Stony Brook University

Theory-Changing Interfaces
and Quantum Algebras

Mykola Dedushenko[1]

[1]Simons Center for Geometry and Physics

06/30/2021

Strings 2021

Based on work with N.Nekrasov To appear in 2107.*****

Photographic evidence:

Introduction

The topic of this talk lies at the intersection of several general ideas.

- QFTs come in families.
  - Interfaces between different members of the family,[1] their OPE:

$$\mathrm{QFT}_1 \,\big|\, \mathrm{QFT}_2 \,\big|\, \mathrm{QFT}_3 \dashrightarrow \mathrm{QFT}_1 \,\big|\, \mathrm{QFT}_3$$

  - "Symmetries" acting on families. They can act between different QFTs, with the generators realized via interfaces.
    - Extend usual symmetries acting within a theory (via codim-1 defects).
  - SUSY interfaces $\Rightarrow$ maps in the Q-cohomology.
    - More generally, derived structures, higher operations, etc.

[1] Interface for us is any codimention-1 defect.

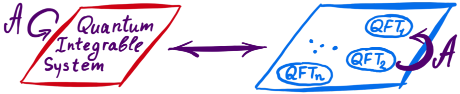
Introduction

A few ideas that play role:

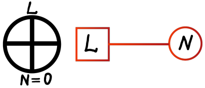
- Wall-crossing.
  - How spaces of SUSY vacua transform as we vary mass/FI parameters across walls.
- Mirror symmetry and symplectic duality in 3d $\mathcal{N} = 4$.
  - Interfaces (walls) between Higgs, Coulomb, mixed, and CFT phases.

Introduction

- Main motivation: Bethe-Gauge correspondence [Nekrasov-Shatashvili]



  - Hilbert space on the left $\cong \oplus_i$ (SUSY vacua in QFT_i) on the right.
  - $\cong$ rep of $\mathcal{A}$ (spectrum-generating algebra, $Y_\hbar(\mathfrak{g})$, $U_\hbar(L\mathfrak{g})$, $U_\hbar(E\mathfrak{g})$)
  - Can we realize the action of $\mathcal{A}$ on the family of QFTs via interfaces?
  - The main example of a family of QFTs:



$\to$ XXX, XXY, or XYZ spin chain (depending on d and Q)

Introduction

- Another big motivation comes from geometric constructions:
  - Stable envelopes [Maulik-Okounkov'12, Aganagic-Okounkov'16]
    - Geometric building blocks, from which R-matrices can be constructed.
    - Once R-matrix interfaces are known, you know the $\mathcal{A}$-action.
  - For X a Nakajima quiver variety, such that the quiver diagram determines an algebra $\mathfrak{g}$:
    - Nakajima constructed the action of $U_\hbar(L\mathfrak{g})$ on $\oplus_i K_T(X_i)$
    - Varagnolo constructed the action of $Y_\hbar(\mathfrak{g})$ on $\oplus_i H_T(X_i)$
    - Both constructions are via Lagrangian correspondences $L \subset X_i \times X_j$, which look like (B, A, A) branes.

Introduction

- Some literature:
  - [Nakajima'00] and [Varagnolo'00] constructions as precursors.
  - Bethe-Gauge correspondence [Nekrasov-Shatashvili'09]. The idea to realize $\mathcal{A}$ via branes, already proposed in [Nekrasov, Strings-2009 talk].
  - A big advance: geometric construction of [Maulik-Okounkov'12]; elliptic case in [Aganagic-Okounkov'16]; K-theoretic case e.g. in [Okounkov-Smirnov'16]. Physics construction remained open.
  - Theory-changing interfaces played role in [Gaiotto-Moore-Witten'15], who explored structures relevant to $\mathcal{Q}_A$ in 2d.
  - See [Bullimore-Kim-Lukowski'17] for a discussion on R-matrices in the context of Bethe/Gauge correspondence.
  - Connection to half-indices in 3d and factorization [Beem-Dimofte-Pasquetti'12,Gadde-Gukov-Putrov'13,Dimofte-Gaiotto-Paquette'17,Bullimore-Crew-Zhang-Dorey'20,Okazaki'20]
  - Some ideas from [MD'18,"Gluing II"] initially played role.
  - Use some tools from [Bullimore-Dimofte-Gaiotto-Hilburn'16], as well as earlier [Hori-Iqbal-Vafa'00]. Also relation to [Cecotti-Vafa'10].

Basic setup

Gauge theory with eight supercharges. The flavor group is G_H. Fix $\mathbf{A} \subset G_H$ – a maximal torus; $U(1)_\hbar$ – R-symmetry that is flavor symmetry for theory with four supercharges. $\mathbf{T} = \mathbf{A} \times U(1)_\hbar$.

3d $\mathcal{N} = 4$ on $\mathbb{E}_\tau \times \overbrace{\mathbb{R}}^{\text{time}}$. Interface is wrapped on the elliptic curve $\mathbb{E}_\tau$, acts on the space of ground states $\mathcal{V}[\mathbb{E}_\tau] \subset \mathcal{H}[\mathbb{E}_\tau]$.

2d $\mathcal{N} = (4,4)$ on $S^1 \times \mathbb{R}$. Interface on S^1 acts on $\mathcal{V}[S^1] \subset \mathcal{H}[S^1]$.

1d $\mathcal{N} = 8$ on $\mathbb{R}$. Interface is a local operator acting on $\mathcal{V}[\mathrm{pt}] \subset \mathcal{H}[\mathrm{pt}]$.

In 3d: also "Coulomb" G_C; the maximal torus $\mathbf{A}' \subset G_C$ is given by topological symmetries, whose currents, for every $U(1)$ gauge group, are

$$J = \star F.$$

Basic setup

The structures we study exist in the $\mathcal{Q}$-cohomology. Which $\mathcal{Q}$?

- 2d $(2,2)$ supercharges:
  - $\bar{Q}_+$ and $\mathcal{Q}_B \to$ "holomorphic-topological" Q in 3d[2]
  - $\mathcal{Q}_A \to$ lifts to "3d A-twist" in 3d[3]
  - $\mathcal{Q} = \mathcal{Q}_A + \mathcal{Q}_A^\dagger = \mathcal{Q}_B + \mathcal{Q}_B^\dagger$ — the Omega-deformation Q

$$\text{In 3d:} \qquad \mathcal{Q}^2 = \underbrace{2D_{\bar{z}}}_{\text{along } \mathbb{E}_\tau} .$$

Operators in the $\mathcal{Q}$-cohomology are interfaces on $\mathbb{E}_\tau$.

One more description: $\mathcal{Q}$ is a 3d $\mathcal{N}=1$ supercharge.

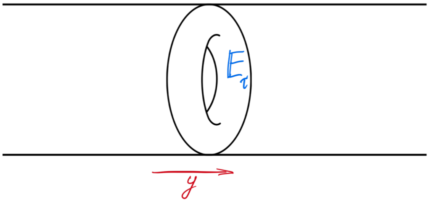
Today: realize stable envelopes as such interfaces.

---

[2] known from [Aganagic-Costello-McNamara-Vafa'17, Costello-Dimofte-Gaiotto'20]

[3] [Benini-Zaffaroni, Closset-Kim-Willett, Baulieu-Losev-Nekrasov]

Basic setup



Turn on flat connections on $\mathbb{E}_\tau$:

$$\overbrace{x \in \mathrm{Hom}(\pi_1(\mathbb{E}_\tau), G_H)/G_H; \quad \hbar \in \mathrm{Hom}(\pi_1(\mathbb{E}_\tau), U(1)_\hbar);}^{\text{Equivariant parameters}}$$
$$\underbrace{z \in \mathrm{Hom}(\pi_1(\mathbb{E}_\tau), G_C)/G_C}_{\text{Kähler parameters}}$$

In 2d: z gets replaced by the θ-angles

In 1d: z completely disappears.

Basic setup

Focus on the Higgs phase, X =Higgs branch. It is well-known that:

- $\mathcal{V}[\mathrm{pt}] \cong H_{\mathbf{T}}(X)$. [Witten'82]
- Similarly, one can identify $\mathcal{V}[S^1]$ with $K_{\mathbf{T}}(X)$.
- $\mathcal{V}[\mathbb{E}_\tau]$ is *related* to the equivariant elliptic cohomology $\mathrm{Ell}_{\mathbf{T}}(X)$.

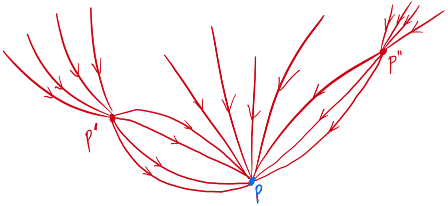
$\mathrm{Ell}_{\mathbf{T}}(X)$ is a scheme; an elliptic generalization of $\mathrm{Spec}\, H_{\mathbf{T}}(X)$ and $\mathrm{Spec}\, K_{\mathbf{T}}(X)$. However, it is not affine, i.e., not a Spec of anything $\Rightarrow$ should study bundles on $\mathrm{Ell}_{\mathbf{T}}(X)$, and $\mathcal{V}[\mathbb{E}_\tau]$ are sections of a "bundle of vacua". No time for this story.

We will focus on the cohomological case in the following. Elliptic generalization (3d lift) comes with two new phenomena: Kähler parameters and boundary anomalies.

Stable envelopes

Let **A** be a torus of flavor group, and $X^{\mathbf{A}} \subset X$ a set of **A**-fixed points.

To $p \in X^{\mathbf{A}}$, attach its *full* **A**-attractor $Attr_p$ (with "broken" trajectories):



There exists a natural map:

$$\mathrm{Stab} : H_{\mathbf{T}}(X^{\mathbf{A}}) \to H_{\mathbf{T}}(X),$$

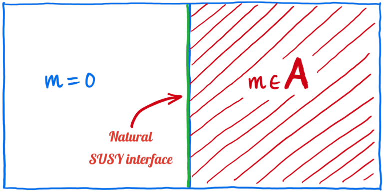
which extends cohomology classes from fixed locus along the full attractor. [Maulik-Okounkov'12]

(Similarly in the K-theoretic case, while in the elliptic case, one extends sections of line bundles on $\mathrm{Ell}_{\mathbf{T}}$)

Janus interface

$X^{\mathbf{A}}$ can be identified with the Higgs branch of the theory with large generic real masses for the flavor symmetry $\mathbf{A}$ turned on.

It is possible to vary *real* masses in the $y \in \mathbb{R}$ direction while preserving half of SUSY.[4] In particular, we can have:



Proposal: such an interface (call it *mass Janus*) gives a physics realization of the map Stab.

[4] This requires an extra term $-\bar{\phi}m'(y)\phi$ in the Euclidean action.

Janus interface

The reason: BPS equations for $\mathcal{Q}$ include $\mathbf{A}_{\mathbb{C}}$ flows parametrized by y.

$$(D_y + \sigma + m(y))\phi = 0, \qquad D_y\sigma = \mu_{\mathbb{R}}.$$

This is a gradient flow for the function

$$\bar{\phi}(m(y) + \sigma)\phi = m(y)\cdot\mu_{\mathbb{R}}^{\mathrm{f}} + \sigma\cdot\mu_{\mathbb{R}}^{\mathrm{g}}$$

On the Higgs branch, it restricts to the Morse function

$$f = \bar{\phi}m(y)\phi.$$

For theories with eight supercharges, all critical points of this function (if isolated) have indices equal to half the target dimension.

Remark: Such gradient trajectories do not contribute to the differential of the MSW complex, i.e., critical points give the exact vacua.

Time-dependent Morse function

So, we need to consider SQM as in [Witten'82] (NLSM into the Higgs branch + the Morse function f representing the effect of masses.[5])

But with time-dependent Morse function.

With time-independent f, the action is $\mathcal{Q}$-exact, up to a "topological term", $S = \{\mathcal{Q}, \dots\} - \mathrm{d}f$.
We still want to use this action when f is time-dependent $\Rightarrow$ need to include $-\frac{\partial f}{\partial y}$ in the action.[6]

Variations δf that vanish at $y \to \pm\infty$ correspond to $\mathcal{Q}$-exact deformations.

[5] In practice, all our computations are done in gauge theory.
[6] This is the term $-\bar{\phi} m'(y) \phi$ we added earlier.

Tension between SUSY and unitarity

This modifies the standard formulas:

$$\mathcal{Q} = \mathrm{d}+\mathrm{d}f, \quad \mathcal{Q}^\dagger = \mathrm{d}^*+\iota_{\nabla f}, \quad H = \frac{1}{2}\{\mathcal{Q},\mathcal{Q}^\dagger\}-i\frac{\partial f}{\partial t}, \quad i\{H,\mathcal{Q}\}+\frac{\partial \mathcal{Q}}{\partial t} = 0.$$

Unitary, no SUSY

Without $\frac{\partial f}{\partial t}$ in H, the evolution is unitary, but SUSY is broken if $\frac{\partial f}{\partial t} \neq 0$.

Non-unitary, SUSY preserved

With $\frac{\partial f}{\partial t}$, $\mathcal{Q}$ is conserved, but the evolution is non-unitary if $\frac{\partial f}{\partial t} \neq 0$.

We want a $\mathcal{Q}$-closed interface between theories with different f's. There is no reason for the corresponding operator to be unitary.

Hence, choose option 2. Have to be very careful: make sure the operator is well-defined!

Time-dependent Morse function, cont'd

Conjugate by e^f: $\mathbb{Q} = e^f \mathcal{Q} e^{-f} = \mathrm{d}$, $\mathbb{G} = e^f \mathcal{Q}^\dagger e^{-f} = \mathrm{d}^* + 2\iota_{\nabla f}$,

$$\mathbb{H} = e^f H e^{-f} + i\frac{\partial f}{\partial t}, \text{ so that } \mathbb{H} = \frac{1}{2}\{\mathbb{Q}, \mathbb{G}\}.$$

(This corresponds to dropping the topological term, and using the Q-exact action $S = \delta(\dots) = \frac{1}{2}(\dot{x} + \nabla f)^2 + \dots$)

$\mathbb{H}$ is d-exact, $\Rightarrow$ naively, evolution is trivial in the de Rham cohomology.

This logic is correct if the target manifold is compact.

The story gets much richer if the target is non-compact.

Non-unitary evolution can make an L^2 function unnormalizable. Yet, matrix elements between L^2 functions are well-defined.

Toy example

Let the target be $\mathbb{C}$, with $f = \frac{m}{2}|z|^2$, where $m \in \mathbb{R}$ can have either sign.
Two states are in the kernel of H:

$$\overbrace{\psi_0 = e^{-f}}^{L^2 \text{ only if } m>0}\,,\ \overbrace{\psi_2 = e^f dz \wedge d\bar{z}}^{L^2 \text{ only if } m<0}\,.$$
(ψ_0 is a solution even for time-dependent f!)

Consider $m(t)$, s.t. $m(-\infty) > 0$ and $m(+\infty) = 0$. Start with $\psi_0 = e^{-f}$ in the past. It evolves into $e^{-f}\big|_{t\to+\infty} = 1$, which is not L^2. Are we in trouble?
Equivariance saves the day. Replace d by $D = \mathrm{d} + \iota_{\epsilon\partial_\varphi}$.

$$\mathbb{Q} = \mathrm{d} + \iota_{V_\varepsilon}, \quad \mathbb{G} = \mathrm{d}^* + V_\varepsilon^\flat \wedge + 2\iota_{\nabla f}, \quad V_\varepsilon = \varepsilon\partial_\varphi.$$

Toy example, cont'd

Define $\omega^2 = m^2 + |\varepsilon|^2$. For constant m (including $m = 0$), the normalizable ground state is

$$\Psi^{(m)} = e^{-f}\Omega^{(m)}, \quad \Omega^{(m)} = \frac{\epsilon}{\sqrt{2\pi(\omega - m)}} e^{-\frac{1}{2}(\omega-m)(x^2+y^2) - \frac{\omega-m}{\epsilon} \mathrm{d}x \wedge \mathrm{d}y},$$

where $z = x + iy$. To compute the transition amplitude, use: (1) shape independence of $m(y)$ to assume

$$m(t) = m\Theta(-t);$$

(2) continuity of $\Omega = e^f \Psi$ across the jump. This results in

$$\langle \Omega^{(0)} | \Omega^{(m)} \rangle = \int \star \bar{\Omega}^{(0)} \wedge \Omega^{(m)} = \sqrt{\frac{|\varepsilon|}{\omega - m}}$$

Toy example, cont'd

- We can compute all possible transitions:
  - $[0| + m] \longrightarrow \langle\Omega^{(0)}|\Omega^{(m)}\rangle = \sqrt{\frac{|\varepsilon|}{\omega - m}} = S_m$
  - $[+m|0] \longrightarrow \langle\Omega^{(m)}|e^{-m|z|^2}|\Omega^{(0)}\rangle = \sqrt{\frac{\omega - m}{|\varepsilon|}} = S_m^{-1}$
  - $[0| - m] \longrightarrow \langle\Omega^{(0)}|\Omega^{(-m)}\rangle = \sqrt{\frac{\omega - m}{|\varepsilon|}} = S_{-m}$
  - $[-m|0] \longrightarrow \langle\Omega^{(-m)}|e^{m|z|^2}|\Omega^{(0)}\rangle = \sqrt{\frac{|\varepsilon|}{\omega - m}} = S_{-m}^{-1}$
  - $[-m|m] \longrightarrow \langle\Omega^{(-m)}|e^{m|z|^2}|\Omega^{(m)}\rangle = \frac{|\varepsilon|}{\omega - m} = R_m$

Here $S_{\pm m}$ is the analog of $\mathrm{Stab}_{\pm m}$, and R_m is the analog of R-matrix. The relation:

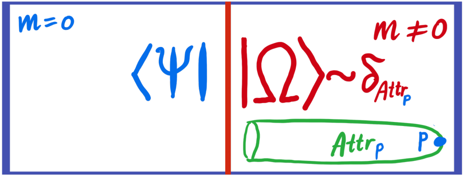
$$R_m = S_{-m}^{-1} \circ S_m$$

is exactly how the R-matrix is built from stable envelopes.

This confirms our expectations, e.g., independence on the shape of $m(t)$.

General Idea

In general, the region with masses prepares an equivariant form, which (in the limit of large masses and metric) looks like a delta-form supported on the attractor of a fixed point.



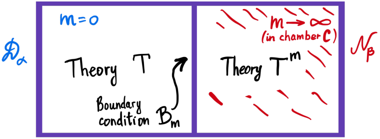
We then compute its overlap with a "probe" equivariant form representing some vacuum in the massless region.

Less simplified view

- What we actually do:
  - Like in the toy example, but the target has many fixed points, S_m are non-trivial upper-triangular matrices.
  - In practice, all computations are done in gauge theory.
  - ... Via SUSY localization.
  - Which is also why the (Euclidean) time $\mathbb{R}$ is replaced by an interval.
  - And the choice of vacua at $t \to \pm\infty$ is represented by special boundary conditions. (Thimble boundary conditions [Hori-Iqbal-Vafa'10,Gaiotto-Moore-Witten'15,Bullimore-Dimofte-Gaiotto-Hilburn'16])

(It is problematic to localize on a non-compact spacetime. Interval is more straightforward, but still can be a challenge, depending on the boundary conditions.)

Executive summary of the interval



- Key ideas:
  - Theory on the left: our gauge theory T.
  - Theory on the right: turn on large real masses $m \in \mathfrak{C}$, integrate out fields that are massive in vacuum β, the remaining gauge theory is T^m.
  - Boundary conditions B_m on fields that "terminate" at the middle are naturally induced by the SUSY jump of masses.
  - On the left: boundary condition corresponding to vacuum α realized via Dirichlet b.c. for gauge fields (exceptional Dirichlet)
  - On the right: vacuum β realized via Neumann boundary conditions for the gauge fields.
  - The $\mathcal{N}_\beta$ boundary contains extra matter to cancel anomalies in 3d case.

Interval index

In $d \geq 2$ spacetime dimensions, we can regard the interval direction as space, and one of the circles as time.

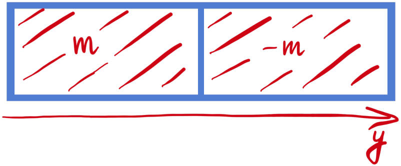
Then the answer can be more conventionally interpreted as the index in a certain soliton sector.

But in 1d, the interval direction can only be time.

R-matrices

R-matrices are realized as $\mathrm{Stab}_{m_2}^{-1} \circ \mathrm{Stab}_{m_1}$, i.e., instead of changing mass from m to 0, we change it from $m_1 \in \mathfrak{C}_1$ to $m_2 \in \mathfrak{C}_2$ (different chambers).

If there is only one mass, and the chambers are $m > 0$, $m < 0$, then the R-matrix is realized by



Let me explain how to construct an interface realizing a raising operator of the $\mathfrak{sl}_2$ Yangian. We want an interface between theories:

$$\boxed{L} \text{———} U(N) \quad \Big| \quad \boxed{L} \text{———} U(N+1)$$

R-matrix and Higgsing

Inspired by [Maulik-Okounkov'12], consider a larger theory T:

$$\boxed{L+1} \text{——} \; U(N+1)\,.$$

It has an extra $U(1)$ flavor symmetry that rotates the added hyper. Let m be the real mass for it. Then at $m \to \infty$, the theory decomposes into

$$\underbrace{\left[\boxed{L} \text{——} \; U(N+1)\right]}_{A} \oplus \underbrace{\left[\left(\boxed{L} \text{——} \; U(N)\right) \otimes \left(\boxed{1} \text{——} \; U(1)\right)\right]}_{B}$$

In sector B, $U(N+1)$ is broken to $U(N) \times U(1)$ via Higgsing.

R-matrix

Consider an R-matrix R_m constructed as above via changing mass from $m \gg 0$ to $m \ll 0$. It has a block form:

$$
\begin{array}{cc} & \begin{array}{cc} A & B \end{array} \\ R_m(u) = \begin{array}{c} A \\ B \end{array} & \left[\begin{array}{c|c} * & * \\ \hline * & * \end{array}\right] \end{array}
$$

The AB and BA blocks represent interfaces that change the gauge group (we can "forget" the $\boxed{1} - U(1)$ factor). They provide realization of the basic raising and lowering operators in the Yangian $Y_\hbar(\mathfrak{sl}_2)$ (in the 1d case).

Generalizations are clear...

Further comments

- Didn't have time to talk about:
  - The elliptic case in details.
  - Details of interval computations.
  - Construction of general R-matrices for general quiver varieties.
  - Janus for FI parameters.
  - Half-index. Can stretch the 3d index and half-index, proving that the squashing parameter b is a trivial deformation of the THF using techniques of [Closset-Dumitrescu-Festuccia-Komargodski'13].
  - It connects holomorphic blocks [B-D-P'12] to half-indices [G-G-P'13,D-G-P'17].
  - Our interfaces describe wall-crossing of the half-index. Acting with an interface, one can transport half-index between chambers, or from the Higgs to the Coulomb phase.
  - Brane constructions of our systems via Type IIA on the ALE spaces. Dualities: relation between supercharges and also to 4d CS approach.

What else?

- For the future:
  - We construct interfaces "up to quasi-isomorphism" $\Rightarrow$ can we study derived structure, higher operations?
  - Generalization to fewer supercharges?
  - $\mathcal{Q}$ in d-dim can be lifted to $\mathcal{Q}_A$ in $d+1$. Analogs of our constructions in quantum cohomology theories?
  - Lifting $\mathcal{Q}$ in 1d to $\mathcal{Q}_A$ in 2d, explore connections to [Gaiotto-Moore-Witten'15]?

What else?

- For the future:
  - We construct interfaces "up to quasi-isomorphism" ⇒ can we study derived structure, higher operations?

Thank you!

Questions?

ctions

  - Lifting $\mathcal{Q}$ in 1d to $\mathcal{Q}_A$ in 2d, explore connections to [Gaiotto-Moore-Witten'15]?