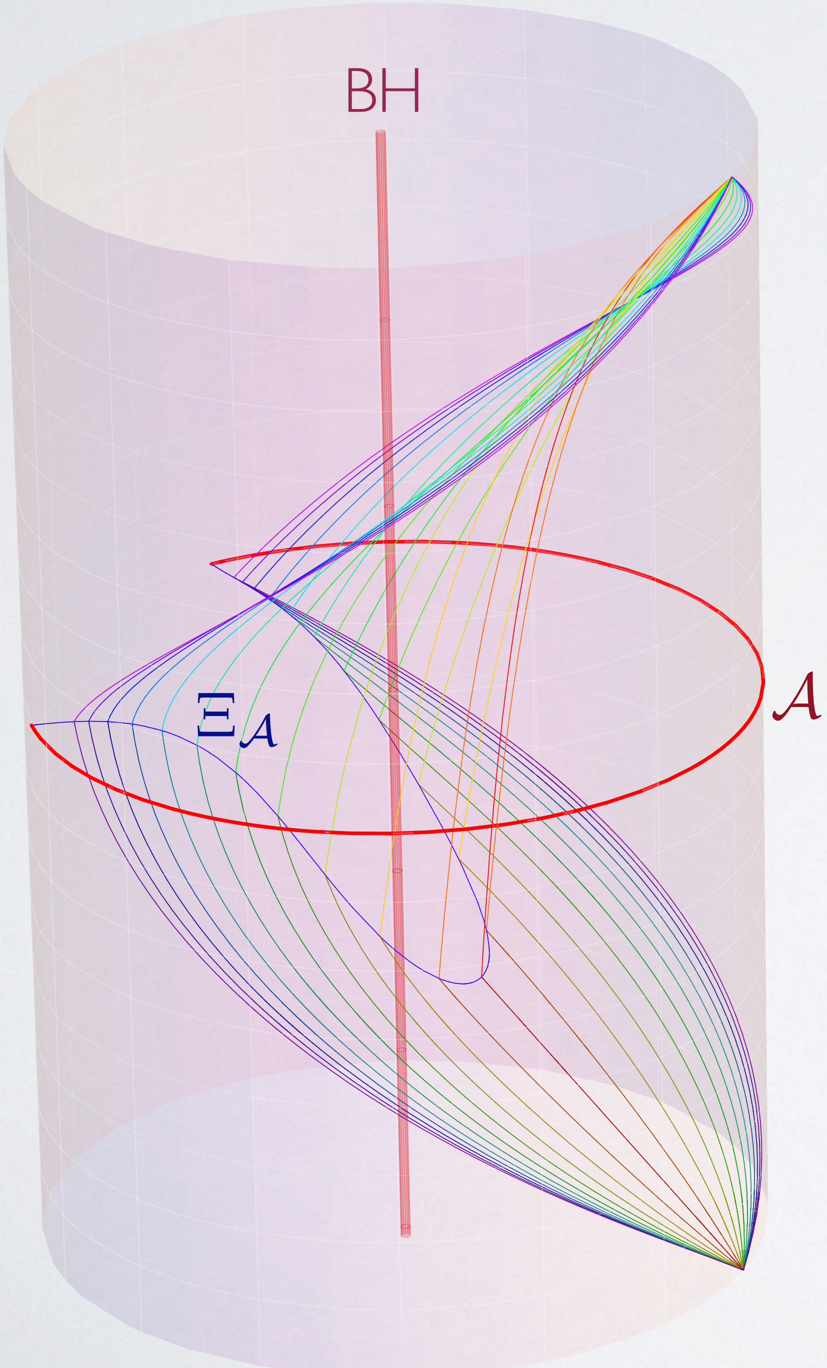
Entanglement wedge example 2

In eternal Schw-AdS doubly-deformed by 4 shells, extremal surface corresponding to $\mathcal{A} = \Sigma_L$ or $\mathcal{A} = \Sigma_R$ lies in middle of 'shadow region'



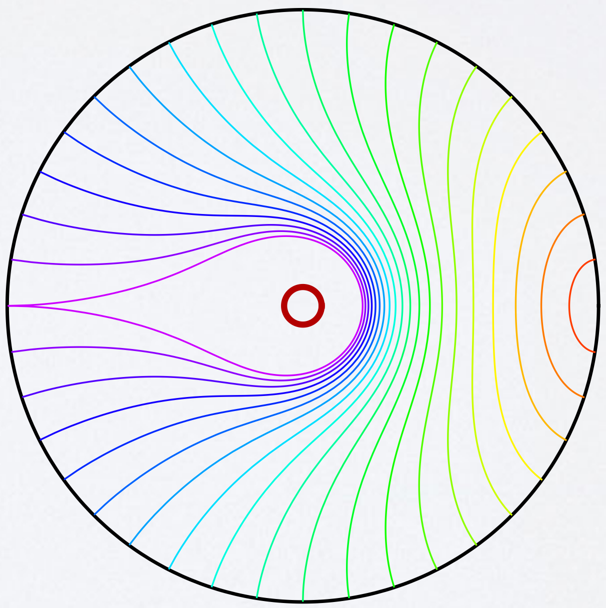
Causal wedge has no holes in 3-d:

- BTZ black hole is never effectively "small" due to low dim.

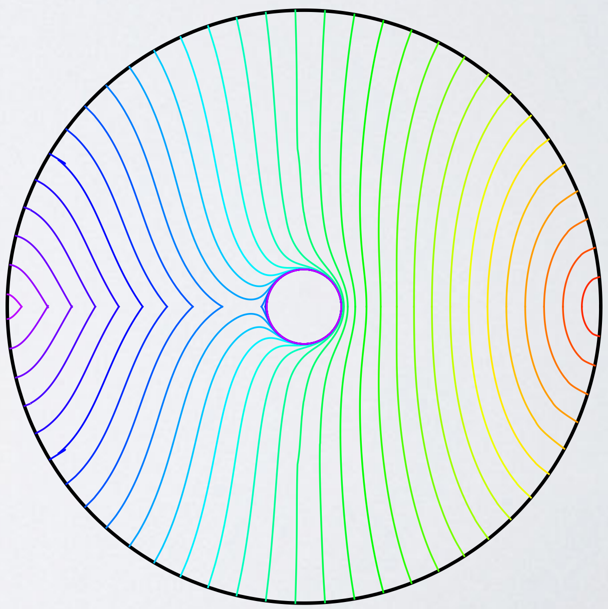


Projection of $\Xi_{\mathcal{A}}$ to Poincare disk
for varying size of $\mathcal{A}$:

3-d



cf. 5-d



Hole-ography: which observers?

- Differential entropy formulated only for time-flip-symmetric case
  - Then static observers preferred; all bdy intervals at same time slice
- But in general static observers don't work:
  - Non-maximal causal wedge & differential entropy formula ill-defined.
- Longest-lived observers optimal; but still ill-defined diff.ent. formula…