Singularities and Gauge Theory Phases

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With
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Other teams

- Kumar, Park, Taylor
- Grimm, Hayashi
- Krause, Mayrhofer, Weigand
- Hayashi, Lawrie, Morisson , Schafer-Nameki

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Love to my other collaborators:

Paolo Aluffi
Patrick Jefferson
Michele Del Zotto, Jonathan Heckman, Cumrun Vafa

At the menu:

- Geometry
- Gauge theories
- Representation theory
- Singularities
- Resolutions of singularities

It also comes as a combo:

…an elliptic fibration

Tale of two worlds

Gauge theories	**Elliptic fibrations**
Gauge algebra	Codimension one singularities
Representations	Codimension two singularities
Yukawa	Codimension three singularities
Coulomb phases	Crepant resolutions
Walls	Partial resolutions
Phase transitions	Flops

5D supersymmetric gauge theories with 8 supercharges

Matter content:

Gravity multiplet
Vector multiplets
Hypermultiplets
Tensor multplets

5D supersymmetric gauge theories with 8 supercharges

Geometry:

Vector multiplets	Very Special Geometry
Hypermultiplets	Quaternionic-Kähler
	Hyperkähler

Coulomb branch of 5D gauge theory

- Vector multiplets-> Weyl chamber
- Massless Hypers at singularities -> sub-chamber structure

$$\mathcal{F}(\phi) = \frac{1}{2} m_0 \mathrm{Tr}(\phi^2) + \frac{c_{cl}}{6} \mathrm{Tr}(\phi^3) + \frac{1}{12} \left(\sum_{\alpha:\mathrm{roots}} |(\phi, \alpha)|^3 - \sum_{w:\mathrm{weights}} |(\phi, w)|^3 \right) .$$

Incidence geometry of a representation R

- g is a Lie algebra with Cartan sub-algebra h
- Roots of g define Weyl chambers
- R : a representation of g
- Weights of R refine the Weyl chambers by adding a sub-chamber structure

Elliptic fibrations

- Elliptic curves are some of the oldest but yet most prominent objects across mathematics
  - Number theory
  - Algebraic geometry
  - Cryptography
  - Geometric design
  - Physics

Elliptic fibrations

$$\begin{array}{ccc} T^2 & \longrightarrow & Y \\ & & \big\downarrow \\ & & B \end{array}$$

Weierstrass model:

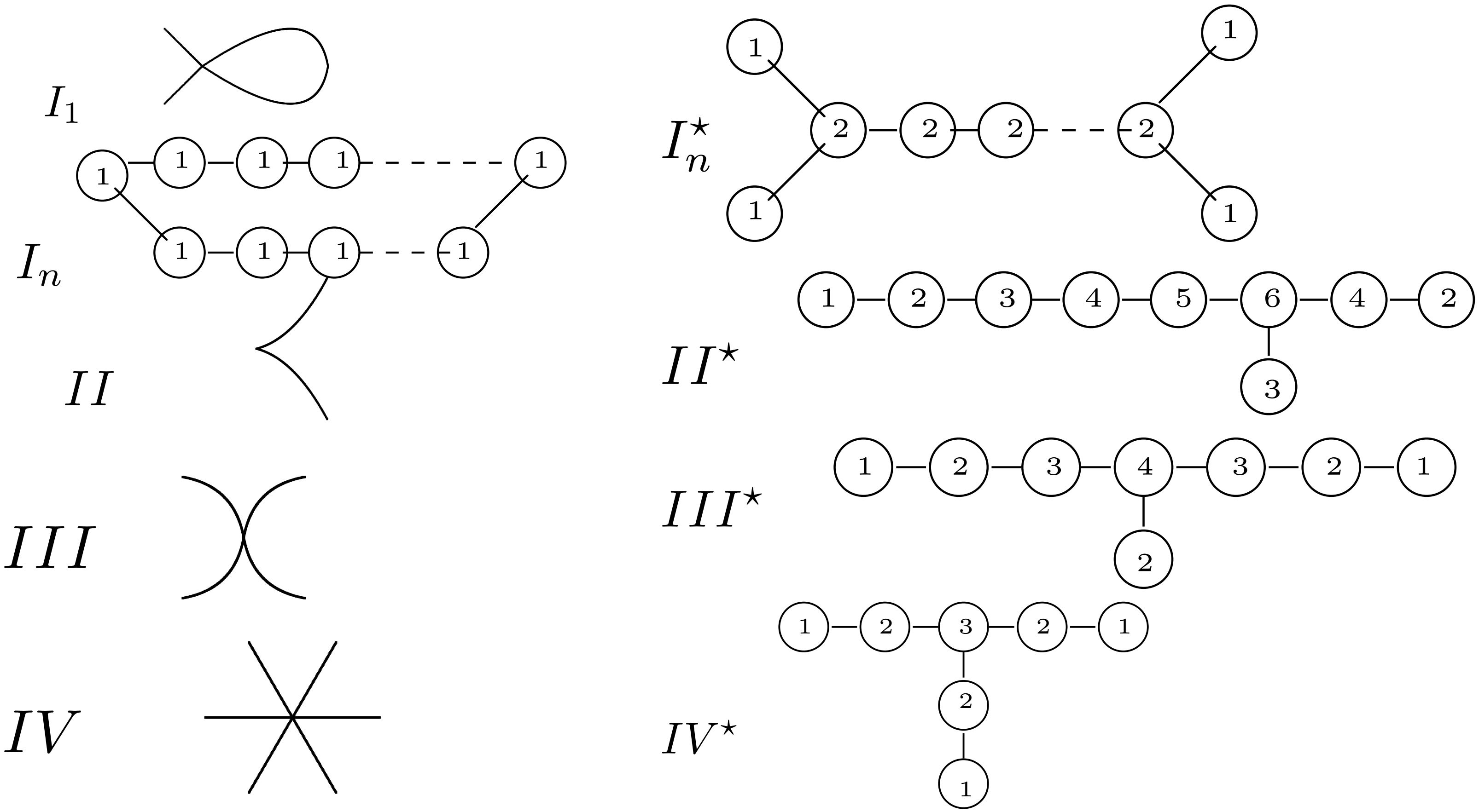
$$Y: \quad y^2 z + a_1 xyz + a_3 yz^2 = x^3 + a_2 x^2 z + a_4 xz^2 + a_6 z^3$$

Singular fibers

- Kodaira
- Weierstrass model
- Néron models
- Tate’s algorithm
- Miranda models (collisions of singularities)
- Szydlo (generalization of Miranda’s model and Tate’s algorithm)

Singular fibers

- Kodaira classification (for elliptic surfaces)



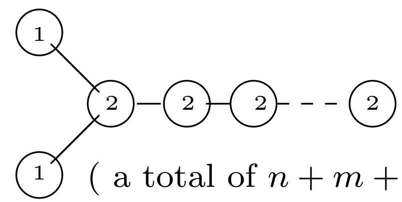
Collision of singularities

(Miranda models)

$J = \infty \quad : \quad I_n + I_m \to I_{n+m}$

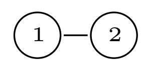
$I_{2n} + I_m^* \to I_{n+m}^*$

$I_{2n+1} + I_m^* \to I_{n+m+1}^{*+}$

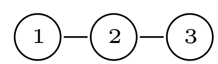


(a total of $n + m + 5$ nodes)

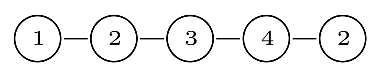
$J = 0 \quad : \quad II + IV \longrightarrow$



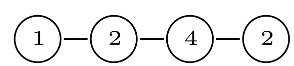
$II + I_0^* \longrightarrow$



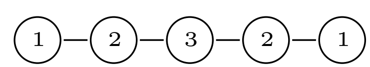
$IV + I_0^* \longrightarrow$



$II + IV^* \longrightarrow$



$J = 1 \quad : \quad III + I_0^* \longrightarrow$

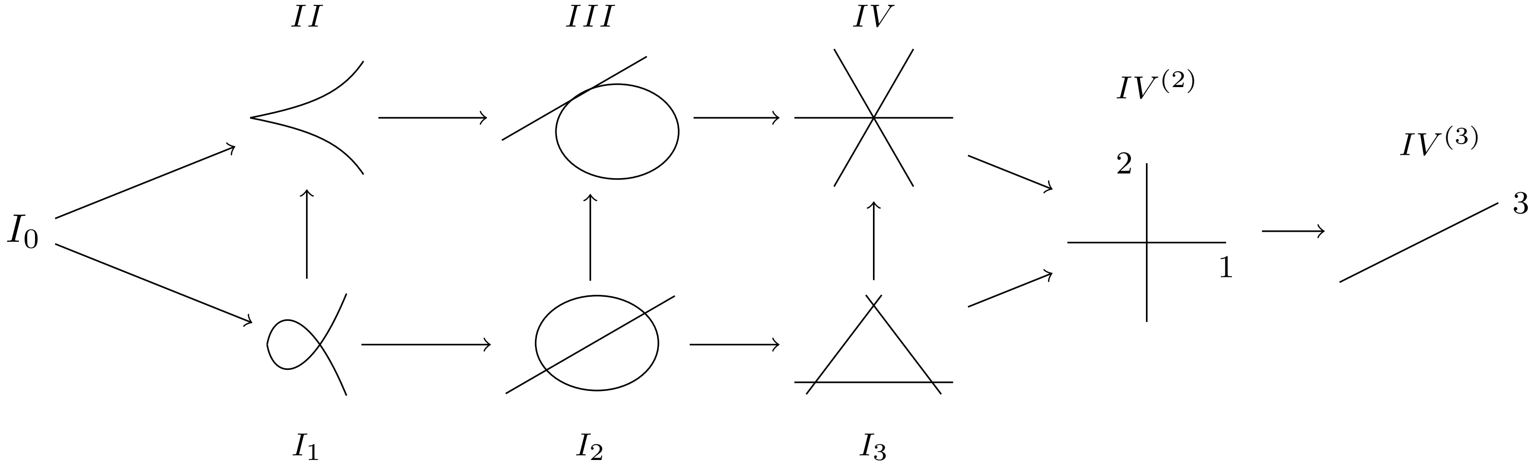


Non-uniqueness of resolutions

(Miranda)	(Szydlo)
$IV + I_0^* :$ 1 — 2 — 4 — 2	$I_0^* + IV :$ 1 — 2 — 3 — 2

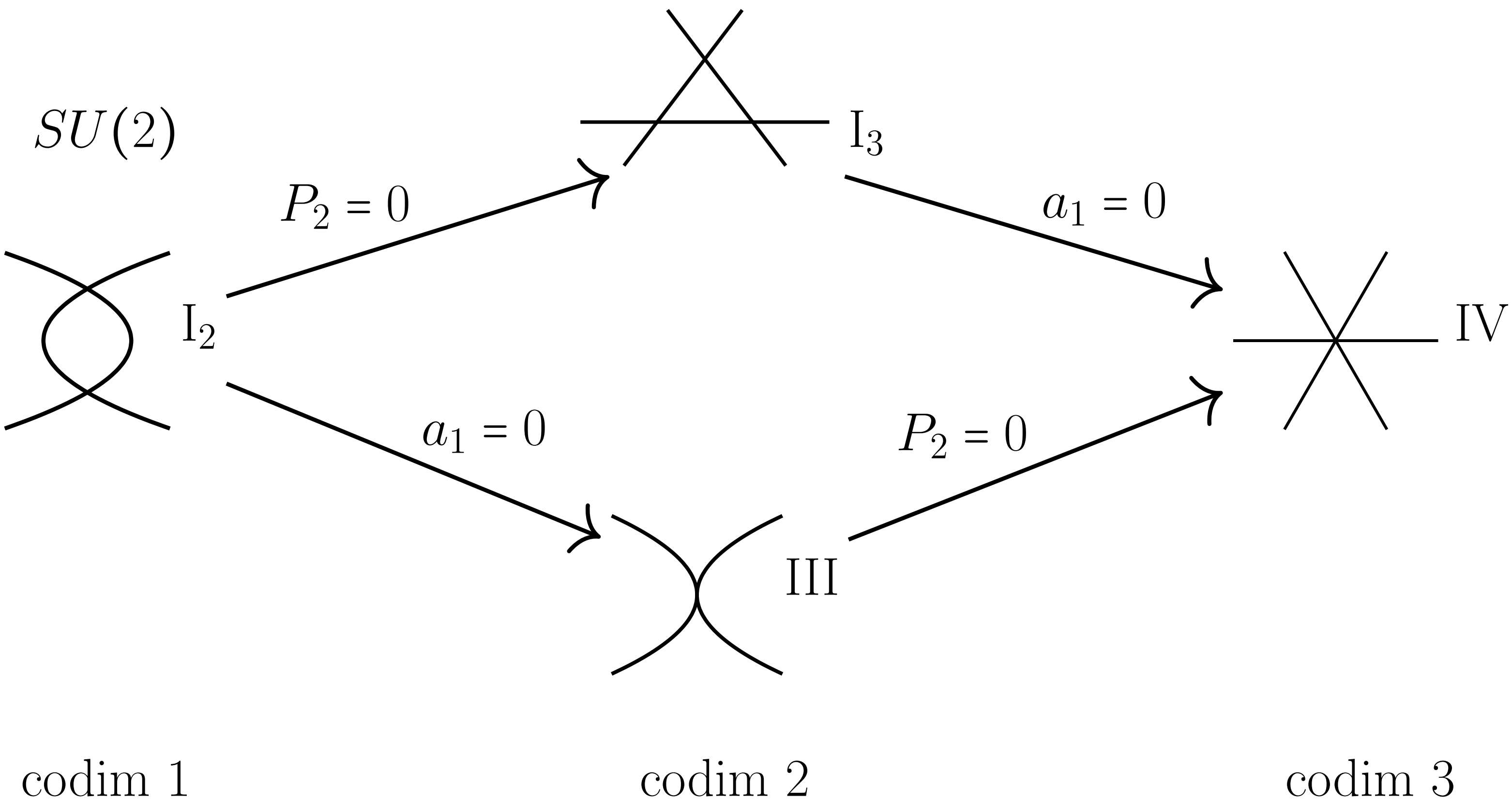
Example of possible fibers of a fibration

(for a general cubic)



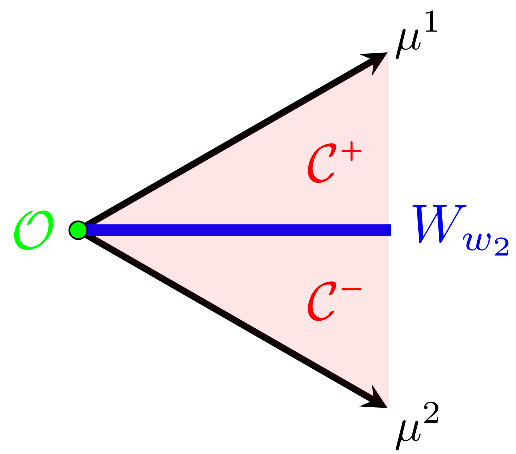
The case of SU(2)

Fiber structure



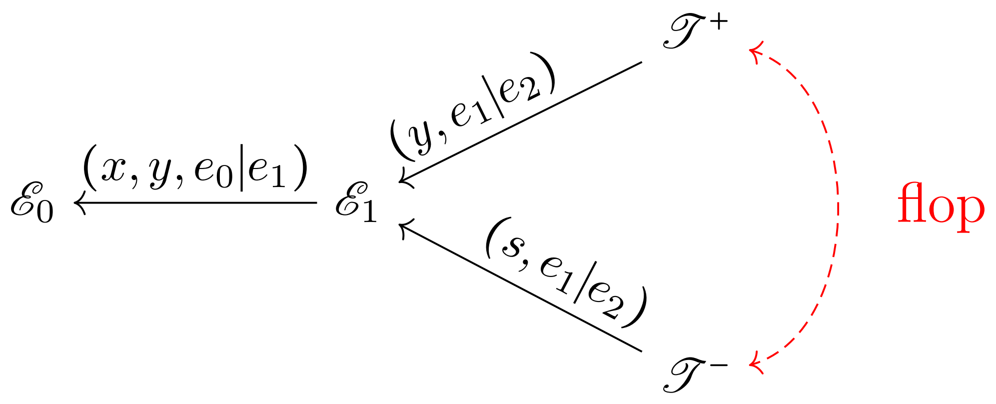
The case of SU(3)

Incidence geometry from the representation theory



The case of SU(3)

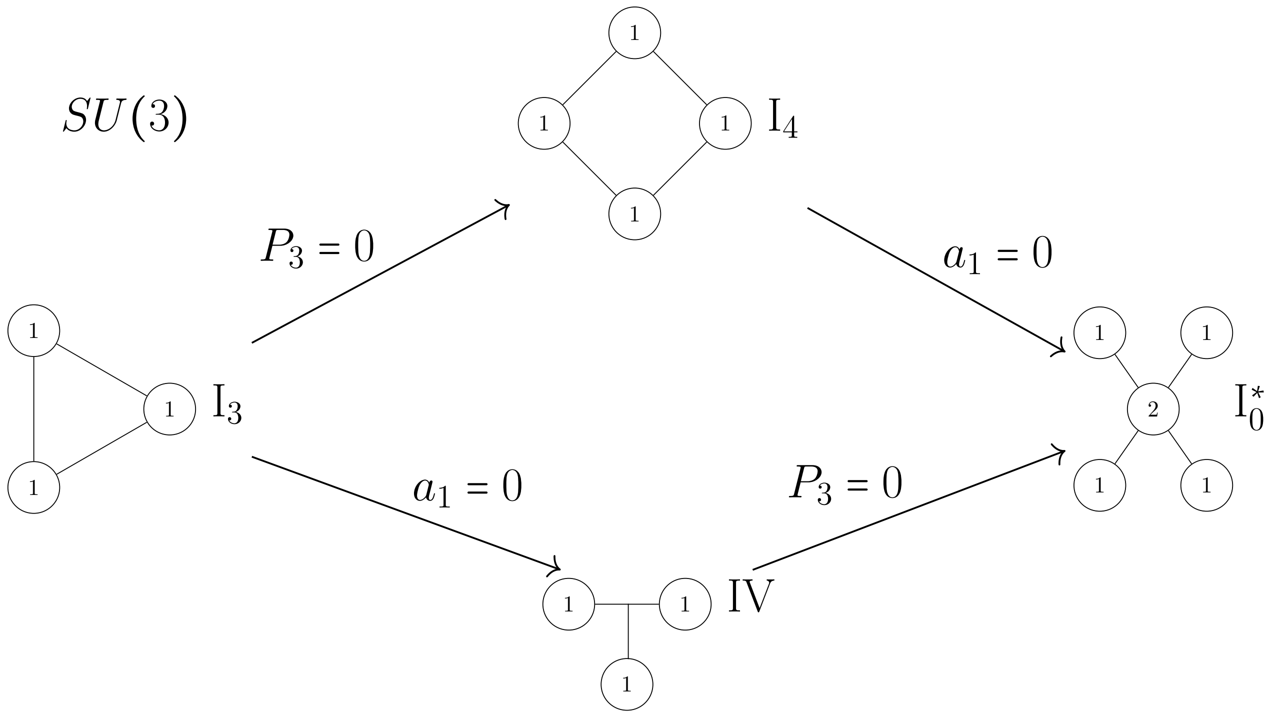
Tree of resolutions



The case of SU(3)

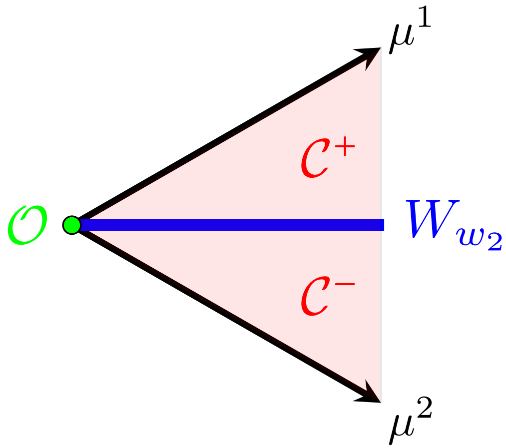
Fiber structure

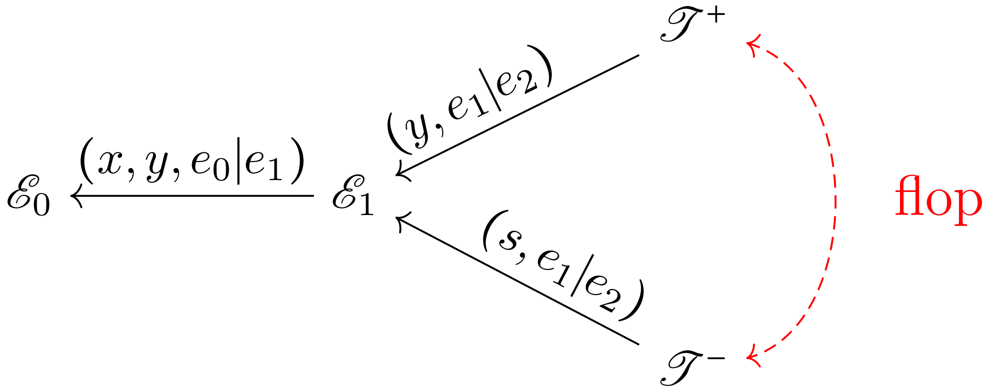
$SU(3)$



codim 1 codim 2 codim 3

The case of SU(3)





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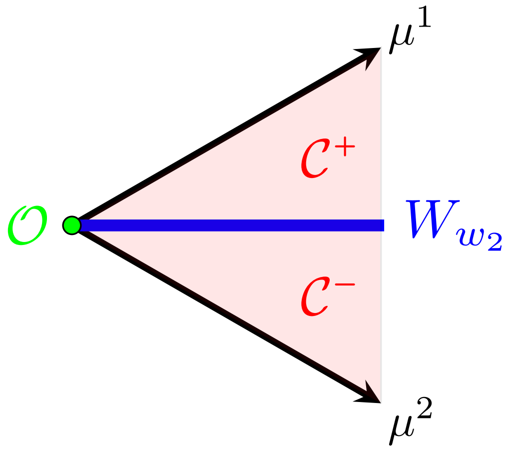
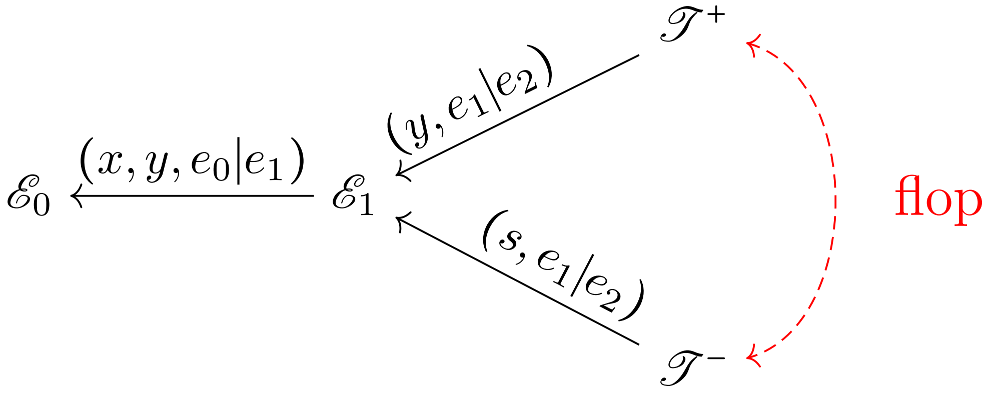
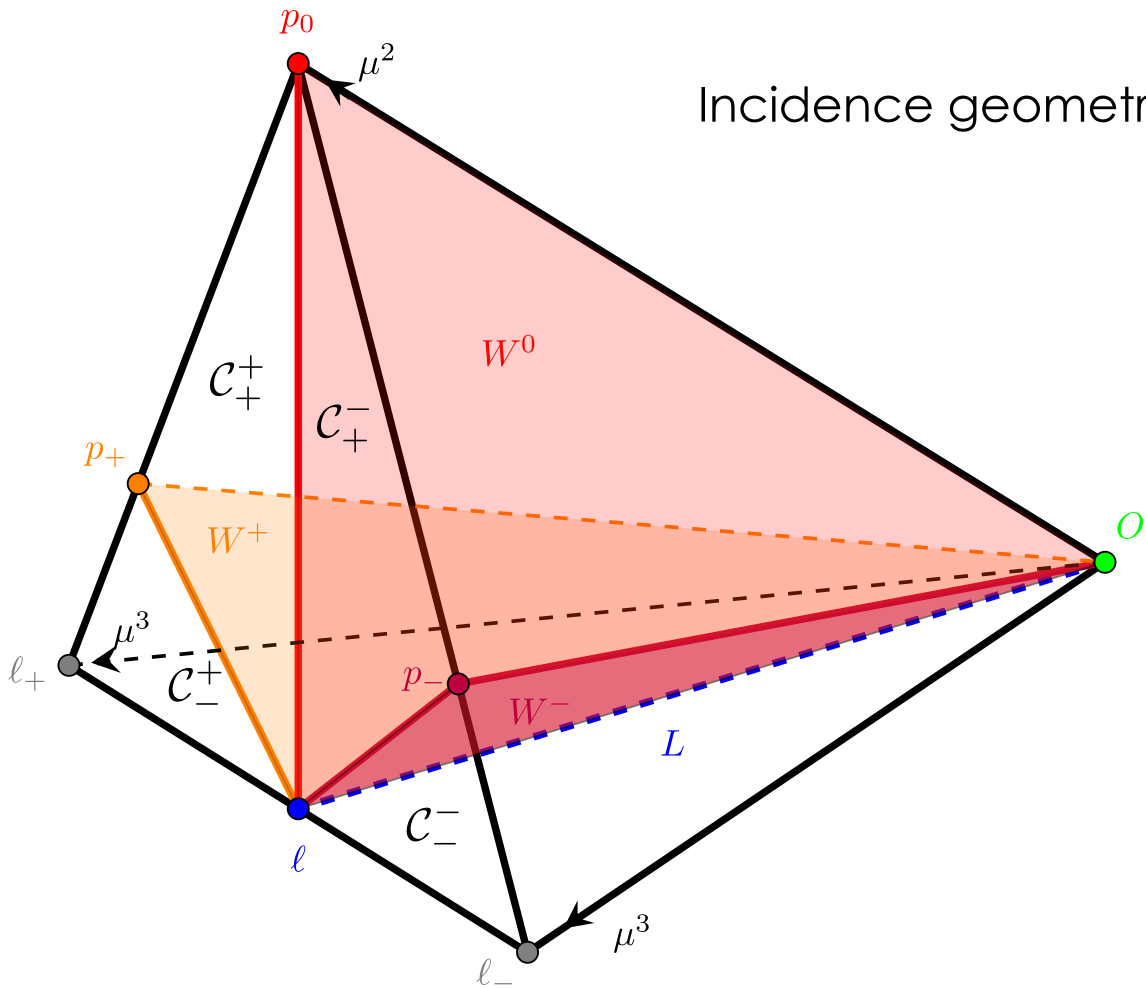




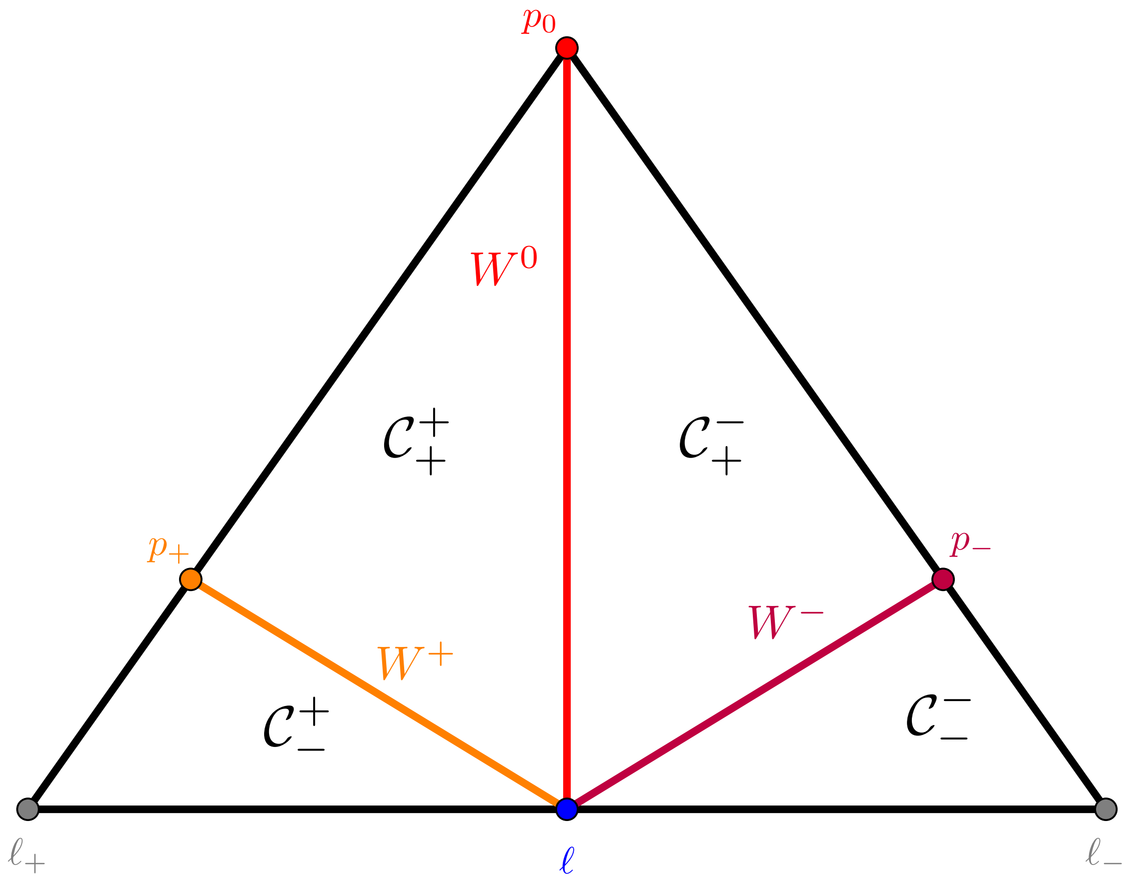
A perfect match!

The case of SU(4)

Incidence geometry

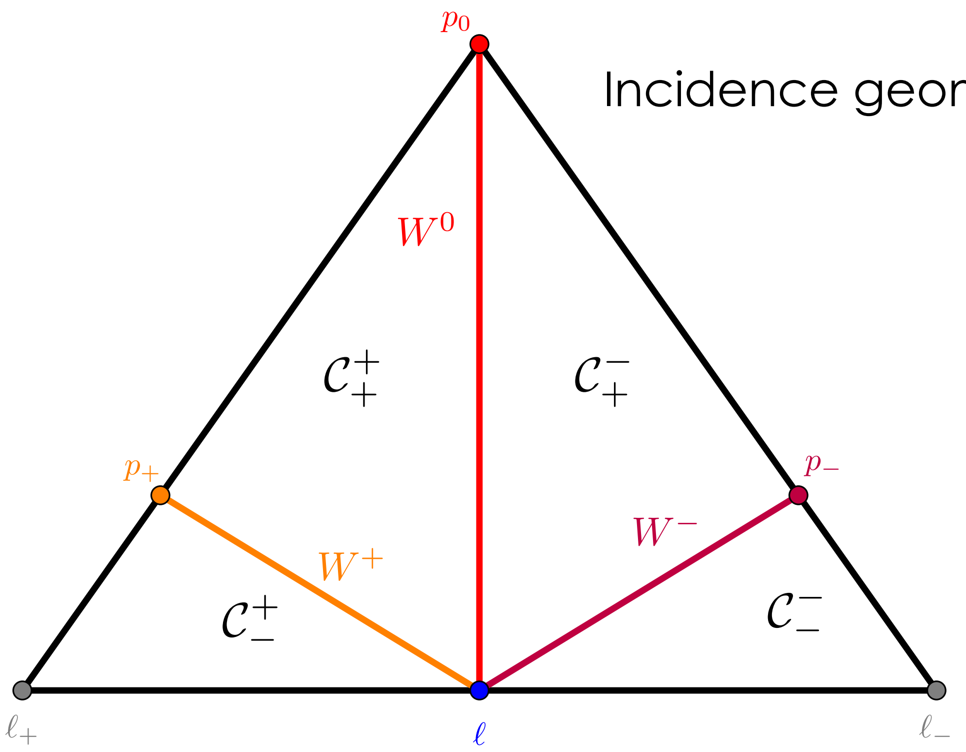


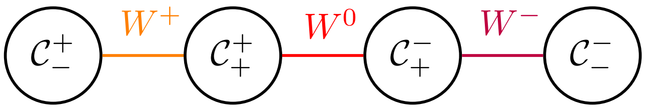
The case of SU(4)

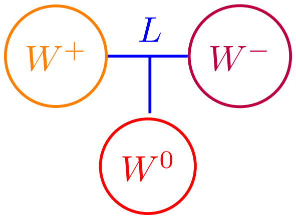


The case of SU(4)

Incidence geometry

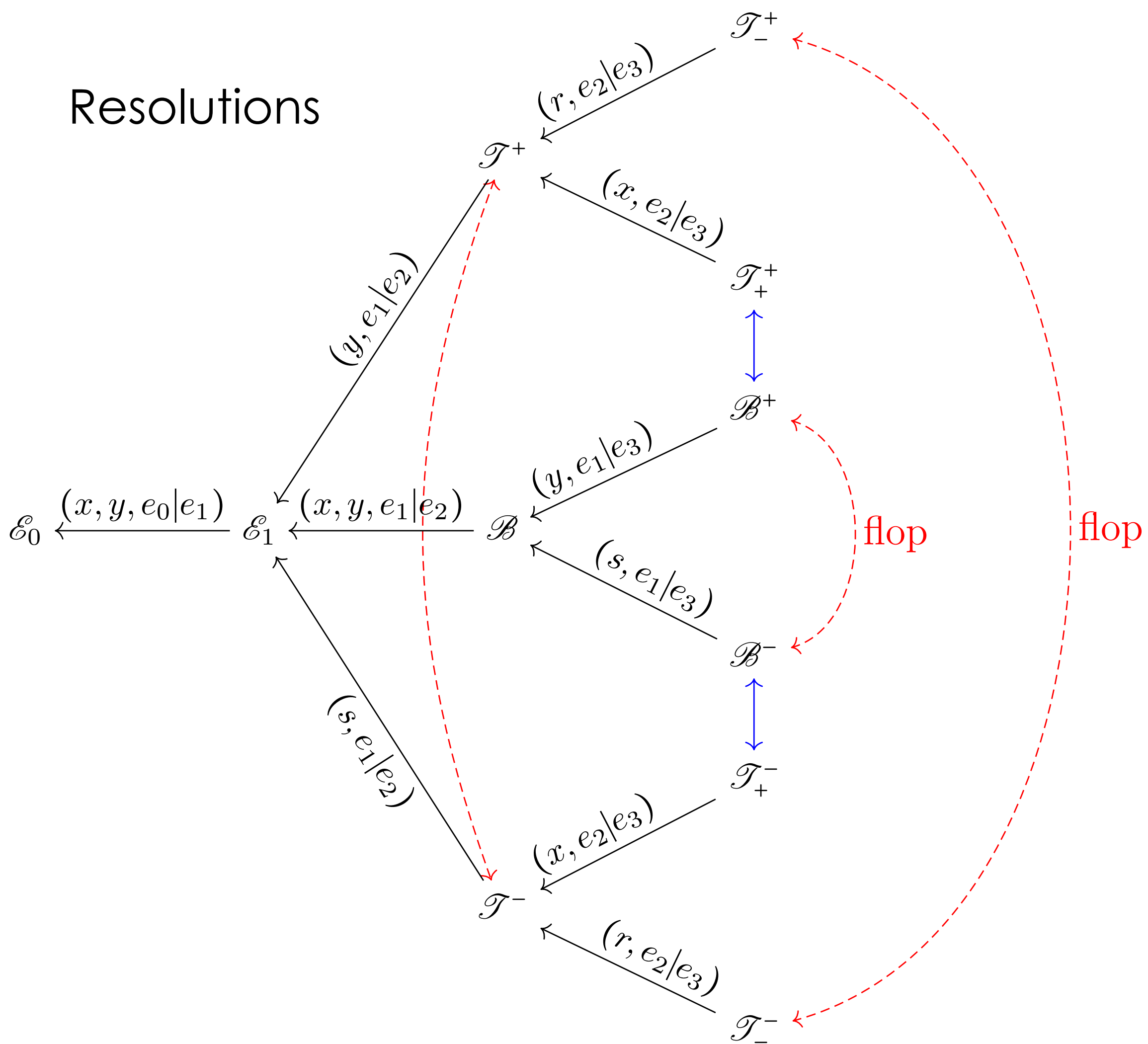






Tree of resolutions for SU(4)

Resolutions



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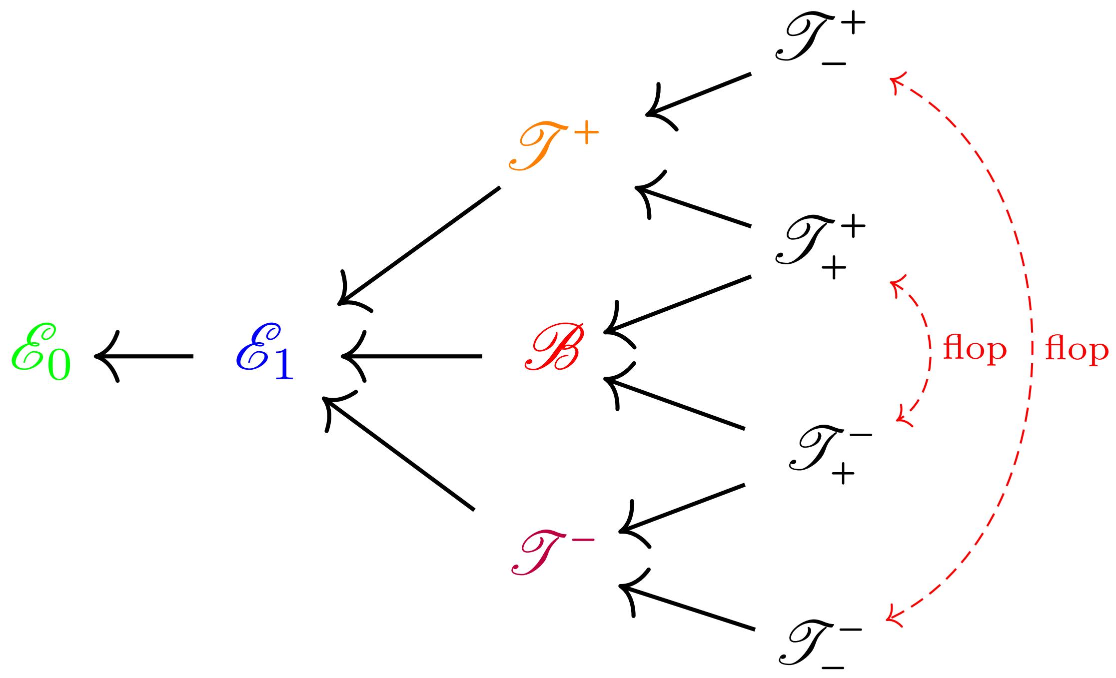

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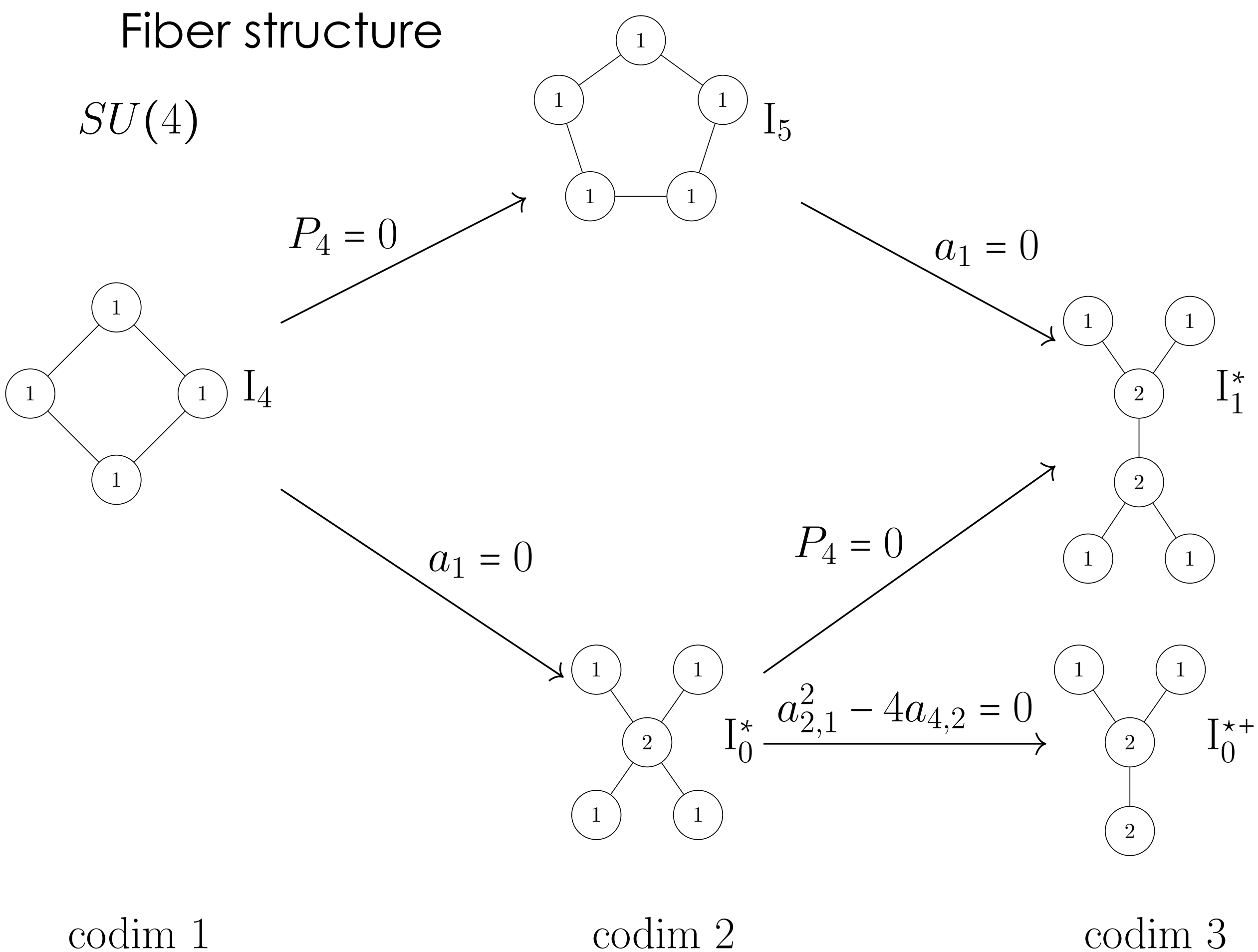
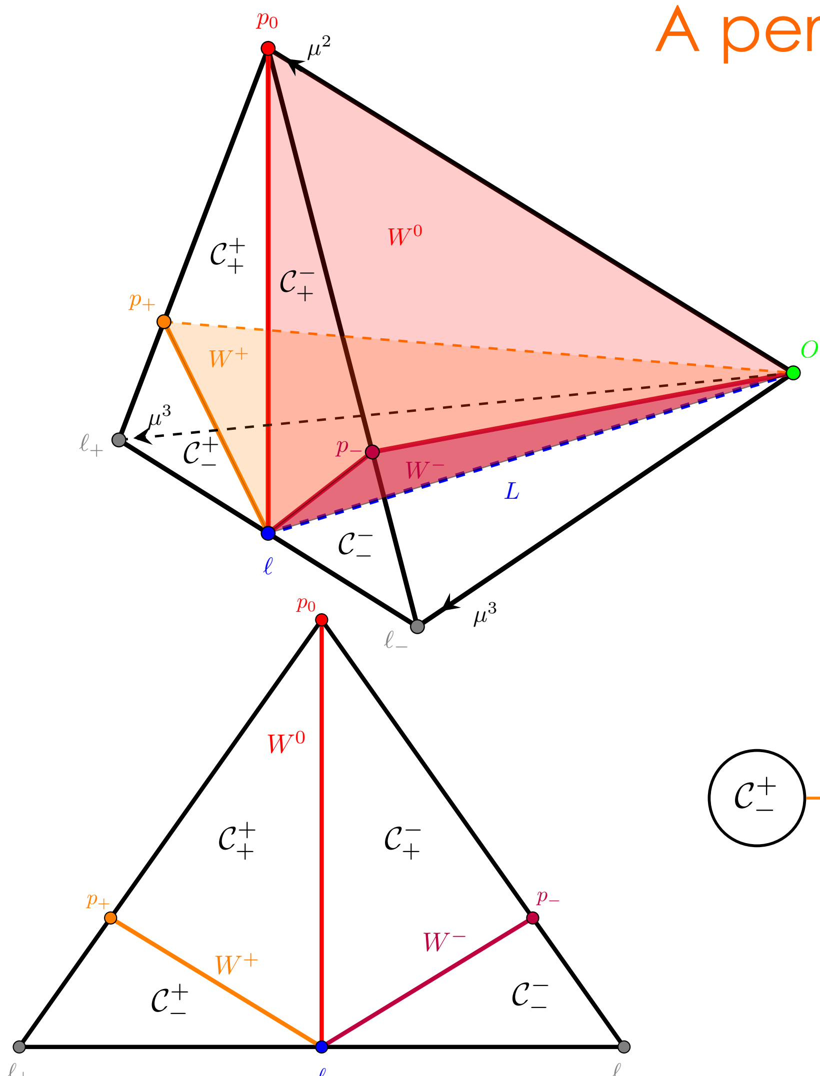
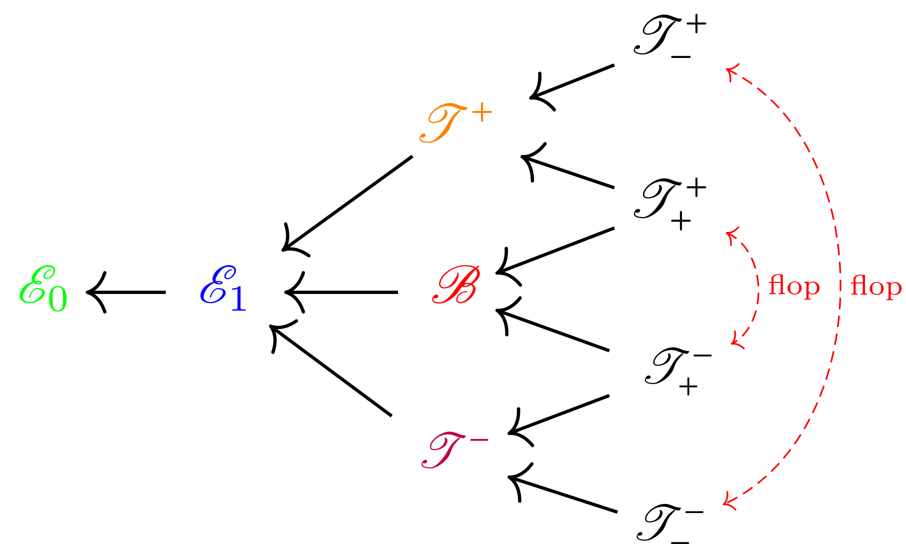
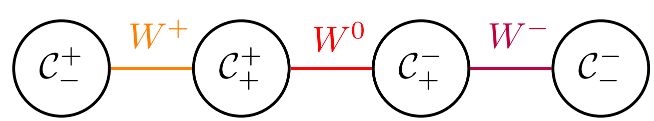


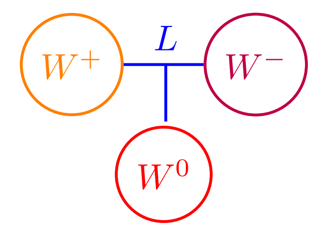
The case of SU(4)

A perfect match!









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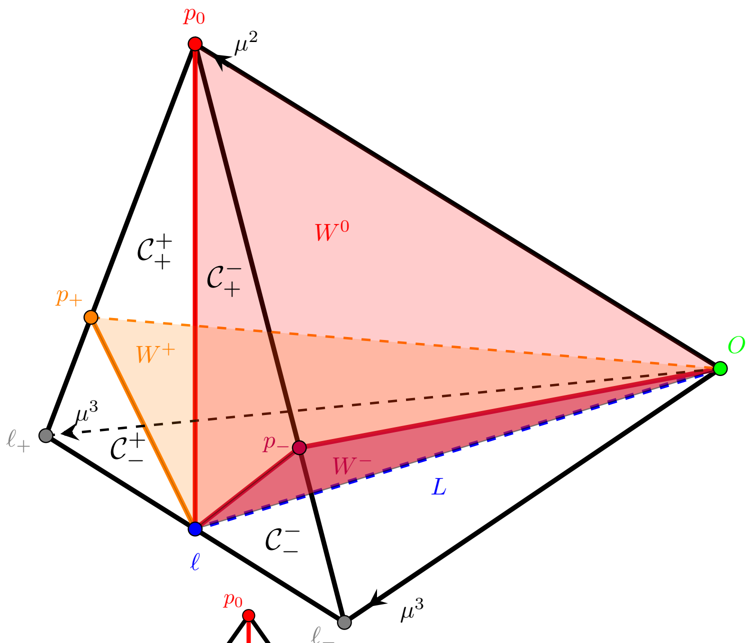

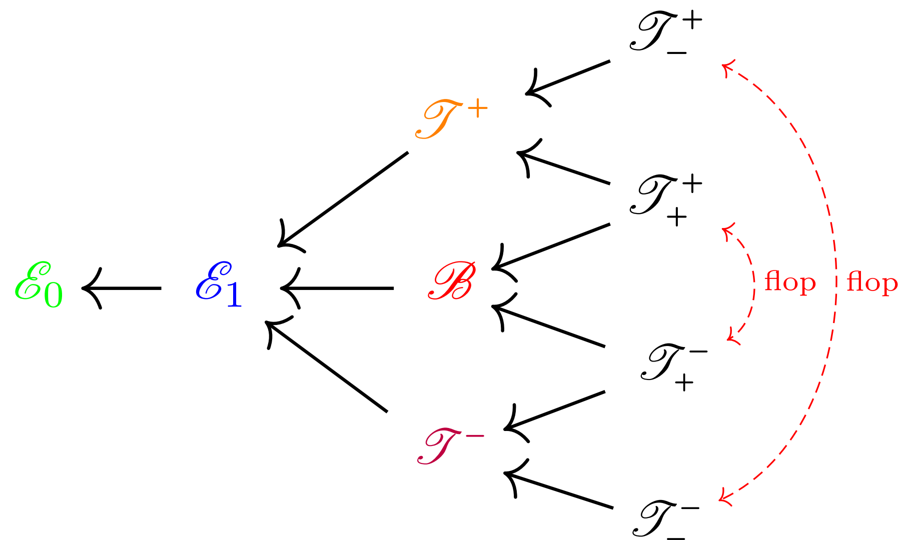

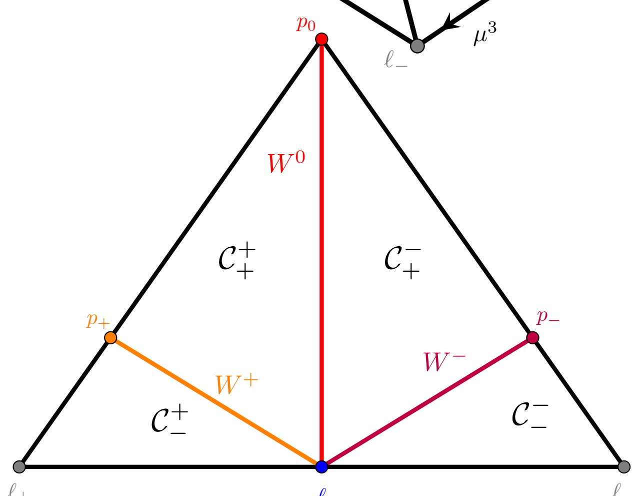

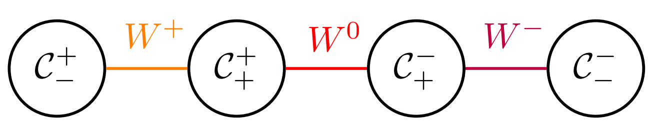

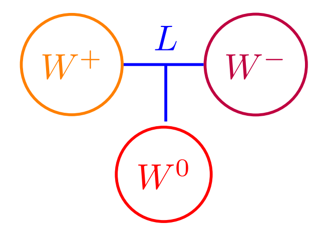

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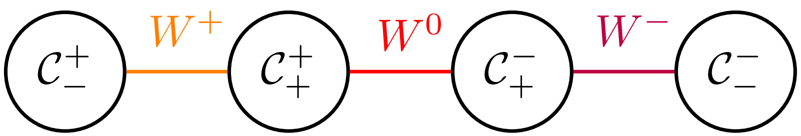
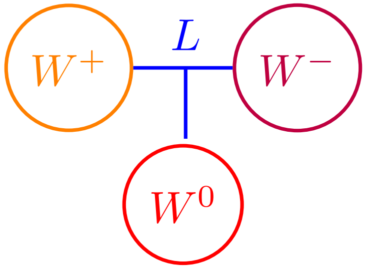
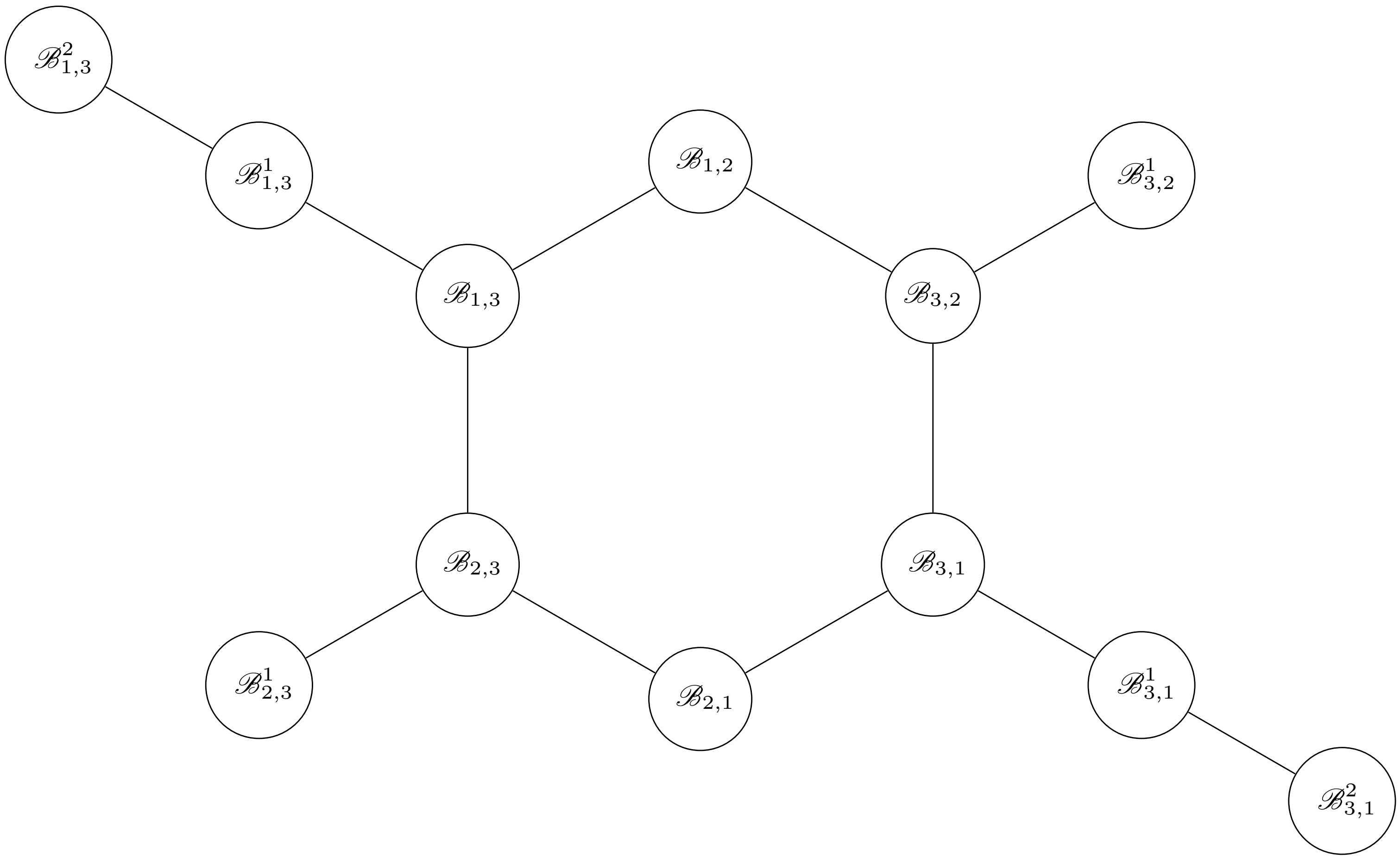




The case of SU(5)



THANK YOU!