Sphere Partition Functions, the Zamolodchikov Metric and Surface Operators

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with Gerchkovitz, Komargodski, arXiv:1405.7271
with Le Floch, to appear

Introduction

- Recent years have seen dramatic progress in the exact computation of partition functions of supersymmetric field theories on curved spaces

- In geometries $S^1 \times \mathcal{M}_d$, the partition function has a standard Hilbert space interpretation as a sum over states

$$Z[S^1 \times \mathcal{M}_d] = \mathrm{Tr}_{\mathcal{H}} \left[(-1)^F e^{-\beta H}\right]$$

1) What does the partition function of a (S)CFT on S^d compute?

- Physical Interpretation
- Ambiguities of Z_{S^d}

2) Sphere partition function $\Longrightarrow$ M2$\subset$M5-brane surface operators

Sphere Partition Function in Conformal Manifold

- Exactly marginal operators $\int d^d x \lambda^i O_i$ define a family of CFTs spanning the conformal manifold $\mathcal{S}$:

$$\lambda^i \text{ are coordinates and } O_i \text{ are vectors fields in } \mathcal{S}$$

- Conformal manifold $\mathcal{S}$ admits Riemannian metric: Zamolodchikov metric

$$\langle O_i(x) O_j(0) \rangle_p = \frac{G_{ij}(p)}{x^{2d}} \qquad p \in \mathcal{S}$$

- CFT can be canonically put on sphere for any $p \in \mathcal{S}$
  - Sphere partition function is an infrared finite observable
  - Z_{S^d} is a probe of the conformal manifold $\mathcal{S}$

- Observable $\langle\mathcal{O}\rangle_\lambda$ defined by expansion around reference CFT

$$\langle\mathcal{O}\rangle_\lambda = \sum_k \frac{1}{k!}\left\langle \mathcal{O}\left(\int d^dx\sqrt{g}\lambda^i O_i(x)\right)^k\right\rangle$$

  - Integrated correlation functions have ultraviolet divergences
  - Need to renormalize so that $\langle\mathcal{O}\rangle_\lambda$ has a continuum limit
  - The structure of divergences of sphere partition function is

$$\log Z_{S^{2n}} = A_1[\lambda^i](r\Lambda_{UV})^{2n}... + A_n[\lambda^i](r\Lambda_{UV})^2 + A[\lambda^i]\log(r\Lambda_{UV}) + F_{2n}[\lambda^i]$$

$$\log Z_{S^{2n+1}} = B_1[\lambda^i](r\Lambda_{UV})^{2n+1}... + B_{n+1}[\lambda^i](r\Lambda_{UV}) + F_{2n+1}[\lambda^i]$$

- Different renormalization schemes differ by diffeomorphism invariant local terms with $\Delta \leq d$ constructed from background fields $g_{mn}(x)$ and $\lambda^i \to \lambda^i(x)$

$$\mathcal{L}(g_{mn}, \lambda^i)$$

- All power-law divergences can be tuned by appropriate counterterms
- In even dimensions:
  - The finite piece $F_{2n}[\lambda^i]$ is ambiguous, there is a finite counterterm

$$\int d^{2n}x\sqrt{g}F_{2n}[\lambda^i]E_{2n}$$

  - There is no local counterterm for the $A[\lambda^i]\log(r\Lambda_{UV})$ term
  - Consistency requires that $A[\lambda^i] = A$, the A-type anomaly
- In odd dimensions:
  - There is no finite counterterm for $\mathrm{Re}(F_{2n+1}[\lambda^i])$
  - Consistency requires that $\mathrm{Re}(F_{2n+1}[\lambda^i]) = \mathrm{Re}(F_{2n+1})$

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Summary

- Unambiguous quantities A and $\mathrm{Re}(F_{2n+1})$ are constant along $\mathcal{S}$
- A and $\mathrm{Re}(F_{2n+1})$ measure entanglement entropy across a sphere in the CFT

Casini,Huerta,Myers

SCFT Sphere Partition Functions

- Regulate the divergences in a supersymmetric way
- Preserve a "massive" subalgebra of superconformal algebra

$$\{Q, Q\} = SO(d+1) \oplus \text{R-symmetry}$$

This is the general supersymmetry algebra of a *massive* theory on S^d

- Counterterms are diffeomorphism and supersymmetric invariant

$$\Longrightarrow \textit{supergravity} \text{ counterterms}$$

  - Realize S^{2n} as supersymmetric background in a supergravity theory

Festuccia,Seiberg

supergravity multiplet: $g_{mn}, \psi_m, \ldots$

  - Represent λ^i as bottom component of a superfield $\Phi^i(x, \Theta)| = \lambda^i(x)$
  - Supergravity invariant constructed from supergravity multiplet and Φ^i

$$\mathcal{L}(g_{mn}, \psi_m, \ldots; \lambda^i, \ldots)$$

Two Dimensional $\mathcal{N}=(2,2)$ SCFTs

- Includes worldsheet description of string theory on Calabi-Yau manifolds
- Conformal manifold $\mathcal{S}$ is Kähler and locally $\mathcal{S}_c \times \mathcal{S}_{tc}$
- $\nexists$ an $\mathcal{N}=(2,2)$ superconformal invariant regulator. $\exists$ two massive $\mathcal{N}=(2,2)$ subalgebras on S^2

$$SU(2|1)_A \overset{\text{mirror}}{\longleftrightarrow} SU(2|1)_B$$

- Defines partition functions Z_A and Z_B Benini,Cremonesi; Doroud,J.G,Le Floch,Lee Doroud,J.G
- Compute the exact Kähler potential K on the conformal manifold Jockers,Kumar,Lapan,Morrison,Romo J.G,Lee

$$Z_A = e^{-K_{tc}} \qquad\qquad Z_B = e^{-K_c}$$

- Partition function subject to ambiguity under Kahler transformations

$$K \to K + \mathcal{F}(\lambda^i) + \bar{\mathcal{F}}(\bar{\lambda}^{\bar{i}})$$

$\mathcal{F}$ is a holomorpic function instead of an arbitrary real function of the moduli

- Kähler ambiguity counterterm in Type A/B 2d $\mathcal{N}=(2,2)$ supergravity. Supergravities gauge either $U(1)_V$ or $U(1)_A$ R-symmetry
  - Coordinates in $\mathcal{S}_c$ are bottom components of chiral multiplets Φ^i
  - Coordinates in $\mathcal{S}_{tc}$ are bottom components of twisted chiral multiplets Ω^i
- The $SU(2|1)_B$ Kähler ambiguity is due to the supergravity coupling

$$\int d^2x d^2\Theta\, \varepsilon R\, \mathcal{F}(\Phi^i) + c.c \quad \supset \; \frac{1}{r^2}\int d^2x\sqrt{g}\,\mathcal{F}(\lambda^i) + c.c$$

$\mathcal{F}$: holomorphic function

R: chiral superfield containing $\mathcal{R}$ as top component

ε: chiral density superspace measure

- The $SU(2|1)_A$ Kähler ambiguity is parametrized by

$$\int d^2x d\Theta^+ d\tilde{\Theta}^- \hat{\varepsilon} F\, \mathcal{F}(\Omega^i) + c.c$$

4d $\mathcal{N} = 2$ SCFTs

- Conformal Manifold $\mathcal{S}$ of 4d $\mathcal{N} = 2$ SCFTs is Kähler
- SCFT on S^4 can be deformed by exactly marginal operators

$$\int d^4x\sqrt{g}\sum_i \left(\tau_i\, C_i + \bar{\tau}_{\bar{i}}\, \bar{C}_{\bar{i}}\right)$$

C_i : top component of 4d $\mathcal{N} = 2$ chiral multiplet with bottom component A_i

τ_i : coordinates on conformal manifold $\mathcal{S}$

- Regulate divergences of Z_{S^4} in an $OSp(2|4) \subset SU(2,2|2)$ invariant way
- Calculate by supersymmetric localization or using Ward identity

$$\partial_i\partial_{\bar{j}} \log Z_{S^4} = \left\langle \int_{S^4} d^4x\sqrt{g}\, C_i(x) \int_{S^4} d^4y\sqrt{g}\, \bar{C}_{\bar{j}}(y) \right\rangle$$

$$= \left\langle A_i(N)\, \bar{A}_{\bar{j}}(S) \right\rangle = G_{i\bar{j}} = \partial_i\partial_{\bar{j}} K$$

- Z_{S^4} of 4d $\mathcal{N} = 2$ SCFTs computes the Kähler potential on $\mathcal{S}$

$$\boxed{Z_{S^4} = e^{K/12}}$$

- How about 4d $\mathcal{N}=1$ SCFTs?
  - Conformal Manifold $\mathcal{S}$ is Kähler
  - Partition Function regulated in an $OSp(1|4) \subset SU(2,2|1)$ invariant way
  - $\exists$ 4d $\mathcal{N}=1$ (old minimal) supergravity finite counterterm

$$\int d^4x \int d^2\Theta \, \varepsilon(\bar{D}^2 - 8R)R\bar{R}F(\Phi^i, \bar{\Phi}^{\bar{i}}) \supset \frac{1}{r^4}\int d^4x\sqrt{g}F(\lambda^i, \bar{\lambda}^{\bar{i}})$$

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Summary

- S^{2n} partition function of SCFTs may have reduced space of ambiguities
- Sphere partition functions of 2d $\mathcal{N}=(2,2)$ and 4d $\mathcal{N}=2$ SCFTs capture the *exact* Kähler potential on their conformal manifold

Surface Operators and M2-branes

to appear J.G, Le Floch

- M2-branes ending on N_f M5-branes

M5	0	1	2	3	4	5	
M2	0	1					6

insert a surface operator in the 6d $\mathcal{N} = (2,0)$ A_{N_f-1} SCFT

- Surface operators labeled by a representation $\mathcal{R}$ of $SU(N_f)$

- M5-branes wrapping a punctured Riemann surface C realize a large class of 4d $\mathcal{N} = 2$ theories (class S) Gaiotto

- M2-branes ending on N_f M5-branes insert a surface operator in the corresponding 4d $\mathcal{N} = 2$ theory

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- Surface operators in 4d gauge theories — Gukov,Witten
  - Order parameters that go beyond the Wilson-'t Hooft criteria
  - Can be described by coupling 2d defect dof to the bulk gauge theory
  - Coupled 4d/2d system can exhibit new dynamics and dualities
- M2-brane surface operators preserve a 2d $\mathcal{N} = (2,2)$ subalgebra of 4d $\mathcal{N} = 2$

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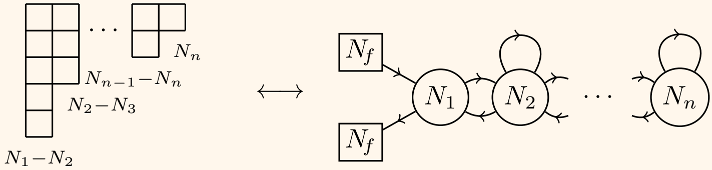
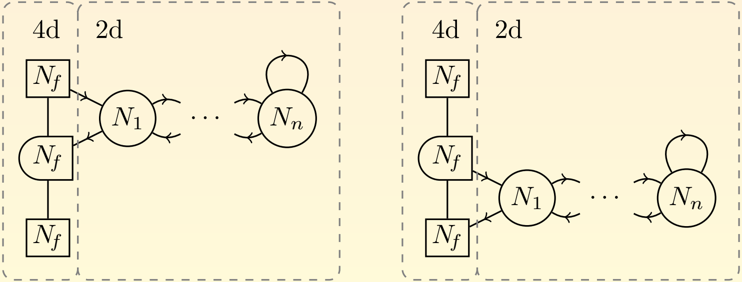
- M2-brane surface operators preserve a 2d $\mathcal{N} = (2,2)$ subalgebra of 4d $\mathcal{N} = 2$
- We have identified the 2d gauge theories corresponding to M2-branes

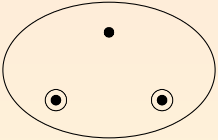


- Surface operator obtained by identifying the $SU(N_f) \times SU(N_f) \times U(1)$ symmetry of the 2d gauge theory with a corresponding gauge or global symmetry of 4d $\mathcal{N} = 2$ theory



- A superpotential on the defect couples 2d fields to 4d fields

- S_b^4 partition function of $\mathcal{T}_C$ is captured by Toda CFT correlator in C — Pestun, AGT

$Z[\mathcal{T}_C] \leftrightarrow$ 

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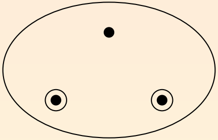
$Z[\mathcal{T}_C] \leftrightarrow$ 

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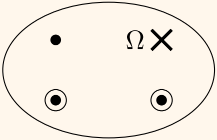
- S_b^4 partition function of $\mathcal{T}_C$ + our 2d gauge theory on S^2 labelled by $\mathcal{R}(\Omega)$ = Toda CFT correlator on C + extra degenerate with momentum $\alpha = -b\Omega$

$Z^{\mathcal{R}[\Omega]}_{S^2 \subset S_b^4} \leftrightarrow$ 

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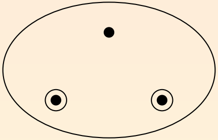
$Z[\mathcal{T}_C] \leftrightarrow$ 

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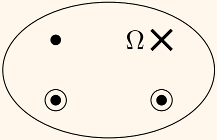
$Z^{\mathcal{R}[\Omega]}_{S^2 \subset S_b^4} \leftrightarrow$ 

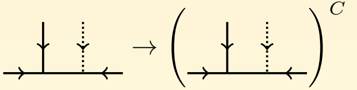
- We $\underline{\text{explicitly verified}}$ this for the 4d $\mathcal{N} = 2$ theory associated to the $\underline{\text{trinion}}$ by using $\underline{\text{exact formulae}}$ for the $\underline{S^2 \text{ partition function}}$ of 2d $\mathcal{N} = (2,2)$ theories

Benini,Cremonesi; Doroud,J.G,Le Floch,Lee

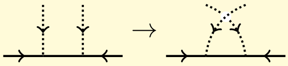
Gauge Theory Dualities as Toda CFT Symmetries

- Through our identification between 2d gauge theories and Toda CFT

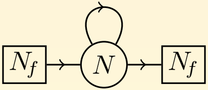
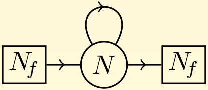
Toda CFT Symmetries $\Longrightarrow$ 4d/2d and 2d Gauge Theory Dualities



2d Seiberg and Kutasov–Schwimmer dualities



2d Seiberg and $(2,2)^*$ dualities for quivers

Duality	Quiver	W	Dual parameters
Seiberg		0	$N^D = N_f - N$ $z^D = z$, $m^D = i/2 - m$
$(2,2)^*$-like		$\sum_t \widetilde{q}_t X^{l_t} q_t$	$N^D = \sum_t l_t - N$ $z^D = z^{-1}$, $m^D = m$
Kutasov–Schwimmer		$\mathrm{Tr} X^{l+1}$	$N^D = lN_f - N$ $z^D = z$, $m^D = i/2 - m$

Conclusion

- In nonsupersymmetric CFTs, F_{2n+1} and A-anomaly are the scheme independent pieces of sphere partition functions

- Sphere partition functions of 2d $\mathcal{N}=(2,2)$ and 4d $\mathcal{N}=2$ SCFTs capture the exact Kähler potential on their conformal manifold

$$Z_A = e^{-K_{tc}} \qquad Z_B = e^{-K_c} \qquad Z_{S^4} = e^{K/12}$$

  - Identified supergravity realization of Kähler transformation ambiguities

- Gave microscopic description of all M2-brane surface operators

- Dualities of 2d $\mathcal{N}=(2,2)$ theories realized in Toda CFT