Precision Tests of the AdS/CFT Correspondence

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Precision Holography –

Agreement between an AdS/CFT result, valid for $N \to \infty, \ \lambda >> 1$, and a gauge theory computation using supersymmetric localization, in same limit.
Match as a function of field theory parameters.

TWO EXAMPLES: (amid growing number of interesting papers)

1. Gravity dual on S^3 of ABJM theory, perturbed by mass terms for its chiral matter fields. arXiv:1302.7310 Pufu + DZF

2. Gravity dual on S^4 of $\mathcal{N} = 2^*$: $\mathcal{N} = 4$ SYM $\to \mathcal{N} = 2$ by mass for hypermultiplet. arXiv:1311.1508 Bobev, Elvang, Pufu, DZF

In both cases we calculate the Free Energy as a function of the mass parameters in perfect agreement with QFT result!

Usual ideas of AdS/CFT duality:

i) global symmetries of gravity dual and bdy. gauge theories agree,

ii) map between classical fields of gravity theory and composite operators of the QFT,

iii) find classical solution of gravity dual with AAdS metric and other fields,

iv) asymptotics at AdS boundary determines sources + vevs for QFT operators,

v) $S_{on-shell}$ of gravity dual is bridge to dual QFT. But it is ∞. Holographic Renormalization determines ∞ CT's for any solution of the EOMs.

vi) The ∞ CT's from Holo. Ren. must be supplemented by a finite CT to satisfy SUSY. (to be emphasized.)

The mass deformation of ABJM

Jafferis, 1012.3210

1. ABJM may be viewed as a $U(N)_k \times U(N)_{-k}$ Chern-Simons theory with 4 bi-fundamental chiral multiplets:
$Y^A(x),\ \chi^A(x),\quad A = 1, .., 4.$

2. To display $\mathcal{L}_{\text{mass}}$ on S^3 of radius a, it is convenient to use 3 traceless diagonal 4×4 matrices:

$$\begin{aligned} T^1 &= \text{diag}(1, 1, -1, -1) \\ T^2 &= \text{diag}(1, -1, 1, -1) \\ T^3 &= \text{diag}(1, -1, -1, 1)\,. \end{aligned}$$

Define 3 bilinear Bose and Fermi operators:

$$\begin{aligned} \mathcal{O}_B^\alpha &= Tr(\tilde{Y} T^\alpha Y) && \Delta = 1 \quad \text{scalar} \\ \mathcal{O}_F^\alpha &= Tr(\tilde{\chi} T^\alpha \chi - \tilde{Y} T^\alpha Y \sigma) && \Delta = 2 \quad \text{pseudoscalar} \end{aligned}$$

3. The mass deformation depends on 3 parameters $\delta_1\,, \delta_2,\ \delta_3$:

$$\mathcal{L}_{\text{mass}} \;=\; \frac{1}{a^2}\sum_{\alpha}(\delta_\alpha + \delta_1\delta_2\delta_3/\delta_\alpha)\mathcal{O}_B^\alpha + \frac{1}{a}\sum_\alpha \delta_\alpha \mathcal{O}_F^\alpha\,.$$

4. The Free Energy of the deformed ABJM theory was calculated by matrix model methods at large N. [Jafferis et al 1103.1181]:

$$F = \frac{4\sqrt{2}\pi N^{3/2}}{3}\sqrt{\prod_A R[Y^A]}\,.$$

The $R[Y^A]$ are deformed R-charges given by

$$R[Y^A] = \frac{1}{2} + (\delta_1 T^1 + \delta_2 T^2 + \delta_3 T^3)_{AA}\,.$$

(It is a curiosity of the "real mass" mechanism on S^3 that mass parameters are related to R-charges.)

Goals of the gravity dual

1) It must source the 3 $\mathcal{O}_B, \quad \Delta = 1$ and the 3 $\mathcal{O}_F \quad \Delta = 2$.

2) An appropriate classical solution must reproduce F.

The gravity dual:

1. After an orgy of group theory we extract a consistent $\mathcal{N} = 2$ truncation from gauged $\mathcal{N} = 8, D = 4$ SG   De Wit-Nicolai

It contains
i) gravity multiplet $g_{\mu\nu}, \psi^i_\mu, A^0_\mu$, and
ii) 3 abelian vector mults $z^\alpha, \chi^{\alpha i}, A^\alpha_\mu$.

Drop all vectors since they vanish in the solution needed to match the S^3-invariant QFT.

With no vectors, we are left with an $\mathcal{N} = 1$ SG.

Bosonic action and potential

$$S = \frac{1}{8\pi G_4} \int d^4x \sqrt{g} \left[-\frac{1}{2} R + \sum_{\alpha=1}^{3} \frac{|\partial_\mu z^\alpha|^2}{(1-|z^\alpha|^2)^2} + \frac{1}{L^2} \left(-3 + \sum_{\alpha=1}^{3} \frac{2}{1-|z^\alpha|^2} \right) \right]$$

Very Simple:

a) scalar $\mathcal{L}_{\rm kin}$ is that of a Kähler σ model on 3 copies of Poincaré disc.

b) Potential gives conformal mass $m^2 L^2 = -2$.

2. Potential in $\mathcal{N} = 1$ SG should be related to a holomorphic $W(z^\alpha)$ by std. formula: $V = e^K \left(\nabla_\alpha W \; K^{\alpha\bar{\beta}} \nabla_{\bar{\beta}} \bar{W} - 3 W \bar{W} \right)$.
We find $W = (1 + z^1 z^2 z^3)/L$.

3. a) Extract fermion trf. rules from $\mathcal{N} = 8$ SG, b) deduce 1st order BPS eqtns for $z^\alpha(\rho),\ B(\rho)$ from these.

Mathematica solves the BPS eqtns. in terms of a conformally flat metric

$$ds^2 = e^{2B(\rho)} \frac{d\rho^2 + \rho^2 d\Omega_3^2}{(1-\rho^2)^2} \,.$$

Solution for the scalars is quite simple

$$z^\alpha(\rho) = c_\alpha f(\rho) \qquad\qquad \tilde{z}^\alpha = \frac{c_1 c_2 c_3}{c_\alpha} f(\rho)$$

with common radial function $f(\rho) = (1-\rho)^2/(1 + c_1 c_2 c_3 \rho^2)$.

i. smooth non-singular solution
ii. 3 arbitrary complex constants c_α
iii. $\tilde{z} \neq z^*$, as expected in Euclidean SUSY
iv. We check that BPS sols. also solve Lagrangian EOM's and find Killing spinors.

4. To extract the physics, we change to usual radial coordinate r.

$$ds^2 = L^2(dr^2 + e^{2A(r)} d\Omega_3^2)\,,$$

with $e^{2A(r)} \sim e^{2r}$ for large r. The bdy. behavior of the z's is

$z^a(r) = a^\alpha e^{-r} + b^\alpha e^{-2r}$ $\qquad$ $\tilde{z}^a(r) = \tilde{a}^\alpha e^{-r} + \tilde{b}^\alpha e^{-2r}\,,$

where $a^\alpha,\ b^\alpha$, etc. are functions of $c_1,\ c_2,\ c_3$.

Puzzle: e^{-r} is usual source rate for $\Delta = 2$ operator, and e^{-2r} is its vev rate. But we need to source 3 $\Delta = 2,\ \mathcal{O}_F$ and 3 $\Delta = 1,\ \mathcal{O}_B$.

Resolution: For $m^2L^2 = -2$, in $D = 4$, SUSY requires Alternate Quantization for either $z + \tilde{z}$ or $z - \tilde{z}$. Then e^{-2r} term becomes source for $\Delta = 1$!

Both $\mathcal{O}_F$ and $z - \tilde{z}$ are pseudoscalar, so we take $a^\alpha - \tilde{a}^\alpha$ as their sources. Conversely, both $\mathcal{O}_B$ and $z + \tilde{z}$ are scalar, so we take $b^\alpha + \tilde{b}^\alpha$ as sources.

This requires using Legendre transform of $S_{\mathrm{on-shell}}$ as generating function for QFT observables. $\qquad$ Klebanov and Witten

5. Holographic Renormalization provides a renormalized action

$$S_{\rm ren} = S_{\rm bulk} + S_{GH} + S_{\rm CT}\,.$$

It is finite, but not satisfactory because it does not respect SUSY for flat-sliced solutions of the same bulk theory.
Diagnostic: $E_{\rm vac} \neq 0$.

We wil use the Bogomolny construction to find the correct CT.

CT's are universal; they govern all solutions of the same theory, so we must use the same CT for S^3-sliced solutions.

A flat-sliced solution has metric $ds^2 = dr^2 + e^{2A(r)} dx^i dx^i$ with $z^\alpha(r)$, i.e. radial dependence only.

Bogomolny for gen. Kahler metric with

$$V = e^K(\nabla_\alpha W \; K^{\alpha\bar{\beta}} \nabla_{\bar{\beta}} \bar{W} - 3W\bar{W}).$$

Start with $S_{\rm bulk}$ and use integration by parts to rewrite it as

$$\begin{aligned} S_{\rm bulk} &= \int d^4x \sqrt{g}[-\frac{1}{2}R + K_{\alpha\bar{\beta}} \partial_r z^\alpha \partial_r^{\bar{\beta}} + V] \\ &= \int d^3x\, dr\, e^{3A} \Big[- (\partial_r A - e^{K/2}|W|)^2 \\ &+ K_{\alpha\bar{\beta}}(\partial_r z^\alpha + \bar{\mathcal{W}}^\alpha)(\partial_r z^{\bar{\beta}} + \mathcal{W}^{\bar{\beta}}) - \frac{\partial}{\partial r}(2e^{3A}|W|) \Big] \end{aligned}$$

Action is stationary if the first order BPS eqtns are satisfied:

$$\partial_r A - e^{K/2}|W| = 0 \qquad \partial_r z^\alpha + \bar{\mathcal{W}}^\alpha = 0.$$

SUSY requires that surface term is cancelled by adding CT

$$S_{\rm SUSY} = \frac{1}{4\pi G_4} \int d^3x\, e^{3A}\, e^{K/2}|W|.$$

evaluated at $r = r_0$, the bdy.

The Bogomolny calculation is exact for flat-sliced solutions so $S_{\rm SUSY}$ contains ∞ terms which match those of $S_{\rm CT}$ plus the finite CT:

$$S_{\rm finite} = \frac{1}{4\pi G_4} \int d^3x \sqrt{h}\, \frac{1}{2}(z^1 z^2 z^3 + \tilde{z}^1 \tilde{z}^2 \tilde{z}^3)$$

Inclusion of this finite correction is crucial for correct calc. of Free Energy!

The Free Energy

1. A SUSY argument to derive the source term from bulk SG: $g_\alpha \mathcal{O}_B^\alpha + f_\alpha \mathcal{O}_F^\alpha$, with $g_\alpha,\ f_\alpha$ functions of the 3 c_α parameters.

Compare with ABJM mass deformation

$$\frac{1}{a^2}\sum_\alpha (\delta_\alpha + \delta_1\delta_2\delta_3/\delta_\alpha)\mathcal{O}_B^\alpha + \frac{1}{a}\sum_\alpha \delta_\alpha \mathcal{O}_F^\alpha\,.$$

Identify the QFT mass parameters

$$\delta_\alpha = n\frac{c_\alpha + c_1c_2c_3/c_\alpha}{1 + c_1c_2c_3}.$$

n is a normalization constant, not usually fixed by AdS/CFT.

2. The Legendre transform of $S_{\rm ren}$ is

$$J = \frac{\pi L^2}{2G_4} \frac{(1-c_1^2)(1-c_2^2)(1-c_3^2)}{(1+c_1 c_2 c_3)^2}.$$

3. Free Energy from localization:

$$F = \frac{4\sqrt{2}\pi N^{3/2}}{3} \sqrt{\prod_A R[Y^A]},$$

An earlier AdS/CFT calc. at the conformal point shows that the coefficients in F matches that of J.

4. Use $R[Y^1] = \frac{1}{2} + \delta_1 + \delta_2 + \delta_3$. Insert the δ_α as functions of the three c_α. For the specific value $n = 1/2$, the argument of the $\sqrt{\ldots}$ becomes a perfect square and matches the rational expression J!!

Part II: $\mathcal{N} = 2^*$ on S^4

A. The hypermultiplet fields are z_1, z_2, χ_1, χ_2 and their formal conjugates.
On flat R^4, the hypermultiplet mass term is

$$\mathcal{L}_{R^4} = m^2\, Tr(z_1\tilde{z}_1 + z_2\tilde{z}_2) + m\, Tr(\chi_1\chi_1 + \chi_2\chi_2 + h.c)$$

SUSY on S^4 requires a third operator (with $\Delta = 2$) Pestun, 2012

$$\mathcal{L}_{\rm mass} = \mathcal{L}_{R^4} + \frac{im}{2a} Tr(z_1^2 + z_2^2 + h.c.)$$

B. Some History: 1. Pilch-Warner, 1985 found truncation of $\mathcal{N} = 8, D = 5$ SG with two scalars ϕ, ψ dual to operators of $\mathcal{L}_{R^4}$. Constructed flat-sliced RG flow.
2. Pestun, 2012 derived the mass deformation on S^4, applied localization, yielding a matrix model.

3. Matrix model solved at large N, $\lambda >> 1$ by Buchel et. al., 1301.1597. Free Energy

$$F_{S^4} = -\frac{N^2}{2}(1 + m^2 a^2)\log[\frac{\lambda(1 + m^2 a^2)e^{2\gamma+1/2}}{16\pi^2}]$$

The third derivative w.r.t ma is scheme independent, so a gravity dual should match

$$\frac{d^3 F_{S^4}}{d(ma)^3} = -2N^2 \frac{ma(m^2 a^2 + 3)}{(m^2 a^2 + 1)^2}$$

4. Motivated by Buchel 1304.5622: Found soltn. on S^4 involving only the two Pilch-Warner scalars. It failed to match $d^3 F$.

Main problem was that the gravity dual on S^4 requires another scalar to source the third operator. We set out to restore the honor of holography!

5. We found new truncation with 3 scalars $\phi,\ \chi,\ \psi$. More simply expressed in terms of $\eta = e^{\phi/\sqrt{6}}$ and $z,\ \tilde{z} = (\chi \pm i\psi)/\sqrt{2}$.

$$
\begin{aligned}
\mathcal{L}_{5D} &= \frac{1}{4\pi G_5}\left[-\frac{R}{4} + 3\frac{\partial_\mu\eta\partial^\mu\eta}{\eta^2} + \frac{\partial_\mu z\partial^\mu\tilde{z}}{(1-z\tilde{z})^2} + V\right] \\
V &= -\left(\frac{1}{\eta^4} + 2\eta^2\frac{1+z\tilde{z}}{1-z\tilde{z}} + \frac{\eta^4}{4}\frac{(z-\tilde{z})^2}{(1-z\tilde{z})^2}\right).
\end{aligned}
$$

i) Simple— e.g. Poincaré disc again.
ii) Expand $V = -3 - \frac{1}{2}(4\phi^2 + 4\chi^2 + 3\psi^2) + \ldots$. Compare with AdS/CFT mass formula $\Delta = 2 + \sqrt{4+m^2}$ to find that the (mass)^2's $-4, -4, -3$ agree with the needed $\Delta = 2,\ 2,\ 3$ for $\phi,\ \chi,\ \psi$.

6. Extract 1st order BPS eqtns from the fermion trf. rules. No analytic solution, so we do the following.
i) UV asymptotics from expansion in e^{-r}, re^{-r} as $r \to \infty$.
Find that the 3 scalars and $A(r)$ depend on two independent parameters: source μ and vev v.
ii) Analysis of IR behavior as $r \to 0$; the four fields depend on one parameter.
iii) A smooth solution that interpolates from IR $\to$ UV will determine $v(\mu)$. From an accurate numerical solution, we extract relation $v(\mu) = -2\mu - \mu \log(1-\mu)^2$.

7. Finite CT: We require SUSY for the truncation of our system to ϕ, ψ of Pilch-Warner. Result is that

$$S_{\rm finite} = \frac{1}{16\pi G_5} \int d^4x \sqrt{h}\, \psi^4$$

must be added to $S_{\rm CT}$.

8. Final Steps: a. calculate $dF/d\mu = dS_{\rm ren}/d\mu$ using chain rule:

$$\frac{dS}{d\mu} = \frac{1}{4\pi G_5} \int d^4x \sqrt{g_0} \left(\langle \mathcal{O}_\psi \rangle \frac{\partial \psi_0}{\partial \mu} + \langle \mathcal{O}_\phi \rangle \frac{\partial \phi_0}{\partial \mu} + \langle \mathcal{O}_\chi \rangle \frac{\partial \chi_0}{\partial \mu} \right) .$$

The $\langle \mathcal{O}_\psi \rangle$, etc are renormalized vevs, and ψ_0, ϕ_0, χ_0 are leading UV source terms.

b. Express these quantities in terms of μ, $v(\mu)$. Use $\frac{1}{4\pi G_5} = \frac{N^2}{2\pi^2}$ to obtain

$$\frac{dS}{d\mu} = \frac{N^2}{2\pi^2} \mathrm{vol}(S^4)(4\mu - 12v(\mu))$$

c. Take two more derivatives using $v(\mu) = -2\mu - \mu \log(1-\mu)^2$:

Result

$$\frac{d^3F}{d\mu^3} = -2N^2\frac{\mu(3-\mu^2)}{(1-\mu^2)^2}$$

Compare with field theory:

$$\frac{d^3F_{S^4}}{d(ma)^3} = -2N^2\frac{ma(3+m^2a^2)}{(1+m^2a^2)^2}$$

Perfect agreement if $\mu = \pm ima$!

Conclusions:

Two different theories give precision tests of
AdS/CFT in a Euclidean, non-conformal setting !