Higher Spin Algebras & Plane Partitions

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Based on work with Shouvik Datta, Rajesh Gopakumar, Wei Li, and Cheng Peng.

Largely motivated by work of Prochazka '15.

The setup

At the tensionless point in moduli space, string theory on AdS is dual to a (nearly) free conformal field theory.

The conserved currents of the free CFT correspond to massless higher spin fields in AdS, and the tensionless string theory contains a Vasiliev higher spin theory as a (closed) subsector.

[Fradkin & Vasiliev, ’87]
[Vasiliev, ’99…]

[Sundborg, ’01], [Witten, ’01], [Mikhailov, ’02],
[Klebanov & Polyakov, ’02], [Sezgin & Sundell, ’03..]

AdS_3 example

[MRG, Gopakumar, '14]

Concrete realisation of this idea in context of AdS_3 :
CFT dual of string theory on $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4$ at tensionless point is

$$\mathrm{Sym}(\mathbb{T}^4) \equiv (\mathbb{T}^4)^{\otimes(N+1)}/S_{N+1}$$

$$\cup$$

$$\mathcal{W}_\infty^{(\mathcal{N}=4)}[0]$$

CFT dual of Vasiliev higher spin theory on AdS_3

Spin chains and Integrability

By comparison with other examples of AdS/CFT we expect that the free point has a spin-chain like description where the integrability of the system is manifest.

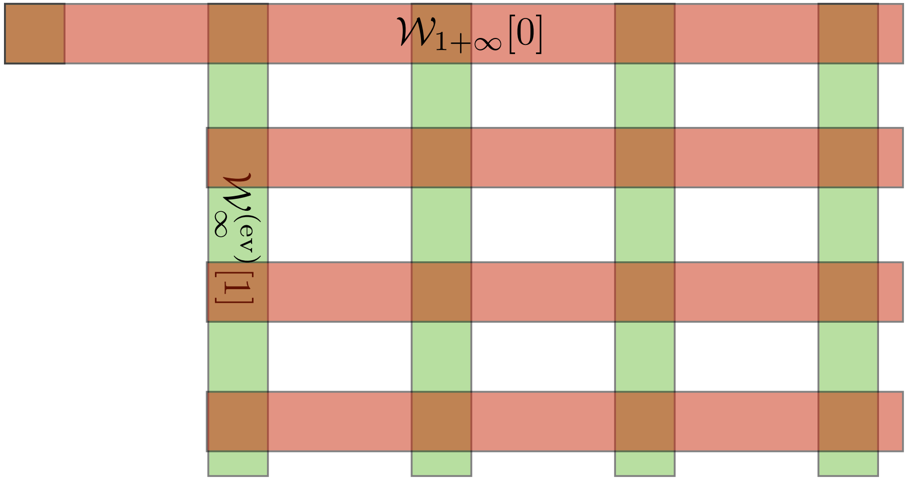
cf. [Babichenko, Stefanski, Zarembo, ‘09]
[Borsato, Ohlsson Sax, Sfondrini, Stefanski, ‘14]

One hallmark of integrability is the appearance of a Yangian symmetry. What is the relation of this expected Yangian symmetry to the higher spin symmetry?

Stringy symmetry

One possible lead is that the actual string theory on AdS_3 has an even bigger symmetry, the Higher Spin Square, whose structure is somewhat reminiscent of a Yangian algebra.

[MRG, Gopakumar, '15]



Bosonic toy model

In the following we shall concentrate on the bosonic version of the 3d/2d HS/CFT duality (whose direct connection to string theory is not yet known).

For this case, the higher spin symmetry is

$$\mathcal{W}_\infty[\lambda] \cong \mathcal{W}_{N,k}$$

[MRG, Gopakumar, '10]
see also [Campoleoni et.al., '10]
and [Henneaux, Rey, '10]

minimal model

where

$$\lambda = \frac{N}{N+k}\,, \qquad c = c_{N,k} = (N-1)\Big(1 - \frac{N(N+1)}{(N+k)(N+k+1)}\Big)\,.$$

Triality symmetry

Actually, the $\mathcal{W}_\infty[\lambda]$ symmetry algebra does not directly depend on λ, but only on the [334] OPE coefficient (as well as on the central charge).

As a consequence, the symmetry algebra exhibits a triality symmetry [MRG, Gopakumar, '12]

$$\mathcal{W}_\infty[\lambda_1] \cong \mathcal{W}_\infty[\lambda_2] \cong \mathcal{W}_\infty[\lambda_3] \qquad \text{at fixed } c$$

In particular,

$$\mathcal{W}_\infty[N] \cong \mathcal{W}_\infty[\tfrac{N}{N+k}] \cong \mathcal{W}_\infty[-\tfrac{N}{N+k+1}] \quad \text{at } c = c_{N,k}$$

u(1) symmetry

In the following we shall often consider the symmetry where an additional u(1) generator is present. The resulting algebra is then

$$\mathcal{W}_{1+\infty}[\lambda] \;\cong\; \mathcal{W}_{\infty}[\lambda] \oplus u(1) \ .$$

can always decouple the u(1) generator
by a coset construction

Affine Yangian

It was argued by [Prochazka, ’15], see also [Feigin, Feigin, Jimbo, Miwa, Mukhin, ’10] that the $\mathcal{W}_{1+\infty}[\lambda]$ algebra is contained in the

affine Yangian of gl(1)

In the following I shall describe this embedding, and the further tests we have performed. It will also become clear that this approach is a powerful method for the description of representations in terms of

plane partitions

The affine Yangian

The affine Yangian is the associative algebra generated by

[Feigin, Tsymbaliuk, '09], [Maulik, Okounkov, '12] [Tsymbaliuk, '14],….

$$e_j\,,\ \ f_j\,,\ \ \psi_j\,,\qquad j=0,1,\ldots$$

subject to a set of commutation & anti-commutation relations. In particular,

$$[\psi_j\,,\psi_k]=0 \qquad\qquad \text{(zero modes)}$$

$$[e_j\,,f_k]=\psi_{j+k}\ \ .$$

(+1)-mode of spin j+1 (-1)-mode of spin k+1

The affine Yangian (ctd)

The generators $\psi_0, \ \psi_1$ are central, and ψ_2 acts diagonally,

$$[\psi_0, e_j] = 0 \ , \quad [\psi_1, e_j] = 0 \ , \quad [\psi_2, e_j] = 2e_j$$
$$[\psi_0, f_k] = 0 \ , \quad [\psi_1, f_k] = 0 \ , \quad [\psi_2, f_k] = -2f_k$$

The eigenvalues of the other ψ_j generators are determined recursively from

$$\begin{aligned} 0 =& [\psi_{j+3}, e_k] - 3[\psi_{j+2}, e_{k+1}] + 3[\psi_{j+1}, e_{k+2}] - [\psi_j, e_{k+3}] \\ &+ \sigma_2 [\psi_{j+1}, e_k] - \sigma_2 [\psi_j, e_{k+1}] - \sigma_3 \{\psi_j, e_k\} \ , \end{aligned}$$

with a similar relation for the f_k .

Affine Yangian (ctd.)

Here $$0 = [\psi_{j+3}, e_k] - 3[\psi_{j+2}, e_{k+1}] + 3[\psi_{j+1}, e_{k+2}] - [\psi_j, e_{k+3}] + \sigma_2[\psi_{j+1}, e_k] - \sigma_2[\psi_j, e_{k+1}] - \sigma_3\{\psi_j, e_k\}$$

parameters characterising affine Yangian

For the following it is convenient to write these parameters via $h_1,\ h_2,\ h_3$ with

$$\begin{aligned} \sigma_1 &= h_1 + h_2 + h_3 = 0 \\ \sigma_2 &= h_1 h_2 + h_1 h_3 + h_2 h_3 \\ \sigma_3 &= h_1 h_2 h_3 \ . \end{aligned}$$

Affine Yangian (ctd.)

The remaining relations are

$$\begin{aligned} 0 =& [e_{j+3}, e_k] - 3[e_{j+2}, e_{k+1}] + 3[e_{j+1}, e_{k+2}] - [e_j, e_{k+3}] \\ &+ \sigma_2 [e_{j+1}, e_k] - \sigma_2 [e_j, e_{k+1}] - \sigma_3 \{e_j, e_k\} \end{aligned}$$

as well as a similar relation for the f_k , and the Serre relations

$$\mathrm{Sym}_{(j_1, j_2, j_3)} [e_{j_1}, [e_{j_2}, e_{j_3+1}]] = 0$$

again with a similar relation for the f_k .

Affine Yangian from CFT

It was argued by [Prochazka, '15] that the relation to CFT is given by

$$h_1 = -\sqrt{\frac{N+k+1}{N+k}} \qquad h_2 = \sqrt{\frac{N+k}{N+k+1}} \qquad h_3 = \frac{1}{\sqrt{(N+k)(N+k+1)}} \ .$$

The other `parameters' of the affine Yangian are the values of the central generators $\psi_0, \ \psi_1$, for which the identification is

$$\underset{\substack{\uparrow \\ \text{central term of u(1)}}}{\psi_0 = -N} \ , \qquad \underset{\substack{\uparrow \\ \text{eigenvalue of u(1) zero mode}}}{\psi_1 = \kappa} \ .$$

Checks of identification

We have checked this identification by constructing the low-lying $\mathcal{W}_{1+\infty}[\lambda]$ generators in terms of affine Yangian generators, e.g.,

[MRG, Gopakumar, Li, Peng, in progress]

$$
\begin{aligned}
J_1 &= e_0 & J_{-1} &= -f_0 \\
L_1 &= -e_1 & L_{-1} &= f_1 \\
V^3_1 &= -e_2 - \tfrac{1}{2}\sigma_3\, e_1\, \psi_0 & V^3_{-1} &= f_2 + \tfrac{1}{2}\sigma_3\, f_1\, \psi_0
\end{aligned}
$$

The V^3_n generators do not come from local fields and need to be adjusted as

Redefining generators

The V_n^3 generators do not come from local fields and need to be adjusted as

$$W_n^3 = V_n^3 + \sigma_3 \tilde{V}_n^3$$

with

$$\tilde{V}_n^3 = \frac{1}{2} \sum_l |l - \tfrac{n}{2}| \, \Theta(l(l-n)) \ : J_{n-l} J_l : + \frac{1}{12} (|n| + 2)(|n| + 1) \, J_n J_0 \ .$$

We have also performed a similar analysis for the spin 4 field; with these identifications we can then calculate the various commutators, in particular the [334] OPE coefficient.

Characteristic coefficient

The relevant coefficient turns out to be

$$\frac{N_4}{N_3^2} = -\frac{75(c+2)(-27+9\sigma_2\psi_0+\sigma_3^2\psi_0^3)}{14(8-4\sigma_2\psi_0-\sigma_3^2\psi_0^3)(17+5\sigma_2\psi_0+5\sigma_3^2\psi_0^3)}$$

This is to be compared with the known $\mathcal{W}_\infty[\lambda]$ formula

$$\frac{N_4}{N_3^2} = \frac{75(c+2)(\lambda-3)\big(c(\lambda+3)+2(4\lambda+3)(\lambda-1)\big)}{14(5c+22)(\lambda-2)\big(c(\lambda+2)+(3\lambda+2)(\lambda-1)\big)}$$

Perfect match upon above identification, i.e.,

$$h_1 = -\sqrt{\frac{N+k+1}{N+k}} \qquad h_2 = \sqrt{\frac{N+k}{N+k+1}} \qquad h_3 = \frac{1}{\sqrt{(N+k)(N+k+1)}} \qquad \psi_0 = -N \ .$$

Isomorphism

The embedding of the $\mathcal{W}_{1+\infty}[\lambda]$ generators into the affine Yangian accounts recursively for all

$$e_j,\ f_j,\ \psi_j, \quad j = 0, 1, \ldots$$

up to normal ordered products of $\mathcal{W}_{1+\infty}[\lambda]$ generators. It therefore demonstrates directly that the affine Yangian of gl(1) is isomorphic to universal enveloping algebra of $\mathcal{W}_{1+\infty}[\lambda]$.

cf. [Feigin, Feigin, Jimbo, Miwa, Mukhin, '10]
[Prochazka, '15]

Free field realisations

We have also found closed form expressions for the affine Yangian generators for the free field realisations at [MRG, Gopakumar, Li, Peng, in progress]

$$\lambda = 0 \quad \cong \quad \text{free fermions}$$

see also [Prochazka, '15]

$$\lambda = 1 \quad \cong \quad \text{free bosons}$$

$$\uparrow$$

in this case only the spin 2,3,… generators appear, i.e. only subalgebra generated by

$$e_r\,,\ f_s\,,\quad r,s \geq 1\ , \qquad \psi_j\,,\ j \geq 2\ .$$

Free field realisations (ctd)

Furthermore, while in both cases $\sigma_3 = 0$, in the bosonic construction

$$\sigma_3 \, \psi_0 = 1 \; (\cong \lambda) \; .$$

$$\sigma_3 = -\frac{1}{\sqrt{(N+k)(N+k+1)}}$$

$$\sigma_3 \psi_0 = \frac{N}{\sqrt{(N+k)(N+k+1)}}$$

As a consequence, the anti-commutator term in, e.g.,

$$\begin{aligned} 0 =& [\psi_{j+3}, e_k] - 3[\psi_{j+2}, e_{k+1}] + 3[\psi_{j+1}, e_{k+2}] - [\psi_j, e_{k+3}] \\ &+ \sigma_2[\psi_{j+1}, e_k] - \sigma_2[\psi_j, e_{k+1}] - \sigma_3\{\psi_j, e_k\} \end{aligned}$$

contributes for j=0

Taking this into account, we again find a **perfect match**!

Triality in affine Yangian

The triality symmetry of the $\mathcal{W}_\infty[\lambda]$ algebra also has a nice interpretation. Triality transformations are generated by

[MRG, Gopakumar, Li, Peng, in progress]

$$\pi_1 : N \mapsto N \ , \qquad k \mapsto -2N - k - 1$$

$$\pi_2 : N \mapsto \frac{N}{N+k} \ , \qquad k \mapsto \frac{1-N}{N+k}$$

The first transformation acts simply on the parameters as

$$\pi_1 : \quad h_1 \longleftrightarrow h_2$$

affine Yangian parameters are invariant!

Triality in affine Yangian (ctd)

On the face of it, the second transformation acts as, e.g.,

$$\pi_2(h_1) = \pi_2\Big(-\sqrt{\frac{N+k+1}{N+k}}\Big) = -\sqrt{N+k+1}\ .$$

However, it also acts non-trivially on $\psi_0 = -N$. In order to absorb this, use scaling symmetry of affine Yangian

$$\psi_j \mapsto \alpha^{j-2}\psi_j\ , \quad e_j \mapsto \alpha^{j-1}e_j\ , \quad f_j \mapsto \alpha^{j-1}f_j$$

under which the h-parameters transform as

$$h_i \mapsto \alpha h_i\ .$$

Triality of affine Yangian (ctd)

Once this is taken into account, one finds

$$\pi_2 : \qquad h_2 \longleftrightarrow h_3$$

affine Yangian parameters are invariant!

Thus the triality symmetry acts by permutations on the three h-parameters!

see also [Prochazka, '15]

The SH^c algebra

The affine Yangian of gl(1) is believed to be isomorphic to the Spherical degenerate double affine Hecke algebra SH^c whose definition is very reminscient of the chiral algebra of the symmetric orbifold.

[Schiffmann, Vasserot, '12]
see also [Kanno, Matsuo, Zhang, '13]

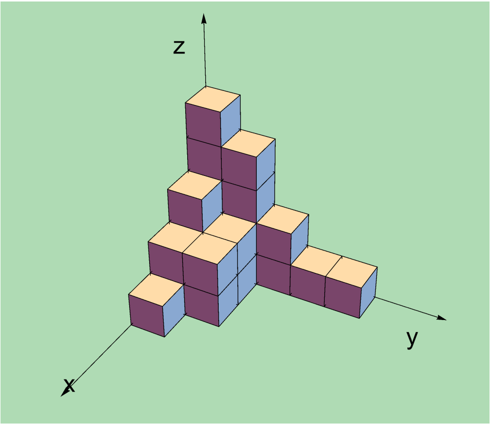
This fits together with the fact that the HSS is (in some sense) contained in the universal enveloping algebra of $\mathcal{W}_{1+\infty}[\lambda]$.

Lie algebra $\longrightarrow$ Lie algebra of universal enveloping algebra
multi-particle $\longrightarrow$ single-particle

Plane Partitions

Maximally degenerate representations of affine Yangian and hence $\mathcal{W}_{1+\infty}[\lambda]$ are described by plane partitions.

[Feigin, Feigin, Jimbo, Miwa, Mukhin, ’10]
[Prochazka, ’15]



trivial asymptotic

Representations labelled by asymptotic behaviour.

Different states of rep. labelled by different configurations with specified asymptotic behaviour.

Vaccum representation

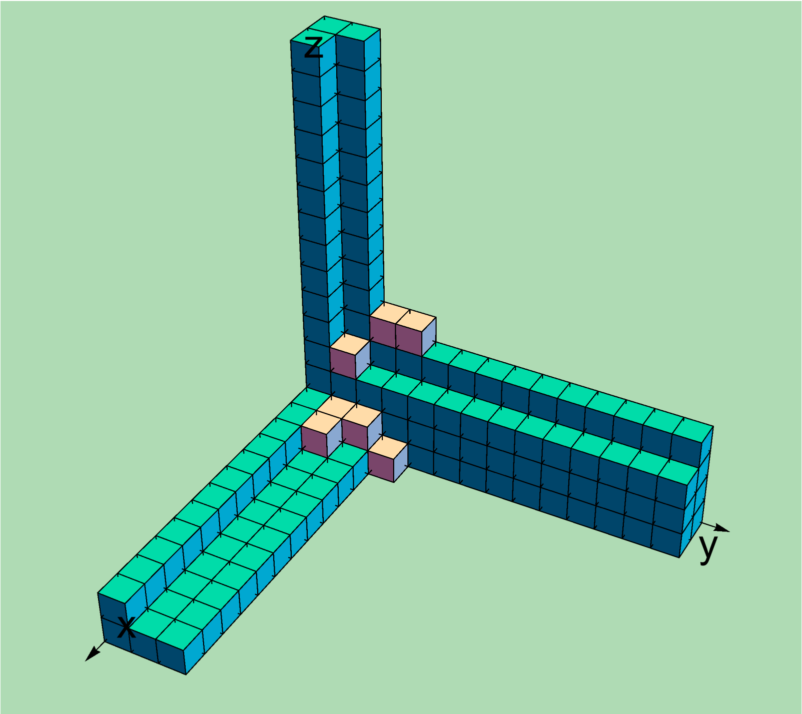
In particular, the states in the vacuum representation corresponds to the plane partitions with trivial asymptotic — counted by MacMahon function

$$\prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n} = \prod_{s=1}^{\infty} \prod_{n=s}^{\infty} \frac{1}{(1-q^n)}$$

Character of $\mathcal{W}_{1+\infty}[\lambda]$
vacuum representation

Non-trivial representations

The non-trivial representations are labelled by their asymptotic behaviour, for example



Additional (brown) boxes: descendants

Charges

The eigenvalues of the ground state with respect to the $\mathcal{W}_{1+\infty}[\lambda]$ zero modes can be calculated combinatorially; the generating function equals

[Prochazka, ’15]

$$1+\sigma_3\sum_{j=0}^{\infty}\frac{\psi_j}{u^{j+1}} = \left(1+\frac{\psi_0\sigma_3}{u}\right)\prod_{\square}\varphi(u-h_{\square})$$

where

$$\varphi(u) = \frac{(u+h_1)(h+h_2)(u+h_3)}{(u-h_1)(u-h_2)(u-h_3)}$$

and

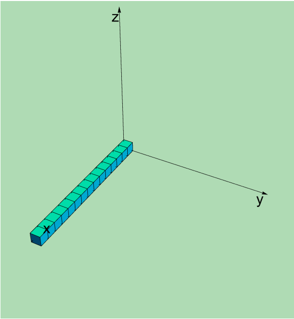
$$h_{\square} = h_1 x(\square) + h_2 y(\square) + h_3 z(\square)$$

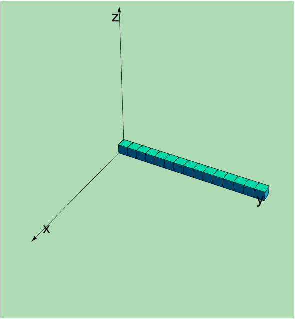
Triality symmetry

Given the action on the h-parameters, the triality symmetry thus permutes the three asymptotic regions.

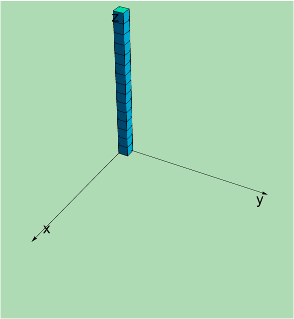
see also [Prochazka, '15]

The usual coset representations are labelled by those plane partitions for which asymptotic behaviour along z-axis, say, is trivial. E.g., the minimal representations correspond to





coset representations



triality image

Characters

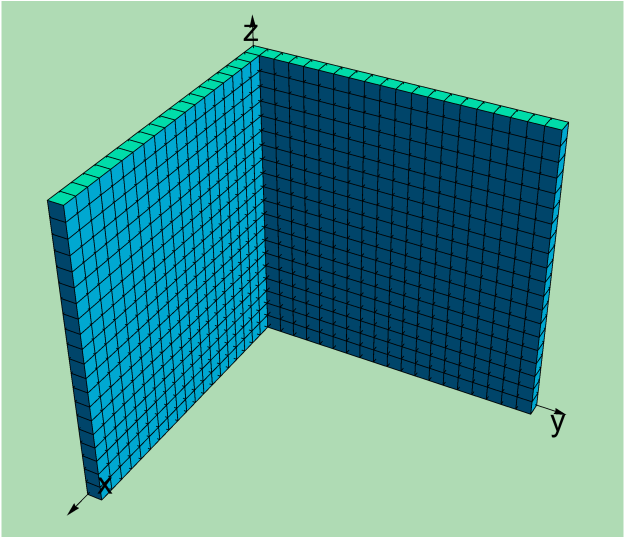
The plane partition viewpoint is a powerful technique for the calculation of the (coset) characters.

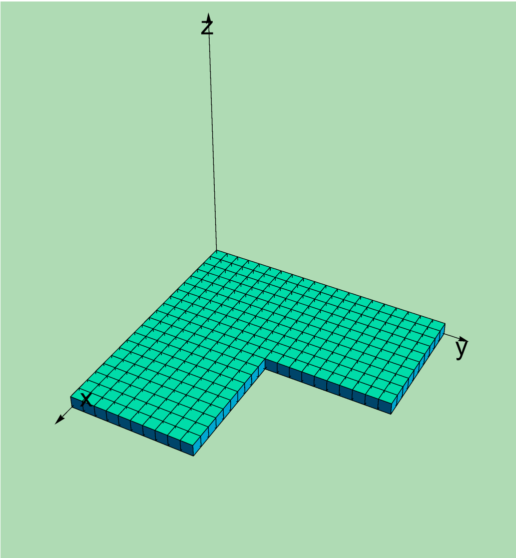
[Datta, MRG, Li, Peng, '16]

For example, we have used this method to find the precise coset representations that correspond to the twisted sectors in the corresponding symmetric orbifold, generalising the analysis of [MRG, Suchanek, '12].

Twisted sectors

The relevant asymptotics are described (in the simplest case) by





free bosons

free fermions

Summary & future directions

Summary:

- Affine Yangian captures $\mathcal{W}_{1+\infty}[\lambda]$ symmetry
- Triality symmetry has natural interpretation
- Powerful method for analysis of (coset) reps

Future directions:

- Spin chain interpretation
- Supersymmetric generalisation

[Babichenko, Stefanski, Zarembo, ‘09]
[Borsato, Ohlsson Sax, Sfondrini, Stefanski, ‘14]

謝謝