Quantum Black Holes and Quantum Holography

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References

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Hurdles for String Theory

- We don’t have a super-LHC to probe the theory directly at Planck scale.
- We don’t even know which phase of the theory may correspond to the real world.

How can we be sure that string theory is the right approach to quantum gravity in the absence of direct experiments?

A useful strategy is to focus on **universal** features that must hold in **all phases** of the theory.

Quantum Black Holes

Any black hole in ***any*** phase of the theory should be interpretable as an ensemble of quantum states ***including finite size effects.***

- ***Universal*** *and* ***extremely stringent*** *constraint*
- *An* ***IR*** *window into the* ***UV***
- Connects to a broader problem of

  Quantum Holography at finite N.

AdS_{p+2}/CFT_{p+1}

- Near horizon of a BPS black hole has AdS_2 factor. More generally, near horizon physics of black p-branes leads to AdS_{p+2}/CFT_{p+1} holography.
- A bulk of the work in holography is in infinite *N* limit, using *classical* gravity to study quantum CFT.
- Our interest will be in *quantum* gravity in the bulk.

After all, a primary motivation for string theory is unification of General Relativity with QM.

Quantum Holography

I will describe three results motivated by these considerations of finite *N* holography.

- AdS_2: *Nonperturbative quantum entropy of black holes including **all finite size corrections**.*
- AdS_4: *New localizing instantons in bulk supergravity for **finite N** Chern-Simons-Matter.*
- AdS_3: *An unexpected connection to the mathematics of **mock modular forms.***

One of the most important clues about quantum gravity is the entropy of a black hole:

What is the **exact quantum generalization** of the celebrated Bekenstein-Hawking formula?

$$S = \frac{\mathbf{A}}{\mathbf{4}} + c_1 \log(A) + c_2 \frac{1}{A} \ldots + e^{-A} + \ldots$$

- How to define it ? How to compute it?
- The exponential of the quantum entropy must yield an **integer**. This is extremely stringent.
- Subleading corrections **depend** sensitively on the phase & provide a window into the **UV structure.**

Defining Quantum Entropy

- The near horizon of a BPS black hole of charge vector Q is AdS_2 so one can use holography.
- Quantum entropy can then be defined as a path integral $W(Q)$ in AdS_2 *over all string fields with appropriate boundary conditions, operator insertion, and a renormalization procedure*.

Sen (09)

- For large charges, logarithm of $W(Q)$ reduces to Bekenstein-Hawking-Wald entropy.

Computing Quantum Entropy.

- Integrate out massive string modes to get a Wilsonian effective action for massless fields.
- Still need to make sense of the formal path integral of supergravity fields. Using it do explicit computations is fraught with danger.
- It helps to have microscopic degeneracies $d(Q)$ from brane counting to compare with:

$$W(Q) = d(Q)$$

One-eighth BPS states in N=8

- Type-II compactified on T^6
- Dyonic states with charge vector ***(Q, P)***
- U-duality invariant $\Delta = Q^2 P^2 - (Q \cdot P)^2$
- Degeneracy given by Fourier coefficients $C(\Delta)$ of

$$\frac{\vartheta(\tau, z)^2}{\eta(\tau)^6}$$

Maldacena Moore Strominger (99)

$$d(\Delta) = (-1)^{\Delta+1} C(\Delta)$$

Hardy-Ramanjuan-Rademacher Expansion

An exact convergent expansion (using modularity)

$$C(\Delta) = N \sum_{c=1}^{\infty} c^{-9/2} \tilde{I}_{7/2}\left(\frac{\pi\sqrt{\Delta}}{c}\right) K_c(\Delta)$$

$$\tilde{I}_{7/2}(z) = \frac{1}{2\pi i}\int_{\epsilon - i\infty}^{\epsilon + i\infty} \frac{ds}{s^{9/2}} \exp\left[s + \frac{z^2}{4s}\right]$$

$$\sim \exp\left[z - 4\log z + \frac{c}{z} + \ldots\right] \qquad z = A/4$$

The c=1 Bessel function sums *all perturbative* (in 1/z) corrections to entropy. The c>1 are non-perturbative

Generalized Kloosterman Sum $K_c(\Delta)$

$$\sum_{\substack{-c\leq d<0;\\ (d,c)=1}} e^{2\pi i \frac{d}{c}(\Delta/4)}\, M^{-1}(\gamma_{c,d})_{\nu 1}\, e^{2\pi i \frac{a}{c}(-1/4)}$$

$$\nu = \Delta \mod 2$$

Relevant only in exponentially subleading nonperturbative corrections . *Even though highly subleading, conceptually very important for* ***integrality.***

New results concerning these nonperturbative phases

Multiplier System

$M^{-1}(\gamma)$ is a particular two-dimensional representation of the $SL(2,\mathbb{Z})$ matrix

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$M^{-1}(T) = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \qquad M^{-1}(S) = \frac{e^{-\frac{\pi i}{4}}}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Use the continued fraction expansion:

$$\gamma = T^{m_t} S \ldots T^{m_1} S$$

Computing $W(\Delta)$

- The structure of the microscopic answer suggests that $W(\Delta)$ should have an expansion

$$W(\Delta) = \sum_{c=1}^{\infty} W_c(\Delta)$$

- Each $W_c(\Delta)$ corresponds to a $\mathbb{Z}_c$ orbifold of the Euclidean near horizon black hole geometry.
- The higher *c* are exponentially subleading. Unless one can evaluate each of them *exactly* it is not particularly meaningful to add them.

Localization enables us to do this.

Path Integral on AdS_2

- Including the M-theory circle, there is a family of geometries that are asymptotically $AdS_2 \times S^1$:

$$ds^2 = (r^2 - \frac{1}{c^2})d\theta^2 + \frac{dr^2}{r^2 - \frac{1}{c^2}} + R^2 \left(dy - \frac{i}{R}(r - \frac{1}{c})d\theta + \frac{d}{c}d\theta\right)^2$$

- These are $\mathbb{Z}_c$ orbifolds of BTZ black hole. These geometries $\mathcal{M}_{c,d}$ all have the same asymptotics and contribute to the path integral.
- Related to the $SL(2,\mathbb{Z})$ family in AdS_3

Maldacena Strominger (98), Sen (09) Pioline Murthy (09)

Localization

- If a supersymmetric integral is invariant under a localizing supersymmetry Q which squares to a compact generator H, then the path integral localizes onto fixed manifold of the symmetry Q.
- We consider localization in N=2 supergravity coupled to n_v vector multiplets whose chiral action is governed by a prepotential *F*.
- *We find the localizing submanifold left invariant by Q and evaluate the renormalized action.*

Off-shell Localizing Solutions

- We found ***off-shell*** localizing instantons in *AdS*$_2$ for supergravity coupled to n_v vector multiplets with scalars X^I and auxiliary fields Y^I

$$X^I = X^I_* + \frac{C^I}{r}\,, \quad Y^I = \frac{C^I}{r^2}\,, \qquad \begin{array}{c} C^I \in \mathbb{R} \\ I = 0, \ldots, n_v \end{array}$$

- These solutions are *universal* in that they are *independent of the physical action* and follow entirely from the off-shell susy transformations.

Renormalized Action

- The renormalized action for prepotential ***F*** is

$$\mathcal{S}_{ren}(\phi, q, p) = -\pi q_I \phi^I + \mathcal{F}(\phi, p)$$

$$\mathcal{F}(\phi, p) = -2\pi i \left[F\Big(\frac{\phi^I + ip^I}{2}\Big) - \bar{F}\Big(\frac{\phi^I - ip^I}{2}\Big) \right]$$

$\frac{1}{2}(\phi^I + ip^I)$ is the off-shell value of X^I at the origin.

- For each orbifold one obtains a Laplace integral of $|Z_{top}|^2$ à la OSV conjecture.

Ooguri Strominger Vafa (04) Lopes Cordosa de Wit Mohaupt (00)

Final integral

The prepotential for the truncated theory is

$$F(X) = -\frac{1}{2}\frac{X^1}{X^0}\sum_{a,b=2}^{7} C_{ab}X^a X^b \qquad (n_v = 7)$$

(dropping the extra gravitini multiplets)

The *path* integral reduces to the Bessel integral

$$W_1(\Delta) = N\int \frac{ds}{s^{9/2}}\exp\left[s + \frac{\pi^2\Delta}{4s}\right]$$

$$W_1(\Delta) = \tilde{I}_{7/2}(\pi\sqrt{\Delta})$$

Degeneracy, Quantum Entropy, Wald Entropy

Δ	$C(\Delta)$	$W_1(\Delta)$	$\exp(\pi\sqrt{\Delta})$
3	8	7.972	230.765
4	-12	12.201	535.492
7	39	38.986	4071.93
8	-56	55.721	7228.35
11	152	152.041	22506.
12	-208	208.455	53252.
15	513	512.958	192401

- Note that $C(\Delta)$ are alternating in sign so that $d(\Delta) = (-1)^{\Delta+1} C(\Delta)$ is strictly positive.

*This is a **prediction from IR** quantum gravity for black holes which is **borne out by the UV**.*

- This explains the Bessel functions for all *c* with correct argument because for each orbifold the action is reduced by a factor of *c*

- *What about the Kloosterman sums?*

It was a long standing puzzle how this intricate number theoretic structure could possibly arise from a supergravity path integral.

Kloosterman from Supergravity

- Our analysis thus far is local and insensitive to global topology. The Chern-Simons terms in the bulk and the boundary terms are sensitive to the global properties of $\mathcal{M}_{c,d}$
- Additional contributions to renormalized action and additional saddle points specified by holonomies of flat connections. Various phases from CS terms for different groups assemble nontrivially into precisely the Kloosterman sum.

Kloosterman and Chern-Simons

$$I(A) = \int_{\mathcal{M}_{c,d}} \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A^3 \right)$$

In our problem we have three relevant groups

$$U(1)^{n_v+1} \qquad SU(2)_L \qquad SU(2)_R$$

$$\sum_{\substack{-c \leq d < 0; \\ (d,c)=1}} e^{2\pi i \frac{d}{c}(\Delta/4)} \, M^{-1}(\gamma_{c,d})_{\nu 1} \, e^{2\pi i \frac{a}{c}(-1/4)}$$

Dehn Twisting

The geometries $\mathcal{M}_{c,d}$ are topologically a solid 2-torus and are related to $\mathcal{M}_{1,0}$ by Dehn-filling.

Relabeling of cycles of the boundary 2-torus:

$$\begin{pmatrix} C_n \\ C_c \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \quad \text{for} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$$

C_1 is contractible and C_2 is noncontractible in $\mathcal{M}_{1,0}$

C_c is contractible and C_n is noncontractible in $\mathcal{M}_{c,d}$

Boundary Conditions and Holonomies

- The cycle C_2 is the M-circle and C_1 is the boundary of AdS_2 for the reference geometry $\mathcal{M}_{1,0}$
- This implies the boundary condition

$$\oint_{C_2} A^I = \text{fixed}\,, \qquad \oint_{C_1} A^I = \text{not fixed}$$

and a boundary term

$$I_b(A) = \int_{T^2} \text{Tr} A_1 A_2 d^2x$$

Elitzur Moore Schwimmer Seiberg (89)

Contribution from $SU(2)_R$

$$\oint_{C_1} A = 2\pi i\gamma\frac{\sigma^3}{2} \qquad \oint_{C_2} A = 2\pi i\delta\frac{\sigma^3}{2}$$

$$\oint_{C_c} A = 2\pi i\alpha\frac{\sigma^3}{2} \qquad \oint_{C_n} A = 2\pi i\beta\frac{\sigma^3}{2}$$

Chern-Simons contribution to the renormalized action is completely determined knowing the holonomies. For abelian the bulk contribution is zero for flat connections and only boundary contributes.

$$I_b[A_R] = 2\pi^2\gamma\delta \qquad I[A_R] = 2\pi^2\alpha\beta$$

Kirk Klassen (90)

$$\oint_{C_1} A_R = -\frac{2\pi i}{c}\frac{\sigma^3}{2}\,, \qquad \oint_{C_2} A_R = 0$$

Supersymmetric $\mathbb{Z}_c$ orbifold $\qquad J_R = 0$

$$\gamma = -1/c\,, \quad \delta = 0\,, \quad \alpha = -1\,, \quad \beta = -a/c$$

(using $\alpha = c\gamma + d\delta\,, \qquad \beta = a\gamma + b\delta$)

The total contribution to renormalized action is

$$S_{ren} = -\frac{2\pi i k_R}{4}\frac{a}{c} \qquad k_R = 1$$

in perfect agreement with a term in Kloosterman.

Multiplier System from $SU(2)_L$

There is an explicit representation of the Multiplier matrices that is suitable for our purposes.

$$M^{-1}(\gamma)_{\nu\mu} = C \sum_{\epsilon=\pm} \sum_{n=0}^{c-1} \epsilon \, e^{\frac{i\pi}{2rc}\left[d(\nu+1)^2 - 2(\nu+1)(2rn+\epsilon(\mu+1)) + a(2rn+\epsilon(\mu+1))^2\right]}$$

Unlike $SU(2)_R$ the holonomies of $SU(2)_L$ are not constrained by supersymmetry and have to be summed over which gives precisely this matrix.

(Assuming usual shift of k going to k +2)

Knot Theory and Kloosterman

- This computation is closely related to knot invariants of Lens space $\mathcal{L}_{c,d}$ using the surgery formula of Witten. *Witten (89) Jeffrey (92)*
- This is not an accident. Lens space is obtained by taking two solid tori and gluing them by Dehn-twisting the boundary of one of them. But Dehn-twisted solid torus is our $\mathcal{M}_{c,d}$
- Intriguing relation between topology and number theory for an appropriate CS theory.

Quantum Entropy: An Assessment

- ✓ Choice of Ensemble: AdS_2 boundary conditions imply a *microcanonical* ensemble. *Sen (09)*
- ✓ Supersymmetry and AdS_2 boundary conditions imply that index = degeneracy and $J_R = 0$

  Sen (10) Dabholkar Gomes Murthy Sen (12)
- ✓ Path integral localizes and the localizing solutions and the renormalized action have simple analytic expressions making it possible to even evaluate the remaining finite ordinary integrals.

- ✓ Contributions from orbifolded localizing instantons can completely account for all nonperturbative corrections to the quantum entropy.
- ✓ All intricate details of Kloosterman sum arise from topological terms in the path integral.
- ✓ (Most) D-terms evaluate to zero on the localizing solutions *de Wit Katamadas Zalk (10) Murthy Rey (13)*

Path integral of quantum gravity (a complex analytic continuous object) can yield a precise integer (a number theoretic discrete object).

$$W(Q) = \int d\Phi e^{-S[\Phi]} = integer$$

Open Problems

- ? We used an N=2 truncation of N=8 supergravity. This should be OK for finding the localizing instantons because the near horizon has N=2 susy. But it's a *truncation*. Fields with mass of the order of the horizon scale are expected to contribute to one-loop determinants.
- ? A more satisfactory treatment of the measure is necessary. Subtleties with gauge fixing from conformal gravity to Poincare gravity.

Off-shell supergravity

- ? We ignored hypermultiplets. Known not to contribute to Wald entropy and from final answer do not seem to contribute to the full quantum entropy either. It would be good to prove this.
- ? It would be useful to have off-shell realization of the two localizing supercharges on all fields of N=8 supermultiplet. Hard technical problem.

Kloosterman sum arising from topological terms should be independent of these subtleties.

An *IR* Window into the *UV*

- The degeneracies *d(Q)* count brane bound states. These are *nonpertubative* states whose masses are much higher than the string scale.
- Our supergravity computation of *W(Q)* can apparently access this information with precision.
- If we did not know the spectrum of branes a priori in the *N=8* theory then we could in principle deduce it. For example, in *N=6* models *d(Q)* is not known but the sugra computation of *W(Q)* seems doable.

Platonic Elephant of M-theory

- Quantum gravity seems more like an equivalent dual description rather than a coarse-graining.
- *It is not only UV-complete (like QCD) but UV-rigid*. E. g. Small change in the effective action of an irrelevant operator will destroy integrality.
- *AdS/CFT* is just one solitonic sector of the theory. It seems unlikely that we can bootstrap to construct the whole theory from a single *CFT* which for a black hole is just a finite dimensional vector space.

AdS_4/CFT_3

- *N* M2-branes in M-theory on $\mathbb{R}^8/\mathbb{Z}_k$
- M-theory on the near horizon $AdS_4 \times S^7/\mathbb{Z}_k$ geometry is holographically dual to ABJM theory.
- The partition function of the CFT is an *Airy function* computed using localization CFT, matrix models methods and resummation. *Kapustin Willett Yaakov (10) Drukker Mariño Putrov (11) Fuji Hirano Moriyama (11) ….*
- Gives the famous $N^{3/2}$ growth of states in the supergravity limit at large 't Hooft coupling.

Airy Function

- $$Z_{CFT} \sim Ai(z) = \int_{\infty e^{-i\frac{\pi}{3}}}^{\infty e^{+i\frac{\pi}{3}}} dt \, \exp\left[\frac{1}{3}t^3 - zt\right]$$
$$\sim \exp\left[\frac{2}{3}z^{3/2} - \log(z^{3/2} + \ldots\right]$$
- $z^{3/2} \sim R^2 \sim N^{3/2}k^{1/2}$ Valid at finite *AdS* radius *R* in 4d Planck units. Ignores M2-brane instantons.
- Can we compute it from bulk quantum gravity? Airy function is very analogous to Bessel function.

Truncation on $S^7/\mathbb{Z}_k$

- Gauged supergravity with two vector multiplets and a square-root prepotential

$$F = \sqrt{X^0 (X^1)^3}$$

Gauntlett Kim Varela Waldram (09)

- We can apply localization methods. There is a two parameter family of off-shell localizing instantons.

Renormalized Action

$$S_{ren} = cF(\phi^I) + N\phi^1 + k\phi^0$$

where c is a simple numerical constant.

- Unlike in the black hole case we obtain something like Z_{top} instead of $|Z_{top}|^2$.
- We get the Airy function if we assume flat measure for the variables $(u = \sqrt{\phi^0}, \mu = \phi^1)$ At present we are not able to derive the measure.

Possible Relation to Topological String

- The boundary matrix model also gives a Laplace integral of the topological string partition function for local $P^1 \times P^1$ with a cubic prepotential.
- $\mathbb{CP}^3$ is not a Calabi Yau and we have gauged supergravity but the truncation also has only two vector multiplets and the square-root prepotential is related to the cubic one by an electric-magnetic duality. Perhaps the two are related in some limit.

AdS_3/CFT_2 and Mock Modular Forms

- Euclidean AdS_3 has a 2-torus boundary and we expect modular symmetry for the partition function.
- Often the *asymptotic* degeneracy includes contributions from not only single-centered black holes but also multi-centered bound states of black holes. Related to *Wall-crossing phenomenon.*
- Isolating the single-centered contribution microscopically is subtle. The counting function is then no longer modular as expected.

Modular symmetry is apparently lost!

- Loss of modularity is a ***serious*** problem. It means loss of general coordinate invariance in the context of AdS3/CFT2. It is far from clear if and how modular symmetry can be restored.
- We obtained a complete solution to this problem for black strings with N=4 supersymmetry. It naturally involves ***mock*** modular forms.
- We obtained a number of new results in the mathematics of mock modular forms motivated by this physics.
- Signifies ***noncompactness*** of boundary CFT.

Decomposition Theorem

- The counting function of single-centered black hole is a *mock Jacobi form*.
- The counting function of multi-centered black holes is an *Appel Lerch sum.*
- Neither is modular but both admit a modular completion by an additive ***nonholomorphic correction term*** restoring modular symmetry!

Dabholkar Murthy Zagier (12)