Ten dimensional symmetry of $\mathcal{N} = 4$ SYM correlators

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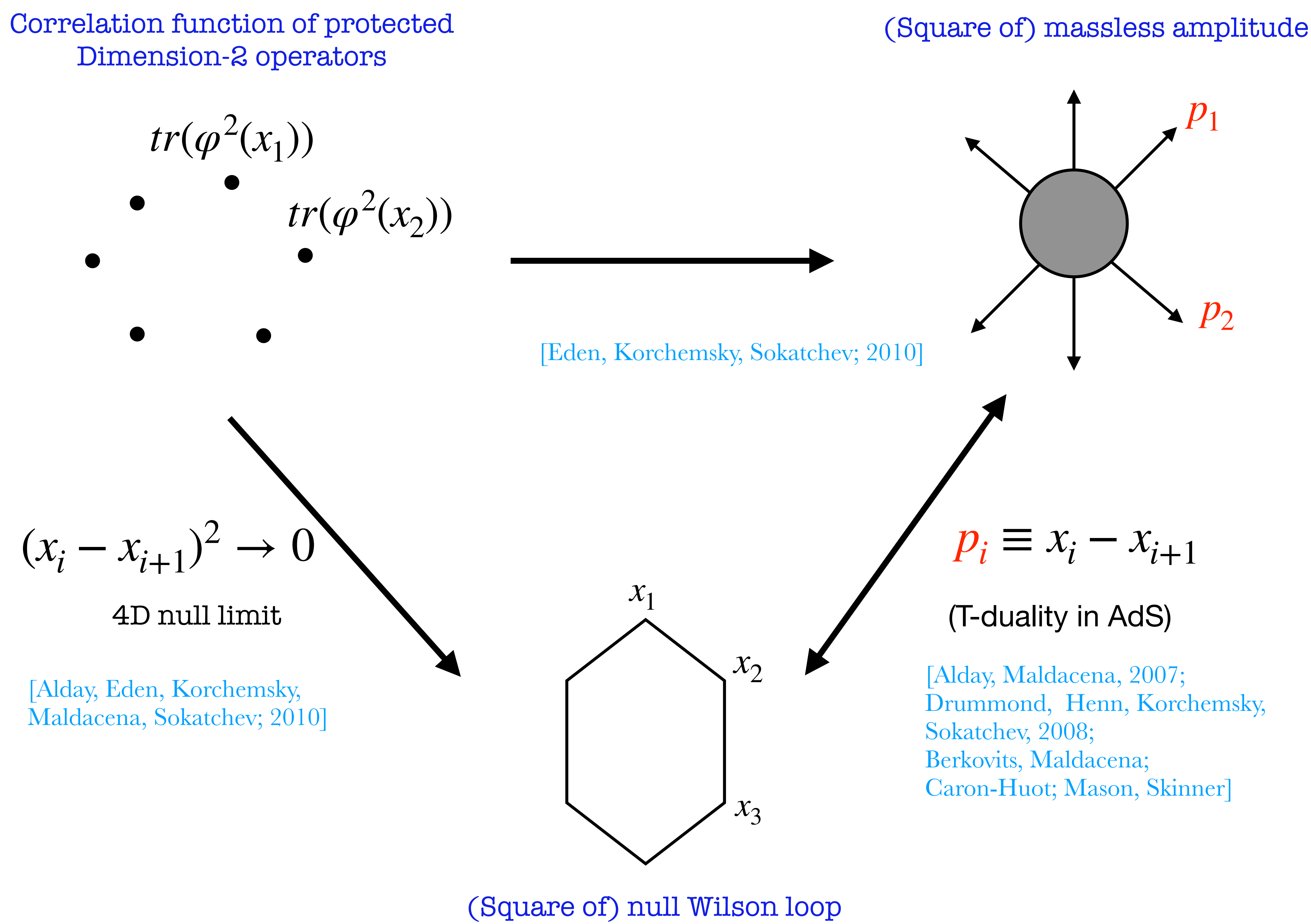
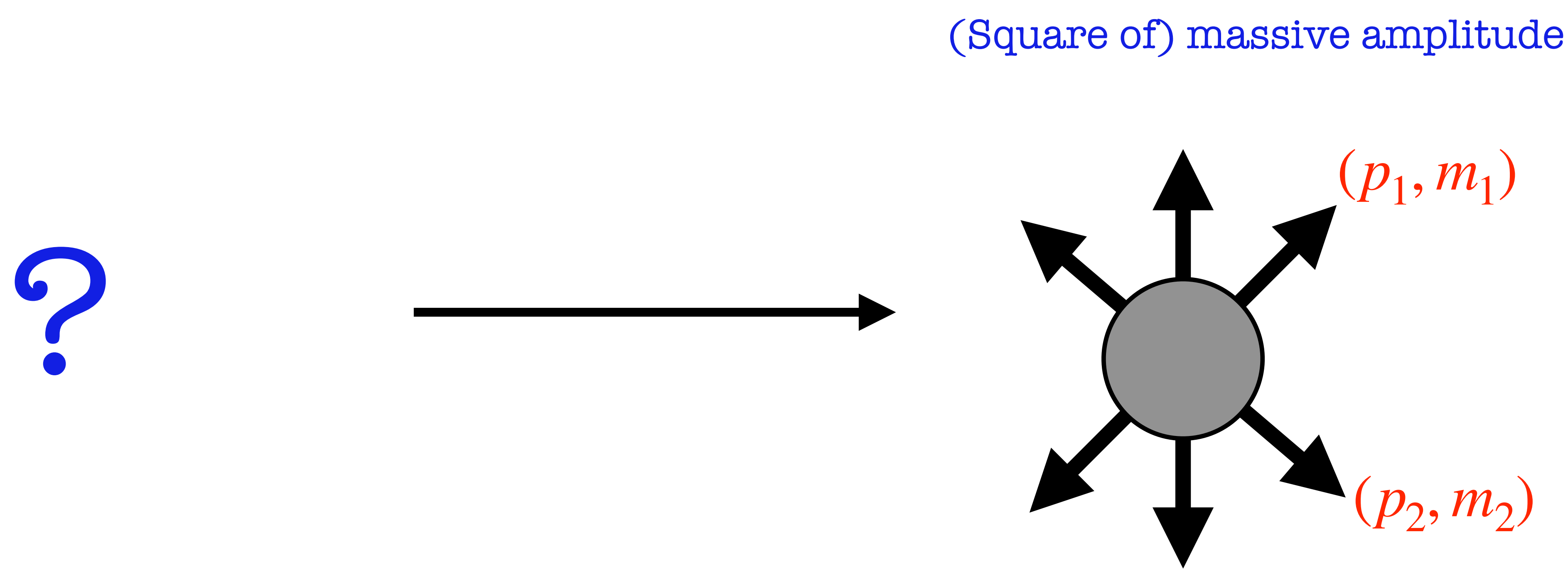
A triality in planar $\mathcal{N} = 4$ SYM



Generalization for massive amplitude



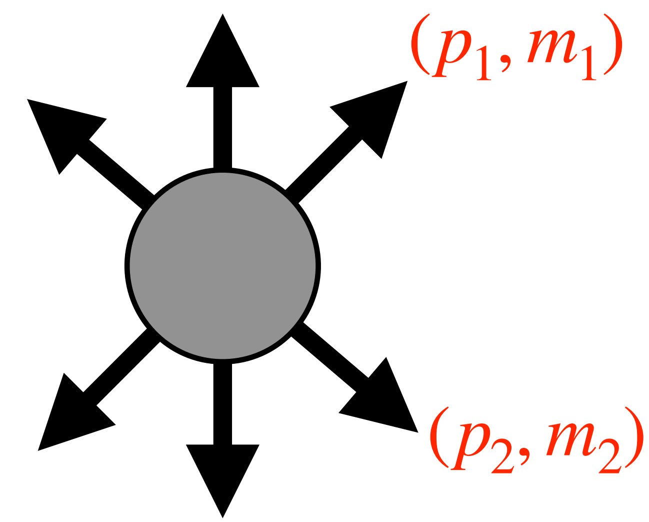
- Massive amplitude in the Coulomb branch (turned on VEVs for scalar fields).

- Amplitude (integrand) has a higher dimensional symmetry that acts on the vector (p_i, m_i) . [Alday, Henn, Plefka, Schuster; Caron-Huot,O'Connell; Bern, Carrasco, Dennen, Huang, Ita]

Generalization for massive amplitude

?

(Square of) massive amplitude



- Need object with higher-dimensional structure
- Candidates: BPS operators dual to KK modes in $AdS_5 \times S_5$

$$\mathcal{O}_k(x,\, y) \;=\; \frac{1}{k}\mathrm{Tr}\left(y.\Phi(x)\right)^k$$

- $\mathcal{O}_k$: # fields = Scaling dimension
- x: 4D position
- y: 6D null polarization vector, $y\,.\,y = 0$
- Φ: Vector of six scalars

- Massive amplitude in the Coulomb branch (turned on VEVs for scalar fields).
- Amplitude (integrand) has a higher dimensional symmetry that acts on the vector (p_i, m_i) .

- In SUGRA a 10D symmetry emerges when summing all (four-point) correlators of $\mathcal{O}_k(x,\, y)$

[Caron-Huot, Trinh, 2018; Aprile, Drummond, Heslop, Paul]

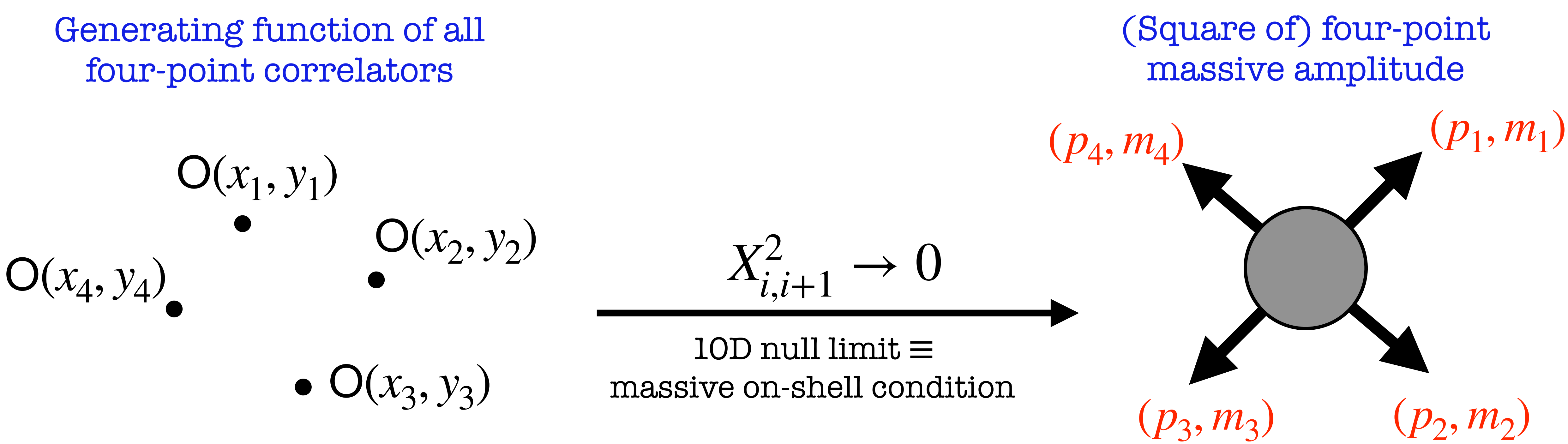
- This talk: similar 10D structure in a different coupling regime.

Generalization of correlator/massive amplitude

- The four-point function of the "master operator" $\mathrm{O}(x,y) \equiv \sum_{k}^{\infty} \mathcal{O}_k(x,y)$ has an emergent 10-dimensional structure that combines spacetime and R-charge distances:

$$X_{i,i+1}^2 \equiv (x_i - x_{i+1})^2 + (y_i - y_{i+1})^2 \overset{duality}{=} p_i^2 + m_i^2$$

- The 10D null limit of the "master" correlator is equal to a massive amplitude in the Coulomb branch.



- Checked at various loop orders.

Outline

- Ten dimensional structure of free correlators.
- 10D symmetry of loop integrands.

Weak Coupling

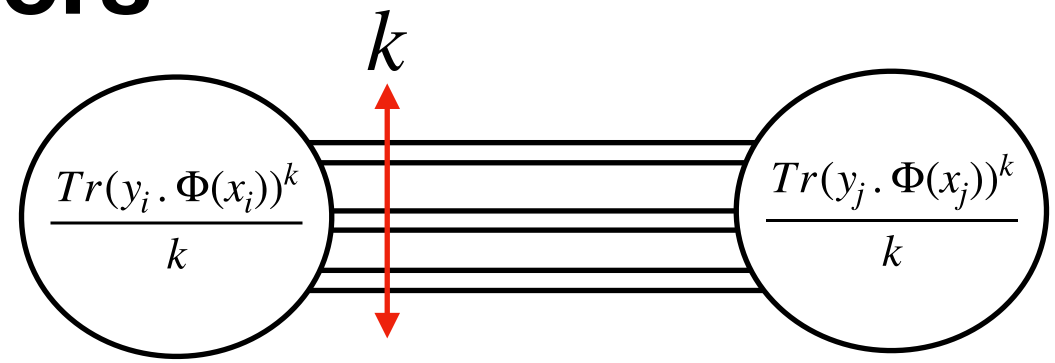
- 10D null limit:
  massive amplitude = large R-charge correlator (octagon).
- Amplitude/octagon from integrability and massless limit.

Free correlators

- Computed by Wick contractions:

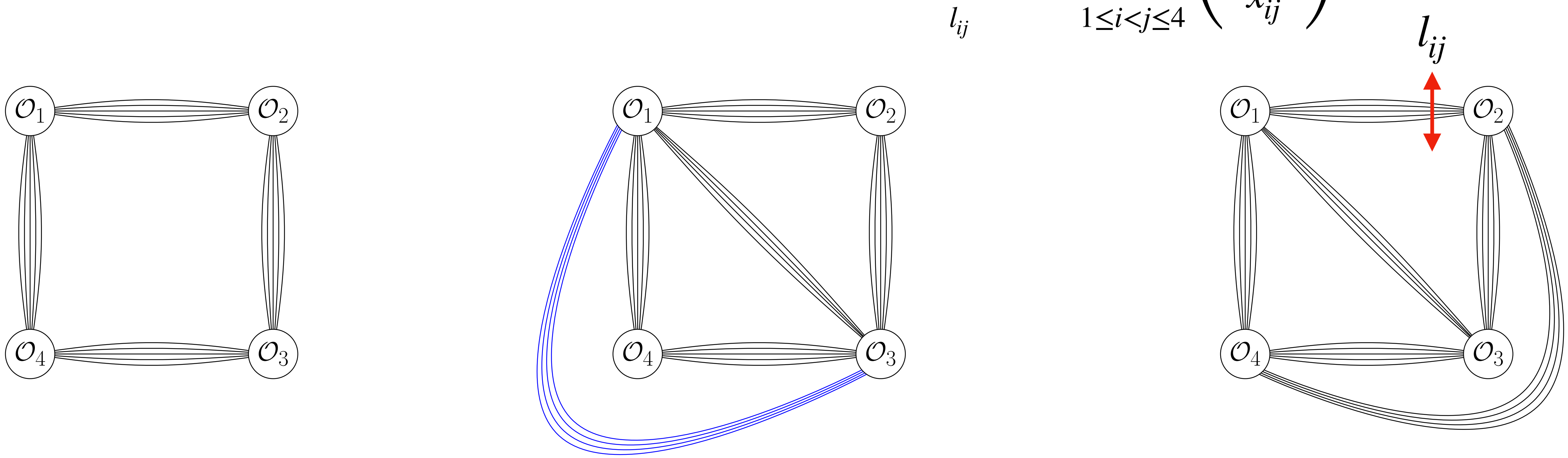
$$\langle \mathcal{O}_k(x_i, y_i)\, \mathcal{O}_k(x_j, y_j)\rangle = \frac{1}{k}\left(\frac{-y_{ij}^2}{x_{ij}^2}\right)^k + O(1/N_c^2)$$

$x_{ij}^2 \equiv (x_i - x_j)^2 \,, y_{ij}^2 \equiv (y_i - y_j)^2$



- The free four-point correlator of the "master operator" $\mathrm{O}(x, y) = \sum_k^\infty \mathcal{O}_k(x, y)$

$$G^{free} \equiv \langle \mathrm{O}(x_1, y_1)\, \mathrm{O}(x_2, y_2)\, \mathrm{O}(x_3, y_3)\, \mathrm{O}(x_4, y_4)\rangle^{(0)} = \sum_{l_{ij}}^\infty C_{\{l_{ij}\}} \prod_{1\le i<j\le 4}\left(\frac{-y_{ij}^2}{x_{ij}^2}\right)^{l_{ij}}$$



$$G^{free} = D_{12}D_{23}D_{34}D_{41} + D_{12}D_{23}D_{34}D_{41}(2D_{13} + D_{13}^2) + 2\, D_{12}D_{13}D_{14}D_{23}D_{24}D_{34} + \textbf{perm.}$$

- **Emergente 10D structure :** $$D_{ij} \equiv \frac{-y_{ij}^2}{x_{ij}^2 + y_{ij}^2} = \sum_{k=1}^\infty \left(\frac{-y_{ij}^2}{x_{ij}^2}\right)^k$$

Loop integrands

- Perturbative series in the 't Hooft coupling $g^2 \equiv \frac{g_{\rm YM}^2 N_c}{16\pi^2}$

$$\langle \mathcal{O}_{k_1}\mathcal{O}_{k_2}\mathcal{O}_{k_3}\mathcal{O}_{k_4}\rangle_{\rm c} = G^{\rm free}_{k_1k_2k_3k_4} + \sum_{\ell=1}^{\infty} G^{(\ell)}_{k_1k_2k_3k_4} + O(1/N_c^2)$$

- We can define an integrand by the Lagrangian insertion method:

$$G^{(\ell)}_{k_1k_2k_3k_4} = \frac{(-g^2)^\ell}{\ell!}\int \frac{d^4x_5}{\pi^2}\cdots\frac{d^4x_{4+\ell}}{\pi^2}\,\mathcal{G}^{(\ell)}_{k_1k_2k_3k_4}\,,$$

- The ℓ-loop integrand is a $(4+\ell)$-point correlator evaluated at leading order:

$$\mathcal{G}^{(\ell)}_{k_1k_2k_3k_4} = \langle \mathcal{O}_{k_1}\mathcal{O}_{k_2}\mathcal{O}_{k_3}\mathcal{O}_{k_4}\mathcal{L}(x_5)\cdots\mathcal{L}(x_{4+\ell})\rangle^{(0)}$$

$$\overset{SUSY}{=} R_{1234}\left(2\,x_{12}^2x_{13}^2x_{14}^2x_{23}^2x_{24}^2x_{34}^2\right)\mathcal{H}^{(\ell)}_{k_1k_2k_3k_4}\,.$$

$$R_{1234} = \frac{(y_{13}^2\,y_{24}^2)^2}{x_{13}^2\,x_{24}^2} + \frac{y_{12}^2\,y_{23}^2\,y_{34}^2\,y_{41}^2}{x_{12}^2\,x_{23}^2\,x_{34}^2\,x_{41}^2}\,(x_{13}^2x_{24}^2 - x_{12}^2x_{34}^2 - x_{14}^2x_{23}^2) + (1\leftrightarrow 2) + (1\leftrightarrow 4)\,.$$

[Eden, Petkou, Schubert. Sokatchev]

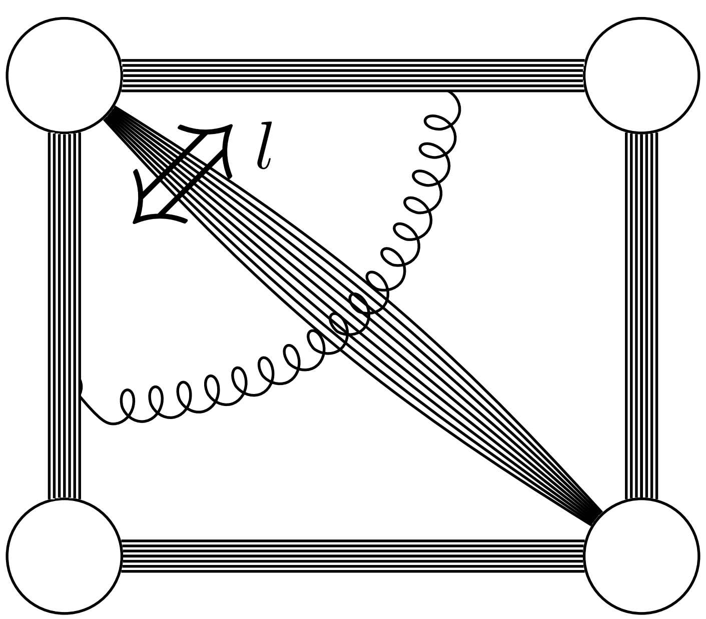
- Advantage: integrand is a rational function with simple poles. It treats external and integration points almost in the same footing (e.g. $\mathcal{H}_{2222}$ has a full permutation symmetry). [Eden, Heslop, Korchemsky, Sokatchev, 2011]

- **Decomposition in R-charge:**

$$\mathcal{H}^{(\ell)}_{k_1k_2k_3k_4} = \sum_{k_i-2=\sum_j b_{ij}} \mathcal{F}^{(\ell)}_{\{b_{ij}\}}(x^2_{ij}) \times \prod_{1\le i<j\le 4} \left(\frac{-y^2_{ij}}{x^2_{ij}}\right)^{b_{ij}}$$

The number of inequivalent structures $\mathcal{F}^{(\ell)}_{\{b_{ij}\}}$ is finite and depends on the loop order.

- **Saturation:** thanks to planarity, a bridge becomes uncrossable when the number of propagators is larger than the loop order.



$$\sim (g^2)^l$$

$$\mathcal{F}^{(\ell)}_{\{b_{12},\cdots\}} \equiv \mathcal{F}^{(\ell)}_{\{\ell-1,\cdots\}}$$

$$\text{if } b_{12} \ge \ell - 1\,.$$

[Chicherin, Drummond, Heslop, Sokatchev, 2015]

- After saturation, we have an infinite tail forming a geometric series.

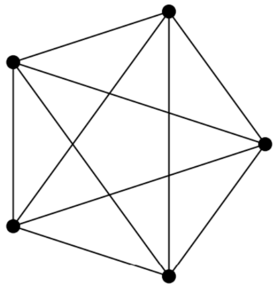
One-loop integrands

- At one loop, saturation implies that all R-charge structures are identical:

$$\mathcal{F}^{(1)}_{\{b_{ij}\}} = \mathcal{F}^{(1)}_{\{0,0,0,0,0,0\}}$$

- The reduced integrands:

$$\mathcal{H}^{(1)}_{2222} = \mathcal{F}^{(1)}_{\{0,0,0,0,0,0\}} = \frac{1}{\prod\limits_{1\le i<j\le 5} x_{ij}^2}$$



$$\mathcal{H}^{(1)}_{k_1k_2k_3k_4} = \sum_{\substack{\{b_{ij}\} \\ k_i-2=\sum_j b_{ij}}} \frac{1}{\prod\limits_{1\le i<j\le 5} x_{ij}^2} \times \prod_{1\le i<j\le 4} \left(\frac{-y_{ij}^2}{x_{ij}^2}\right)^{b_{ij}}$$

- Resumming the geometric series:

$$\mathcal{H}^{(1)} = \sum_{k_i\ge 2} \mathcal{H}^{(1)}_{k_1k_2k_3k_4} = \frac{1}{\prod\limits_{1\le i<j\le 5} (x_{ij}^2 + y_{ij}^2)} \qquad \text{with } y_{5i}^2 = 0$$

- Higher-loop data shows similar pattern.

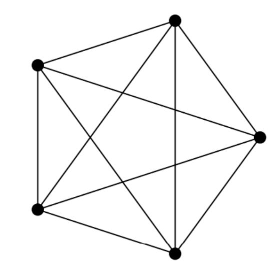
10D symmetry of loop integrands

- At each loop order, all (reduced) integrands form a geometric series that resums into a function which depends only on $X_{ij}^2 \equiv x_{ij}^2 + y_{ij}^2$.

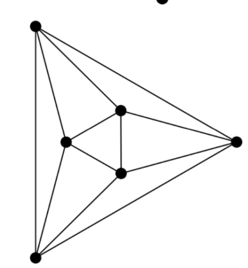
$$\mathcal{H}^{(\ell)}_{k_1 k_2 k_3 k_4}(x_{ij}^2, y_{ij}^2) = \text{coefficient of } \left(\prod_{i=1}^4 \beta_i^{k_i - 2}\right) \text{ in } \mathcal{H}^{(\ell)}(X_{ij}^2)\,\Big|_{y_{ij}^2 \to \beta_i \beta_j\, y_{ij}^2} \,,$$

- This generating function can be uplifted from the known case $\mathcal{H}_{2222}$ by replacing all four-dimensional distances x_{ij}^2 by ten-dimensional ones X_{ij}^2

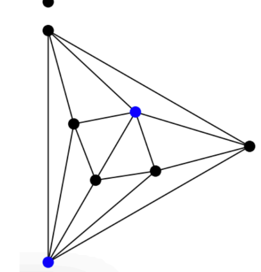
$$\mathcal{H}^{(1)} = \frac{1}{\prod_{1\le i<j\le 5} X_{ij}^2}\,,$$



$$\mathcal{H}^{(2)} = \frac{1}{48}\,\frac{X_{12}^2 X_{34}^2 X_{56}^2 \;+\; S_6\,\text{permutations}}{\prod_{1\le i<j\le 6} X_{ij}^2}\,,$$



$$\mathcal{H}^{(3)} = \frac{1}{20}\,\frac{(X_{12}^2)^2\,(X_{34}^2 X_{45}^2 X_{56}^2 X_{67}^2 X_{73}^2) \;+\; S_7\,\text{permutations}}{\prod_{1\le i<j\le 7} X_{ij}^2}\,.$$



[Eden, Heslop, Korchemsky, Sokatchev; 2012]

- It inherits the full permutation symmetry of $\mathcal{H}_{2222}$.

  The dimension-2 operator and the chiral Lagrangian belong to the stress-tensor super-multiplet.

10D structure of four-point correlators

- We set the 6D null condition for the external points and turn off the R-charge of the internal points.

$$y_i . y_i = 0 \quad \text{when } i = 1,2,3,4 \quad \text{and} \quad y_i = 0 \quad \text{when} \quad i = 5, \cdots, 4+\ell$$

and integrate:
$$G^{(\ell)} \equiv \sum_{k_i \geq 2} G^{(\ell)}_{k_1 k_2 k_3 k_4} = \frac{(-g^2)^\ell}{\ell!} R_{1234} \left(2\, x_{12}^2 x_{13}^2 x_{14}^2 x_{23}^2 x_{24}^2 x_{34}^2\right) \int \frac{dx_5^4}{\pi^2} \cdots \frac{dx_{4+\ell}^4}{\pi^2} \mathcal{H}^{(\ell)} .$$

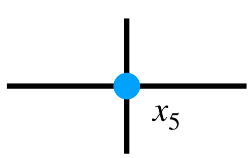
- One and two-loop examples:

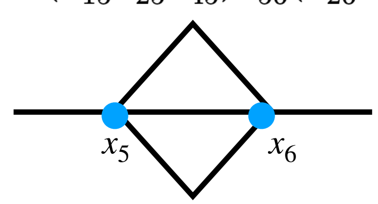
$$G^{(1)} = -2g^2 R_{1234}\, g_{1234} \prod_{1 \leq i < j \leq 4} \frac{1}{1-d_{ij}} ,$$

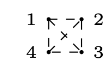
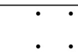
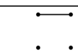
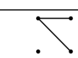
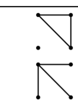
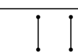
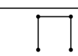
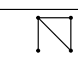
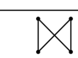
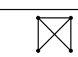
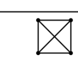
$$G^{(2)} = 2g^4 R_{1234} \left(c_h^1\, h_{12;34} + c_h^2\, h_{13;24} + c_h^3\, h_{14;23} + \frac{1}{2} \left(c_{gg}^1\, x_{12}^2 x_{34}^2 + c_{gg}^2\, x_{13}^2 x_{24}^2 + c_{gg}^3\, x_{14}^2 x_{23}^2 \right) [g_{1234}]^2 \right)$$

$$c_h^1 = \frac{(1-d_{12}) + (1-d_{34})}{\prod_{1 \leq i < j \leq 4} (1-d_{ij})} \quad \text{and} \quad c_{gg}^1 = \frac{(1-d_{12})(1-d_{34})}{\prod_{1 \leq i < j \leq 4} (1-d_{ij})} \quad \text{with} \quad d_{ij} \equiv \frac{-y_{ij}^2}{x_{ij}^2} .$$

$$g_{1234} = \frac{1}{\pi^2} \int \frac{d^4 x_5}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2} \quad \text{and} \quad h_{13;24} = \frac{x_{24}^2}{\pi^4} \int \frac{d^4 x_5\, d^4 x_6}{(x_{15}^2 x_{25}^2 x_{45}^2) x_{56}^2 (x_{26}^2 x_{36}^2 x_{46}^2)} .$$





$\{b_{ij}\}$		c_{gg}^1	c_{gg}^2	c_{gg}^3	c_h^1	c_h^2	c_h^3
$\{0,0,0,0,0,0\}$		1	1	1	2	2	2
$\{\beta_1,0,0,0,0,0\}$		0	1	1	1	2	2
$\{\beta_1,\beta_2,0,0,0,0\}$		0	0	1	1	1	2
$\{\beta_1,\beta_2,0,\beta_3,0,0\}$ $\{\beta_1,\beta_2,\beta_3,0,0,0\}$		0	0	0	1	1	1
$\{0,0,\beta_1,\beta_2,0,0\}$		1	1	0	2	2	0
$\{\beta_1,0,\beta_2,\beta_3,0,0\}$		0	1	0	1	2	0
$\{\beta_1,\beta_2,\beta_3,\beta_4,0,0\}$		0	0	0	1	1	0
$\{0,\beta_1,\beta_2,\beta_3,\beta_4,0\}$		1	0	0	2	0	0
$\{\beta_1,\beta_2,\beta_3,\beta_4,\beta_5,0\}$		0	0	0	1	0	0
$\{\beta_1,\beta_2,\beta_3,\beta_4,\beta_5,\beta_6\}$		0	0	0	0	0	0

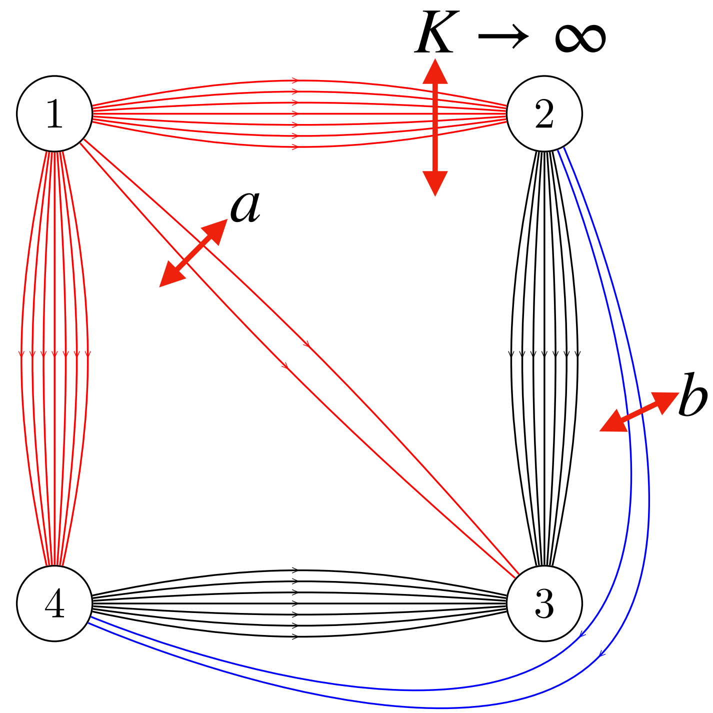
$\beta_i \geq 1$

- Similar checks up to 5 loops. [Chicherin, Georgoudis, Goncalves, Pereira, 2018] [Chicherin, Drummond, Heslop, Sokatchev, 2015]
- Predictions at higher loops (up to ten loops from knowledge of the seed $\mathcal{H}_{2222}$) [Bourjaily, Heslop, Tran, 2016]
- Higher loop integrals are hard to evaluate.
- A tractable problem using integrability: correlators with large R-charge (octagons). [FC 2018]

10D null limit: octagon = amplitude

- The simplest correlators factorized into squares (octagons).

- Their integrands receive contributions only from $\mathcal{F}_{\{a,\infty,\infty,\infty,\infty,b\}}$



$O_1 = \mathrm{Tr}(\bar{X}^{2K+a})$

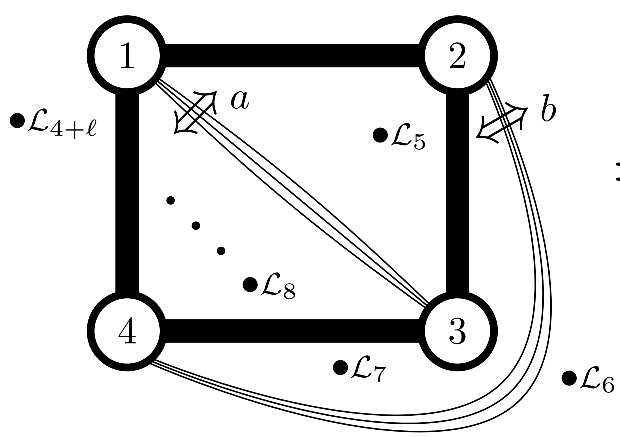
$O_2 = \mathrm{Tr}(X^K \bar{Z}^K \bar{Y}^b) + \text{cyclic permutations}$

$O_3 = \mathrm{Tr}(Z^{2K} X^a) + \text{cyclic permutations}$

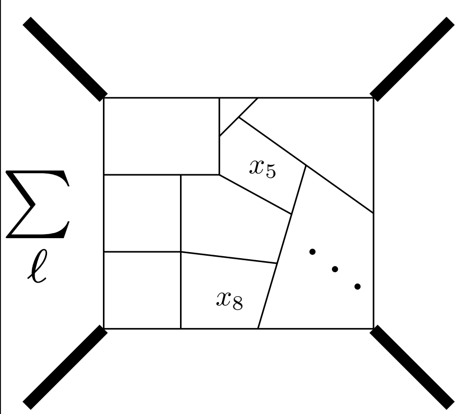
$O_4 = \mathrm{Tr}(Z^K \bar{X}^K Y^b) + \text{cyclic permutations}$

- Only the terms of "G" with the four poles $\dfrac{1}{X_{12}^2 X_{23}^2 X_{34}^2 X_{41}^2}$ contribute to the simplest correlators. We can select them by taking the 10D null limit:

$$\lim_{X_{i,i+1}^2 \to 0} G = \sum_{a,b,\ell}$$



$$= \Bigg[\sum_\ell$$



$$\Bigg]^2$$

$p_i^\mu \equiv x_{i,i+1}^\mu$ and $m_i^2 \equiv y_{i,i+1}^2$

$$= \quad \mathbb{O} \times \mathbb{O} \quad = \quad M \times M$$

- Octagon and amplitude are identical at the integrand and integrated level. They are IR finite.

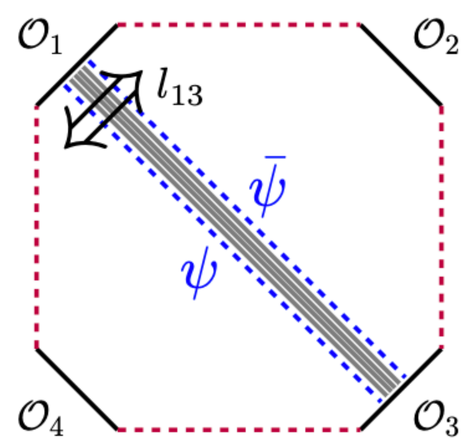
Octagon from Integrability

$$\mathbb{O} = \mathbb{O}_0 + \sum_{l=1}^{\infty} (d_{13})^l \, \mathbb{O}_l + (d_{24})^l \, \mathbb{O}_l$$

Gluing two hexagons by summing over mirror particles:

[Basso, Komatsu, Vieira 2013;
Fleury, Komatsu; Eden, Sfondrini 2016;
FC 2018]

$$\mathbb{O}_l(z, \bar{z}, d_{13}, d_{24}) = \lim_{X^2_{i,i+1} \to 0} \sum_{\psi}$$



ψ : **complete basis of states in the two-dimensional world sheet**

Octagon is given by an infinite determinant

$$\boxed{\mathbb{O}_l = \det(1 - \mathbb{K}_l)}$$

[Kostov, Petkova, Serban;
Belitsky, Korchemsky;
2019-2021]

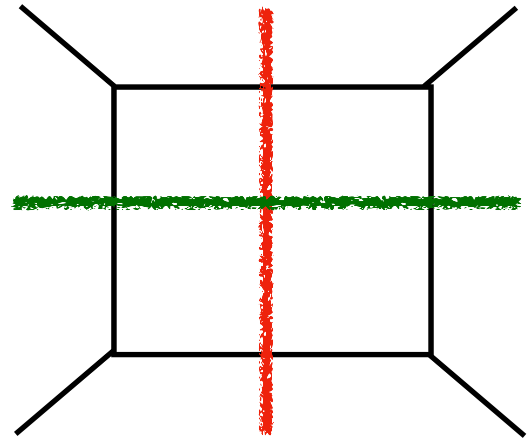
$$(\mathbb{K}_l)_{ij} = (-1)^{i-j}(2j + l - 1) \int_0^{\infty} d\tau \, \chi(\tau) \, \frac{J_{2i+l-1}(2g\tau) \, J_{2j+l-1}(2g\tau)}{\tau}$$

$$\chi(\tau) = \frac{(1 - d_{13}d_{24})}{\sqrt{z\bar{z}(1-z)(1-\bar{z})}} \frac{1}{\cosh(\sqrt{\zeta^2 + \tau^2}) - \cos\phi}, \text{ with } e^{-2\zeta} = \frac{z\bar{z}}{(1-z)(1-\bar{z})}, \, e^{2i\phi} = \frac{z(1-\bar{z})}{\bar{z}(1-z)}.$$

$$z\bar{z} = \frac{x^2_{12}x^2_{34}}{x^2_{13}\, x^2_{24}}, \, (1 - z)(1 - \bar{z}) = \frac{x^2_{14}x^2_{23}}{x^2_{13}\, x^2_{24}}$$

Steinmann condition for amplitudes:

Double discontinuities vanish in overlapping channels



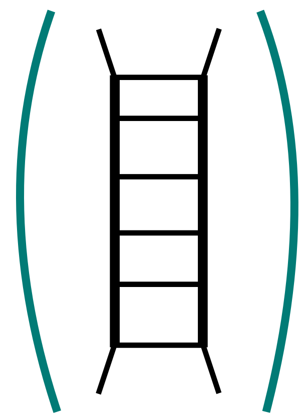
Octagon has a vanishing double discontinuity

$$\text{disc}_s\,\text{disc}_t\,\mathbb{O} = 0$$

with $s = x_{13}^2\,,\, t = x_{24}^2$

Weak coupling:

$\mathbb{O}$ = sum of det

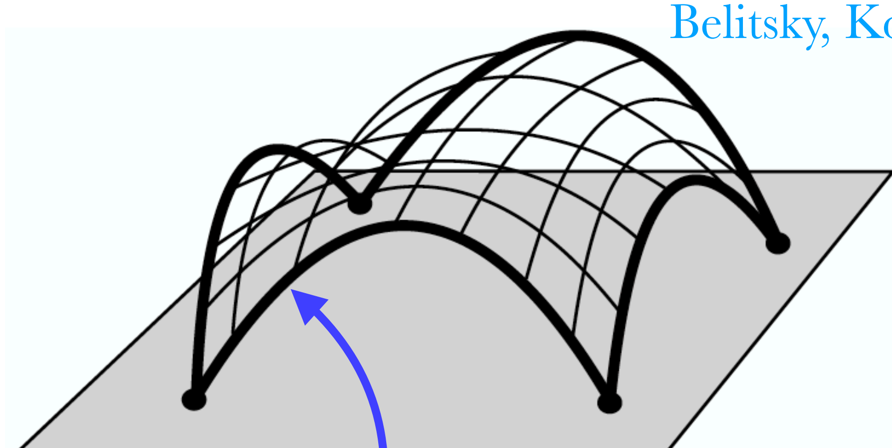


Provides analytic results for (unknown) conformal integrals such as fishnets and deformations

Strong coupling:

$$\mathbb{O} = e^{-g\,Area}$$

[Bargheer, FC, Vieira, 2019; Belitsky, Korchemsky]

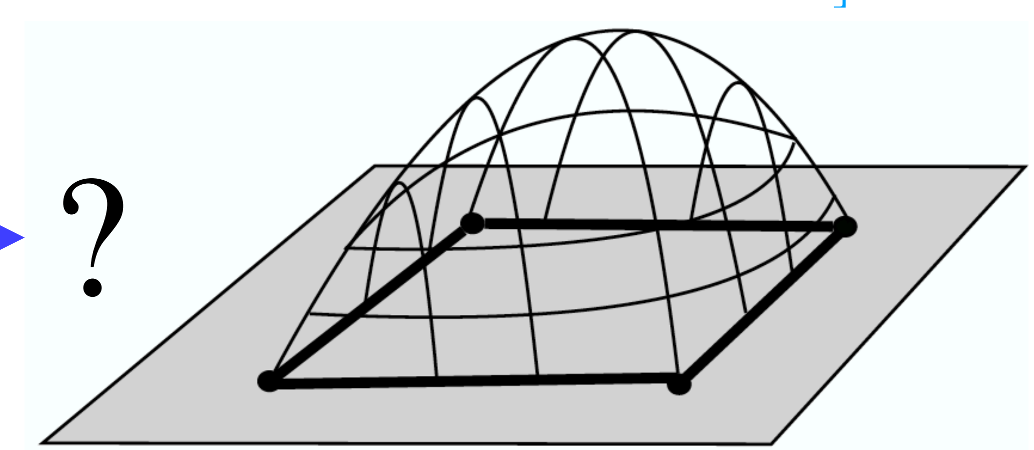


$x_{i,i+1}^2 \to 0$

4D null limit of octagon or massless limit of the amplitude

?

[Alday, Maldacena, 2007; Kruczenski]

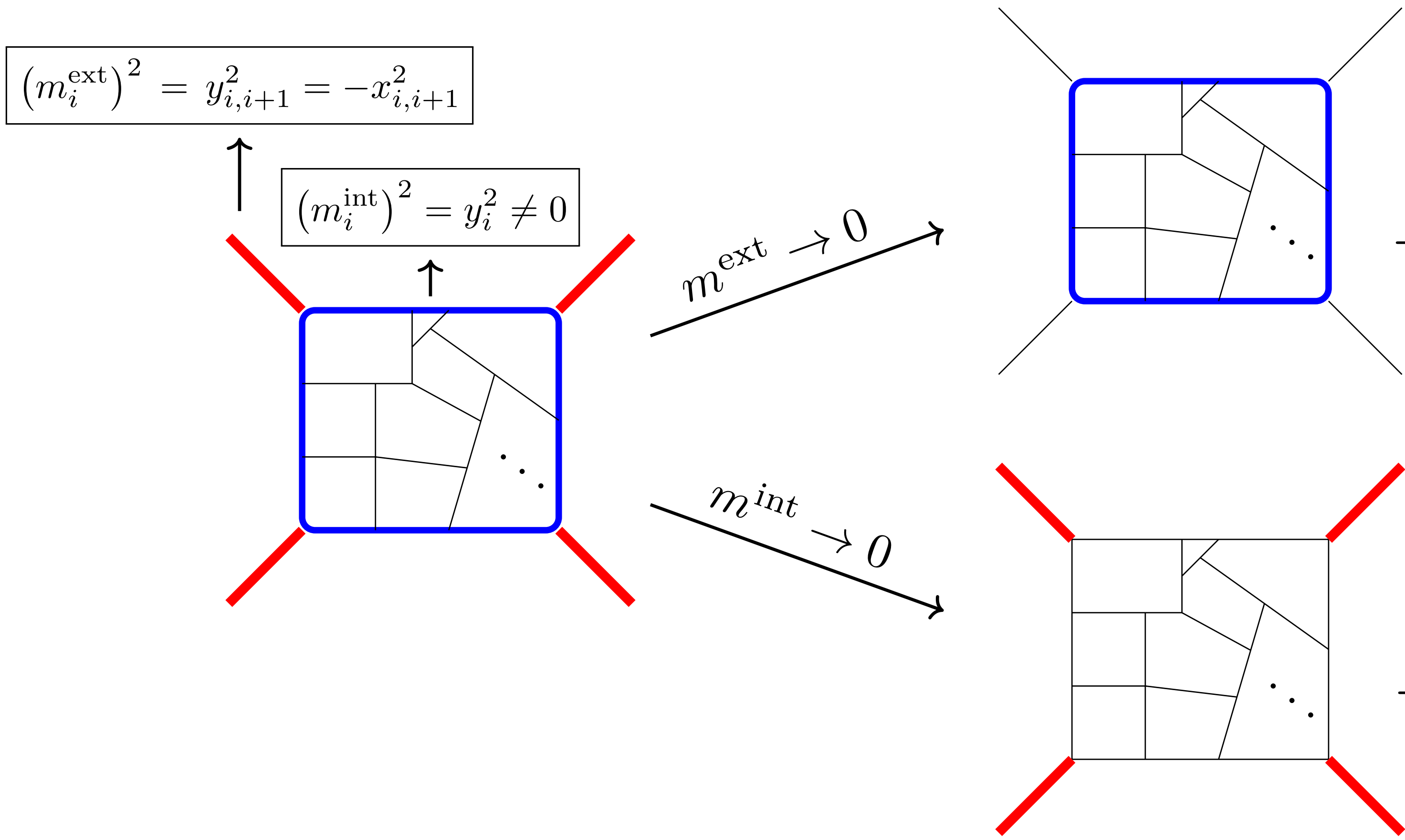


Perimeter is null in $AdS_5 \times S_5$

Four-cusped null Wilson loop

Massless limit

Coulomb-branch amplitudes with double logarithmic scaling:



$\xrightarrow{m^{\text{int}} \to 0} \quad e^{-\frac{1}{2}\Gamma_{\text{cusp}} \log \frac{(m^{\text{int}})^2}{x_{13}^2} \log \frac{(m^{\text{int}})^2}{x_{24}^2}}$

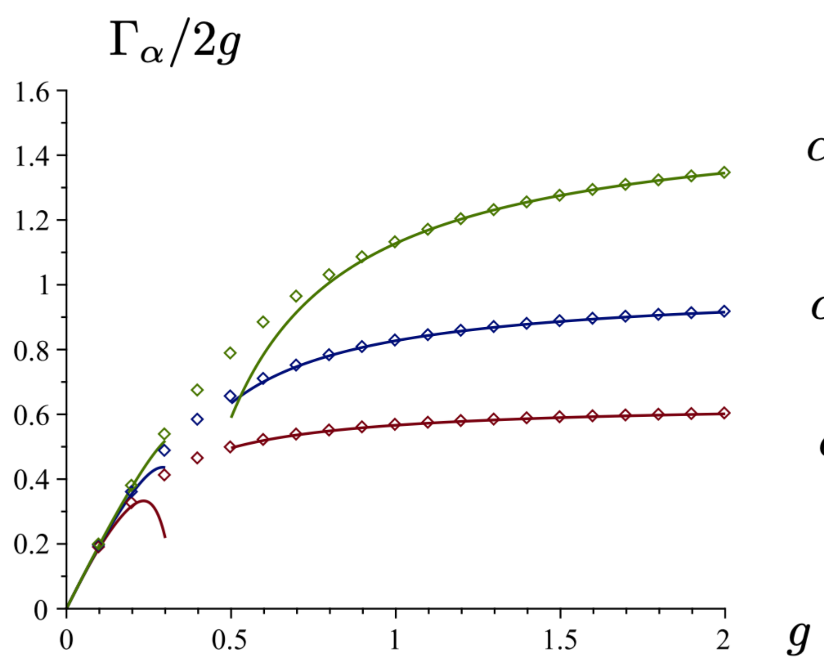
[Alday, Henn, Plefka, Schuster, 2007; Bern, Dixon, Smirnov, 2005]

$\xrightarrow{m^{\text{ext}} \to 0} \quad e^{-\frac{1}{16}\Gamma_{\text{oct}} \log^2\left(\frac{x_{12}^2 x_{23}^2 x_{34}^2 x_{41}^2}{(x_{13}^2 x_{24}^2)^2}\right)}$

[FC, 2018]

Controlled by functions of the coupling satisfying a deformed BES equation:

[Beisert, Eden, Staudacher, 2006]



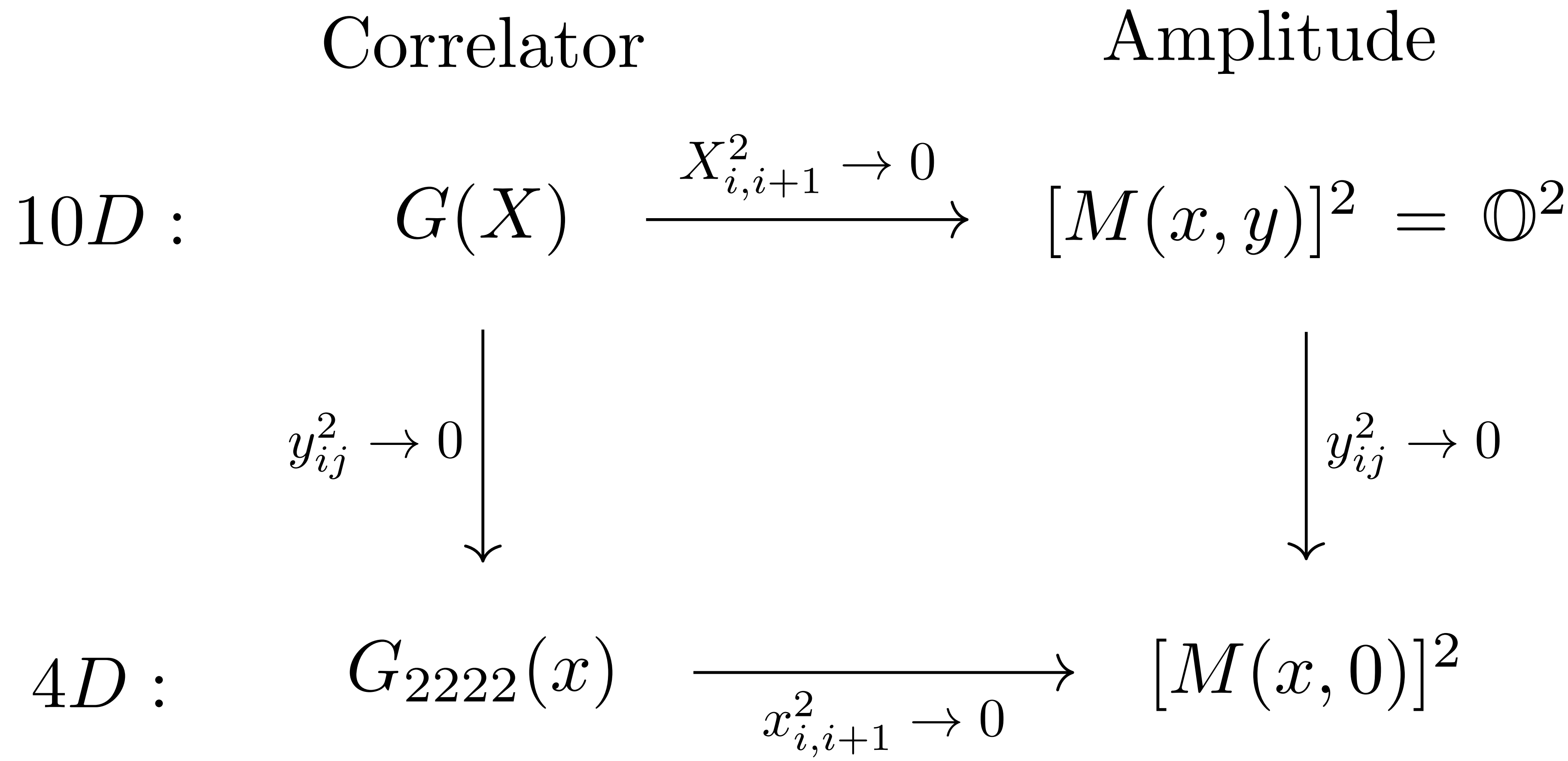
$\alpha = \frac{\pi}{3}$

[Basso, Dixon, Papathanasious, 2020]

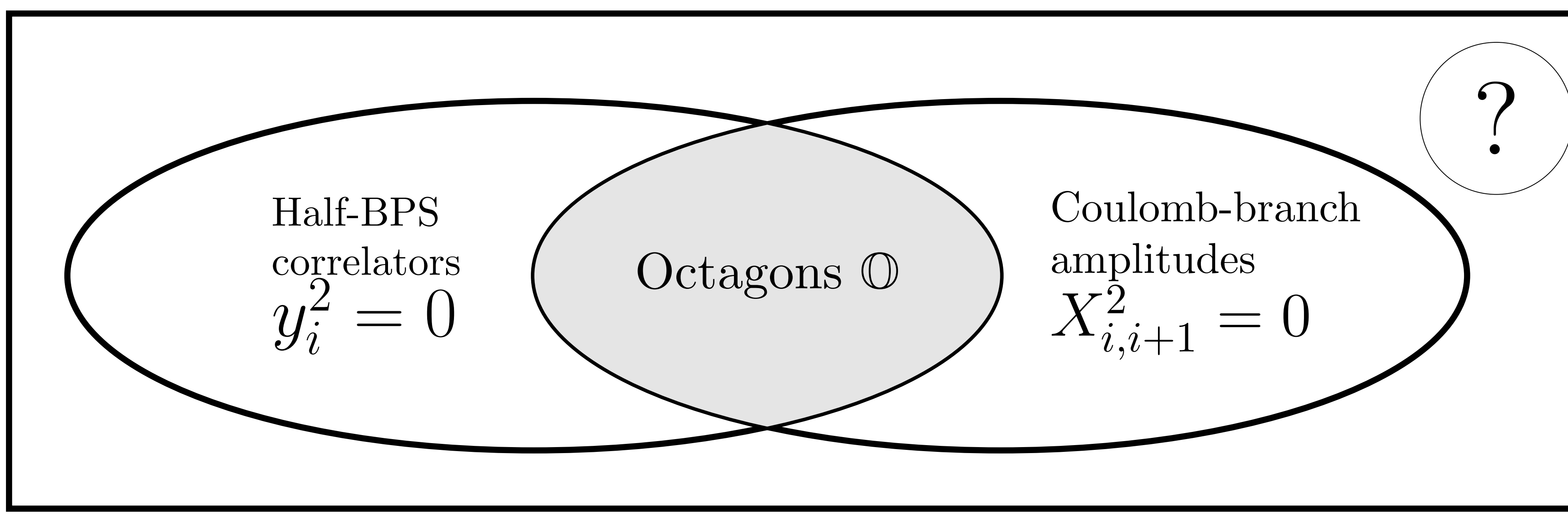
$\alpha = \frac{\pi}{4} \longrightarrow \Gamma_{\text{cusp}}$

$\alpha = 0 \longrightarrow \Gamma_{\text{oct}} = \frac{2}{\pi^2} \log\cosh(2\pi g)$ [Belitsky, Korchemsky, 2019]

Summary



Outlook



- **Can we relax the null condition?**
- **Higher-points?**
- **Relation to 10D sym. in SUGRA?**
- **Integrability in the Coulomb branch?**