Bit Threads and Holographic Entanglement

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Based on arXiv:1604.00354 [hep-th], with Michael Freedman

Contents

1 How should one think about the minimal surface?

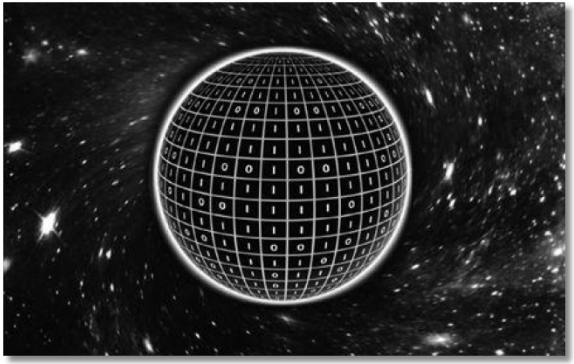
In semiclassical gravity, surface areas are related to entropies

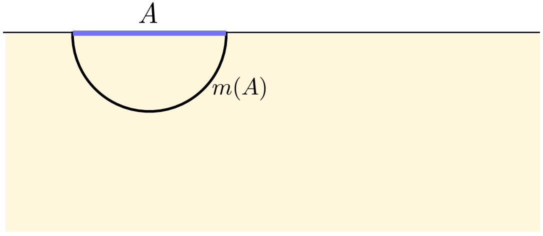
Bekenstein-Hawking ['74]: For black hole

$$S = \frac{1}{4G_{\mathrm{N}}} \mathrm{area}(\text{horizon})$$

Why?

Possible answer: Microstate bits "live" on horizon, at density of 1 bit per 4 Planck areas





Ryu-Takayanagi ['06]: For region in holographic field theory (classical Einstein gravity, static state)

$$S(A) = \frac{1}{4G_N} \mathrm{area}(m(A))$$

$m(A)$ = bulk minimal surface homologous to A

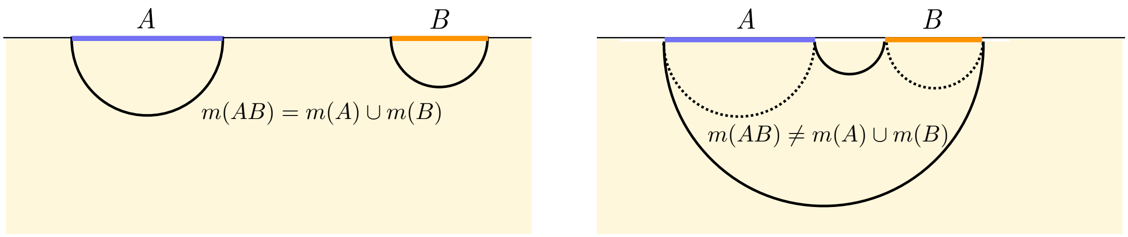
Do microstate bits of A "live" on $m(A)$?

Unlike horizon, $m(A)$ is not a special place; by choosing A, we can put $m(A)$ almost anywhere

Puzzles:

- Under continuous changes in boundary region, minimal surface can jump

  Example: Union of separated regions A, B



- Information-theoretic quantities are given by differences of areas of surfaces passing through different parts of bulk:

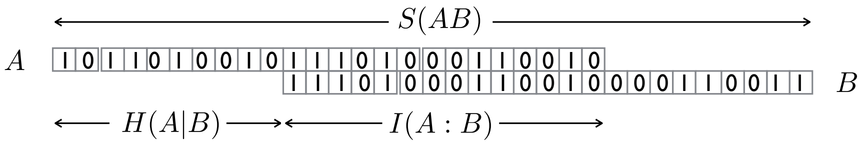
Conditional entropy:	$H(A\|B) = S(AB) - S(B)$
Mutual information:	$I(A : B) = S(A) + S(B) - S(AB)$
Conditional mutual information:	$I(A : B\|C) = S(AB) + S(BC) - S(ABC) - S(C)$

$$H(A|B) = S(AB) - S(B) \qquad\qquad I(A:B) = S(A) + S(B) - S(AB)$$

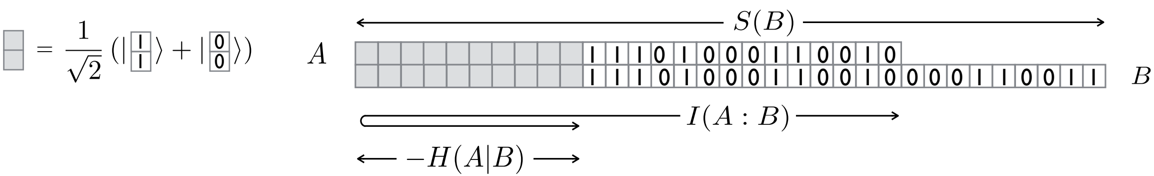
Information-theoretic meaning (heuristically):

Classical: $H(A|B)$ = # of (independent) bits belonging purely to A

$I(A:B)$ = # shared with B



Quantum: Entangled (Bell) pair contributes 2 to $I(A:B)$, -1 to $H(A|B)$; can lead to $H(A|B) < 0$



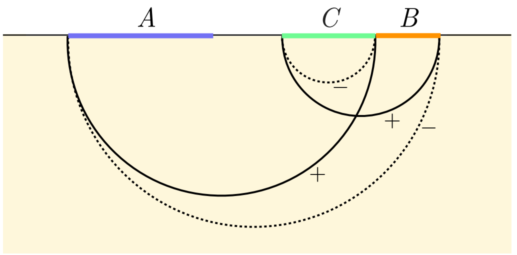
$I(A:B|C) = S(AB) + S(BC) - S(ABC) - S(C)$ = correlation between A & B conditioned on C

What do differences between areas of surfaces, passing through different parts of bulk, have to do with these measures of information?

- RT obeys strong subadditivity [Headrick-Takayanagi '07]

$$I(A:BC) \geq I(A:C)$$

  What does proof (by cutting & gluing minimal surfaces) have to do with information-theoretic meaning of SSA (monotonicity of correlations)?



To answer these questions, I will present a new formulation of RT

- Does not refer to minimal surfaces (demoted to a calculational device)
- Suggests a new way to think about the holographic principle, & about the connection between spacetime geometry and information

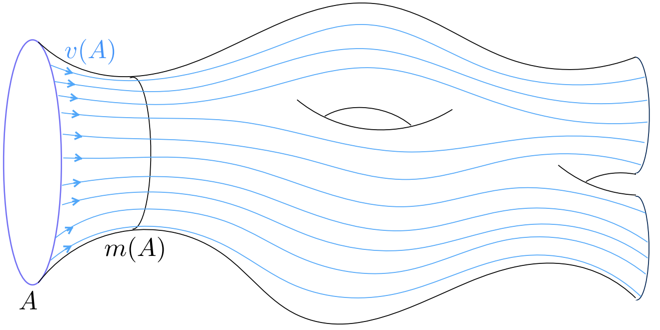
2 Reformulation of RT

Consider a Riemannian manifold with boundary

Define a *flow* as a vector field v obeying $\nabla \cdot v = 0$, $|v| \leq 1$
Think of flow as a set of oriented threads (flow lines) beginning & ending on boundary, with transverse density $= |v| \leq 1$

Let A be a subset of boundary
Max flow-min cut theorem (originally on graphs; Riemannian version: [Federer '74, Strang '83, Nozawa '90]):

$$\max_v \int_A v = \min_{m \sim A} \text{area}(m)$$



Note:

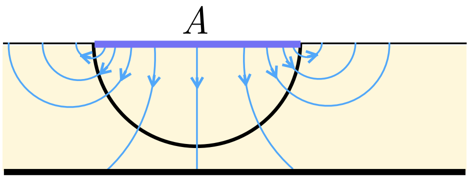
- Max flow is highly non-unique (except on $m(A)$, where $v =$ unit normal)

  Let $v(A)$ denote *any* max flow

- Finding max flow is a linear programming problem

RT version 2.0:

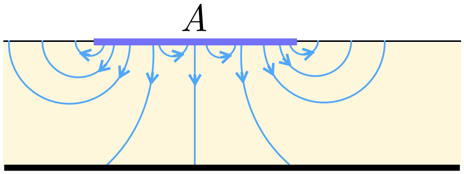
$$S(A) = \max_v \int_A v \qquad (4G_\mathrm{N} = 1)$$
$$= \max \# \text{ of threads beginning on } A$$



Threads can end on A^c or horizon

Each thread has cross section of 4 Planck areas & is identified with 1 (independent) bit of A

Automatically incorporates homology & global minimization conditions of RT



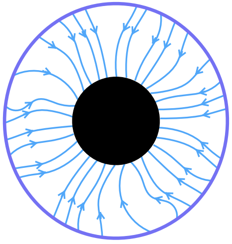
Threads are "floppy": lots of freedom to move them around in bulk & move where they attach to A

Also lots of room near boundary to add extra threads that begin & end on A (don't contribute to $S(A)$)

Role of minimal surface: bottleneck, where threads are maximally packed, hence counted by area

Naturally implements holographic principle: entropy $\propto$ area because bits are carried by one-dimensional objects

Bekenstein-Hawking:

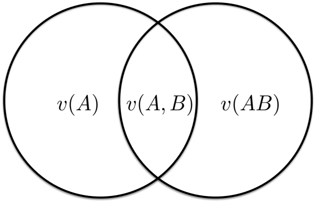


3 Threads & information

Now we address conceptual puzzles with RT raised before

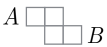
First, $v(A)$ changes continuously with A, even when $m(A)$ jumps

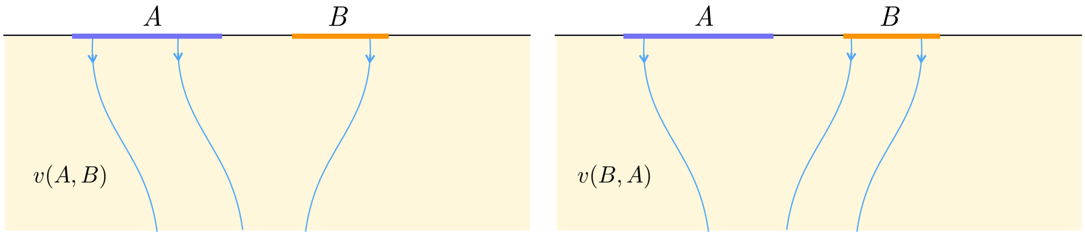
Now consider two regions A, B



We can maximize flux through A *or* B, not in general through both
But we *can* always maximize through A *and* AB (nesting property)
Call such a flow $v(A,B)$

Example 1: $S(A) = S(B) = 2$, $S(AB) = 3 \Rightarrow I(A:B) = 1$, $H(A|B) = 1$



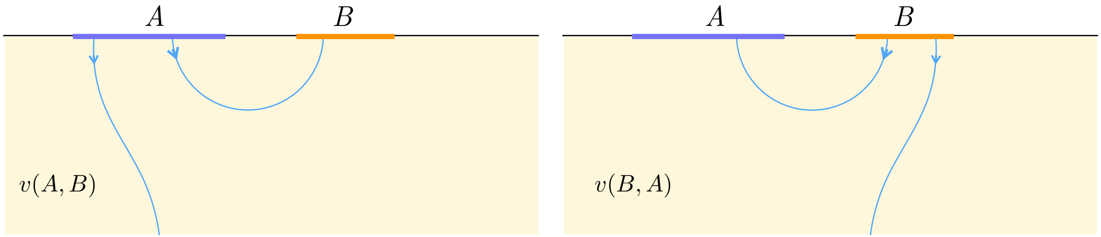


Lesson 1:

- Threads that are stuck on A represent bits unique to A
- Threads that can be moved between A & B represent correlated pairs of bits

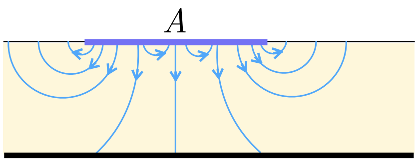
Example 2: $S(A) = S(B) = 2$, $S(AB) = 1 \Rightarrow I(A:B) = 3$, $H(A|B) = -1 \Rightarrow$ entanglement!

One thread leaving A *must* go to B, and vice versa



Lesson 2:

- Threads that connect A & B (switching orientation) represent entangled pairs of bits



Apply lessons to single region:

- freedom to move beginning points around reflects correlations within A
- freedom to add threads that begin & end on A reflects entanglement within A

In equations:

Conditional entropy:

$$\begin{aligned} H(A|B) &= S(AB) - S(B) \\ &= \int_{AB} v(AB) - \int_B v(B) \\ &= \int_{AB} v(B,A) - \int_B v(B,A) \\ &= \int_A v(B,A) \\ &= \text{min flux on } A \text{ (maximizing on } AB) \end{aligned}$$

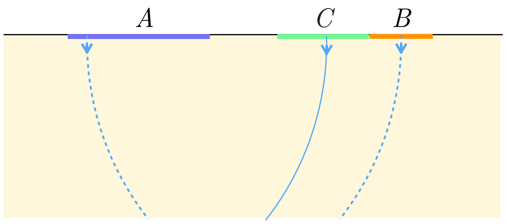
Mutual information:

$$\begin{aligned} I(A:B) &= S(A) - H(A|B) \\ &= \int_A v(A,B) - \int_A v(B,A) \\ &= \text{max} - \text{min flux on } A \text{ (maximizing on } AB) \\ &= \text{flux movable between } A \text{ and } B \text{ (maximizing on } AB) \end{aligned}$$

Max flow can be defined even when flux is infinite: flow that cannot be augmented
Regulator-free definition of mutual information:

$$I(A:B) = \int_A (v(A,B) - v(B,A))$$

Conditional mutual information:



$$
\begin{aligned}
I(A:B|C) &= H(A|C) - H(A|BC) \\
&= \int_A v(C,A,B) - \int_A v(C,B,A) \\
&= \text{max} - \text{min flux on } A \text{ (maximizing on } C \text{ \& } ABC) \\
&= \text{flux movable between } A \text{ \& } B \text{ (maximizing on } C \text{ \& } ABC) \\
&= \text{(flux movable between } A \text{ \& } BC) - \text{(movable between } A \text{ \& } C) \\
&= I(A:BC) - I(A:C)
\end{aligned}
$$

Strong subadditivity ($I(A:B|C) \geq 0$) is clear

In each case, clear connection to information-theoretic meaning of quantity/property

Open problem: Use flows to prove "monogamy of mutual information" property of holographic EEs [Hayden-Headrick-Maloney '12]

$$I(A:BC) \geq I(A:B) + I(A:C)$$

and generalizations to more parties [Bao et al '15]
Flow-based proofs may illuminate the information-theoretic meaning of these inequalities

4 Extensions

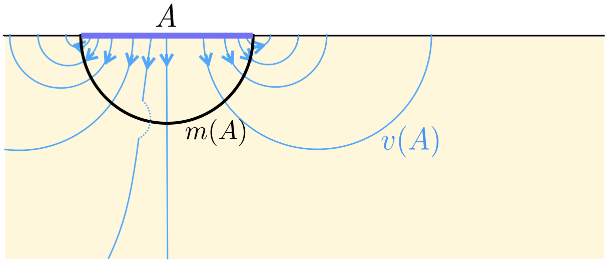
4.1 Emergent geometry

Metric $\longleftrightarrow$ Set of allowed thread configurations

4.2 Quantum corrections

Faulkner-Lewkowycz-Maldacena ['13]: Quantum (order G_N^0) corrections to RT come from entanglement of bulk fields

May be reproduced by allowing threads to jump from one point to another (or tunnel through microscopic wormholes, à la ER = EPR [Maldacena-Susskind '13])



4.3 Covariant bit threads

With Veronika Hubeny (to appear)

Hubeny-Rangamani-Takayanagi ['07] covariant entanglement entropy formula:

$$S(A) = \mathrm{area}(m(A))$$

$m(A)$ = minimal extremal surface homologous to A

Need generalization of max flow-min cut theorem to Lorentzian setting

Define a *flow* as a vector field v (in full Lorentzian spacetime) obeying

- $\nabla \cdot v = 0$
- no flux into or out of singularities
- integrated norm bound: $\forall$ timelike curve C,

$$\int_C ds\, |v_\perp| \leq 1 \qquad\qquad (v_\perp = \text{projection of } v \text{ orthogonal to } C)$$

Any observer sees over their lifetime a total of at most 1 thread per 4 Planck areas

Theorem (assuming NEC, using results of Wall ['12] & Headrick-Hubeny-Lawrence-Rangamani ['14]):

$$\max_v \int_{D(A)} v = \mathrm{area}(m(A)) \qquad\qquad D(A) = \text{boundary causal domain of } A$$

Linearizes problem of finding extremal surface area

HRT version 2.0:

$$S(A) = \max_v \int_{D(A)} v$$

To maximize flux, threads seek out $m(A)$, automatically confining themselves to entanglement wedge

Threads can lie on common Cauchy slice (equivalent to Wall's ['12] maximin by standard max flow-min cut) or spread out in time