Strings 2014, Princeton

Umbral Moonshine and String Theory

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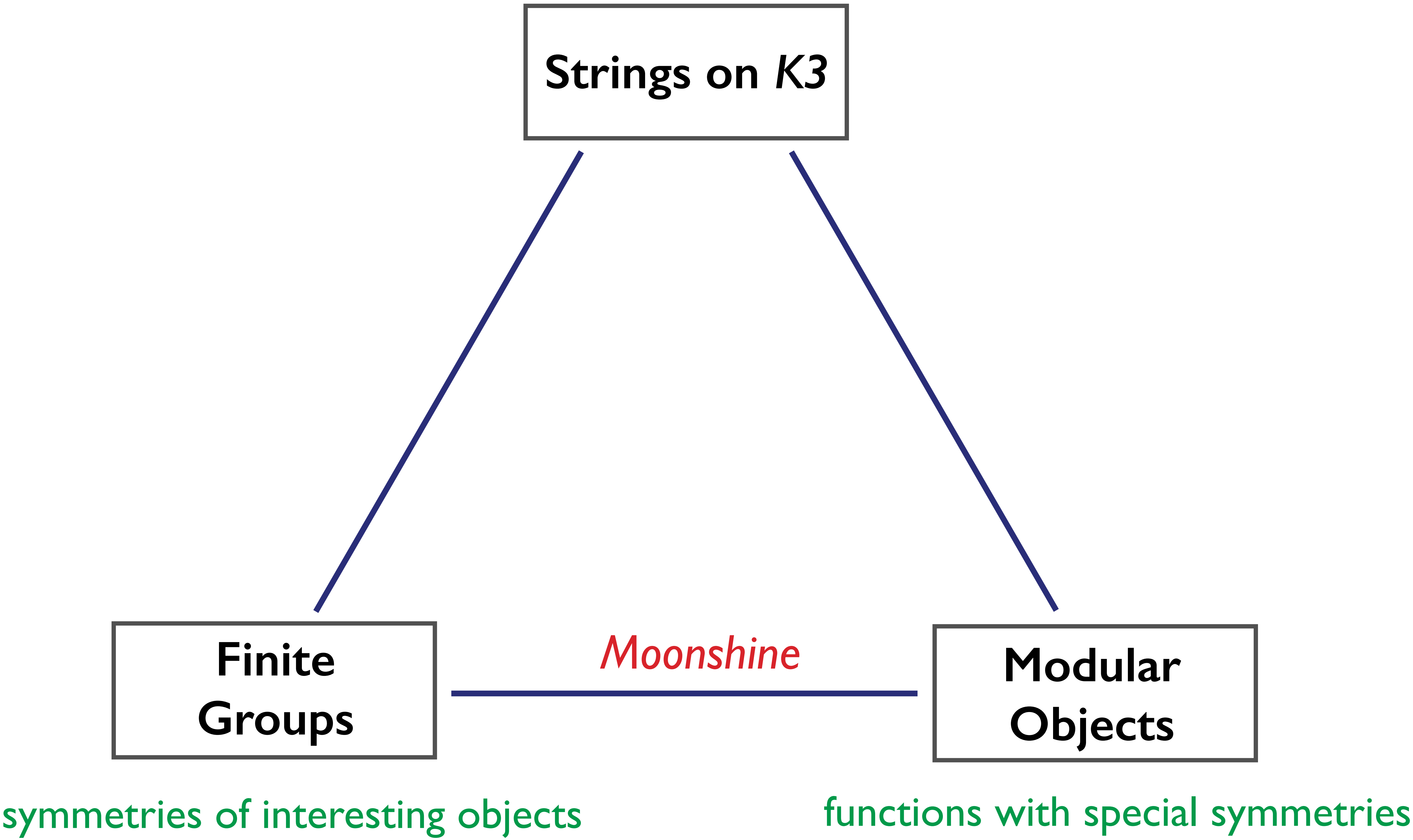
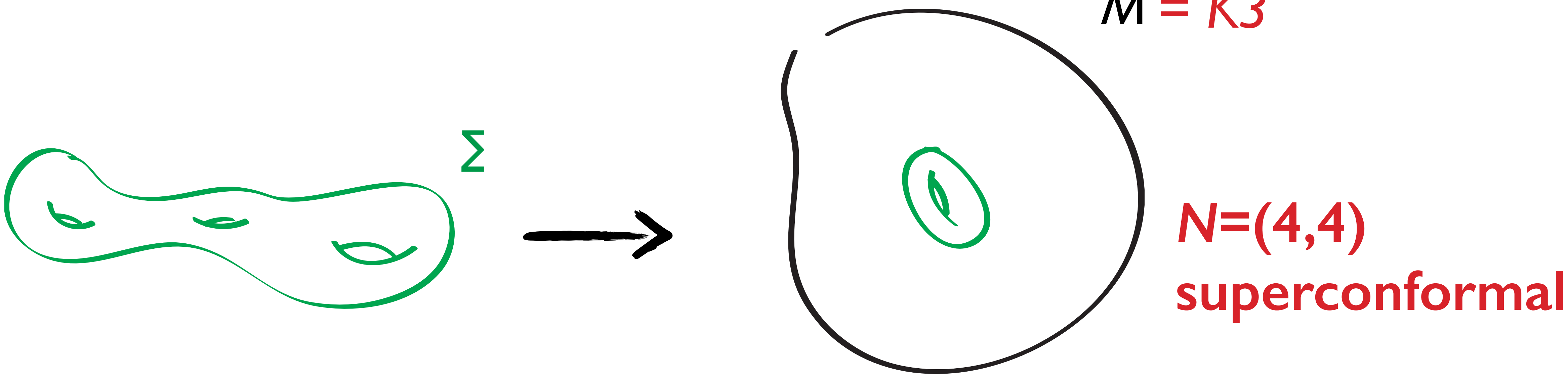
*: on leave from CNRS, France.

A Mysterious Story About



K3 Sigma-Model

2d sigma models: use strings to probe the geometry.



Elliptic Genus of 2d SCFT

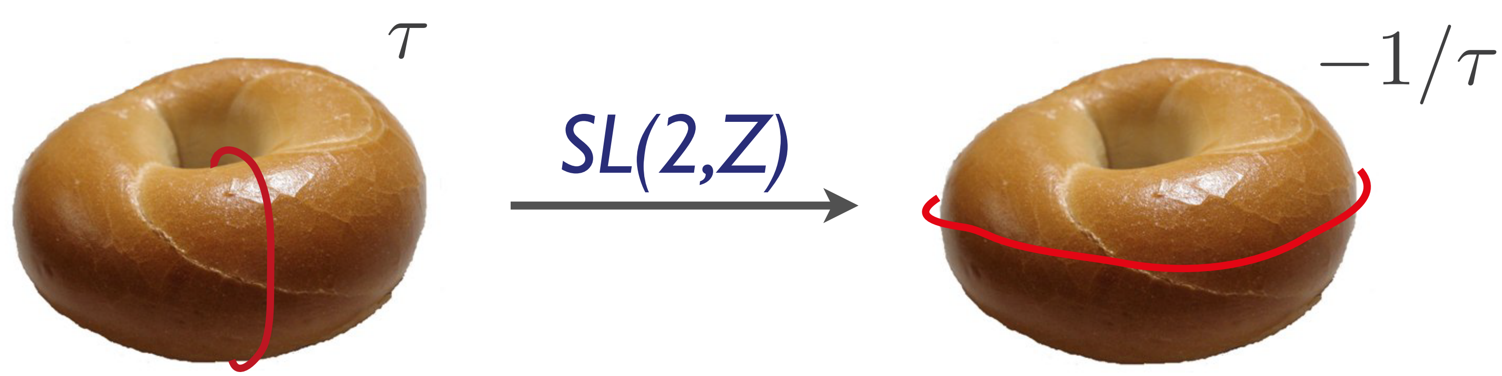
In a 2d *N>=(2,2)* SCFT, susy states are counted by the elliptic genus:

$$\mathbf{EG}(\tau, z; \mathrm{CFT}) = \mathrm{Tr}_{\mathcal{H}_{\mathrm{RR}}}\left((-1)^{J_0+\bar{J}_0} y^{J_0} q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - c/24}\right)$$

$$q = e^{2\pi i \tau}, y = e^{2\pi i z}$$

[Schellekens–Warner, Witten '87]

- holomorphic
- modular



- topological

EG () = **EG** ()

K3 Sigma-Model

2d sigma model on *K3* is a *N=(4,4)* SCFT.

⇒ The spectrum fall into irred. representations of the *N=4* SCA.

$$\mathbf{EG}(\tau,z;K3)=\mathrm{Tr}_{\mathcal{H}_{RR}}\left((-1)^{J_0+\bar{J}_0}y^{J_0}q^{L_0-c/24}\bar{q}^{L_0-c/24}\right)=8\sum_{i=2}^{4}\left(\frac{\theta_i(\tau,z)}{\theta_i(\tau,0)}\right)^2$$

$$= 24 \text{ massless multiplets } + \infty\text{−tower of massive multiplets}$$

$$=\frac{\theta_1^2(\tau,z)}{\eta^3(\tau)}\left(24\,\mu(\tau,z)+2\,q^{-1/8}(-1+45\,q+231\,q^2+770\,q^3+\dots)\right)$$

"Appell–Lerch sum"

numbers of massive N=4 multiplets

also dimensions of irreps of *M24*,

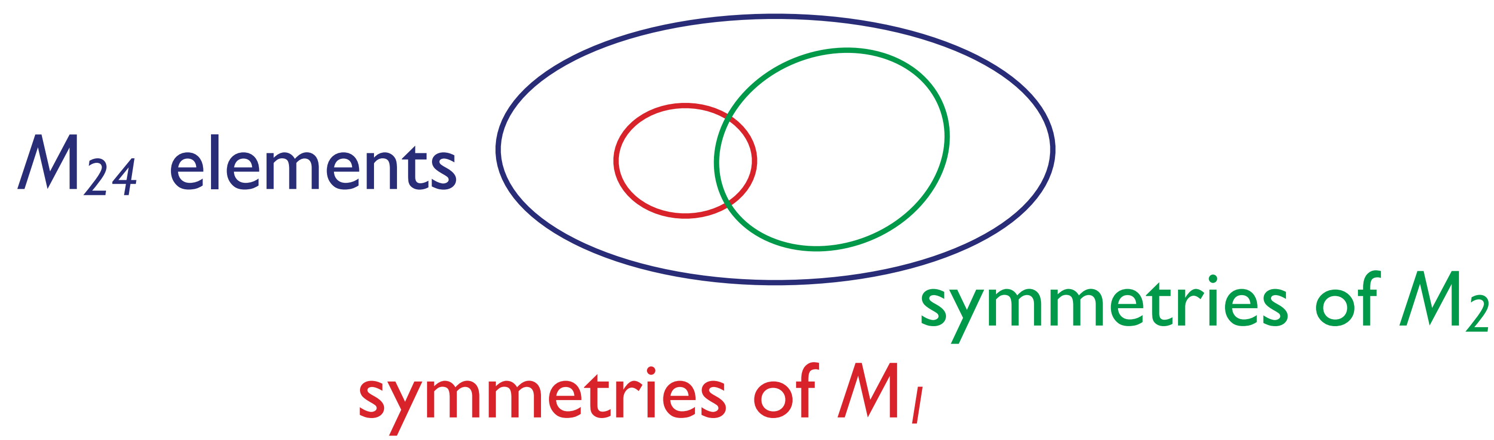
an interesting finite group with ~10^8 elements

[Eguchi–Ooguri–Tachikawa '10]

Why EG($K3$) $\longleftrightarrow$ M_{24}?

Q: Is there a $K3$ surface M whose symmetry (that preserves the hyperKähler structure) is M_{24}? [Mukai '88, Kondo '98]

No!



Q: Is there a $K3$ sigma model whose symmetry is M_{24}?

[Gaberdiel–Hohenegger–Volpato '11]

No!

Why EG($K3$) $\longleftrightarrow$ M_{24}?

- We do not have a precise answer yet.

- Plenty of evidence from both the CFT viewpoint

[See for instance: Taormina–Wendland, Gaberdiel–Persson–Volpato
, Creutzig–Höhn....]

as well as (heterotic, non-perturbative) string theory viewpoint

[See for instance: MC '10, MC–Duncan '12, Persson–Volpato, He–McKay,
MC–Dong–Duncan–Harvey–Kachru–Wrase '13]

that it is **not just a coincidence** that we see here.

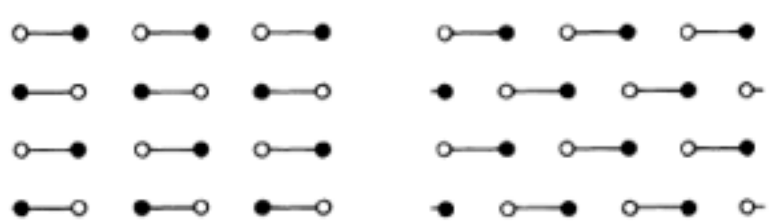
- An active topic of research. Some ideas include: we need a new way to study the **symmetries of BPS states**, and/or we need to unravel certain **hidden symmetries in K3 compactifications**.

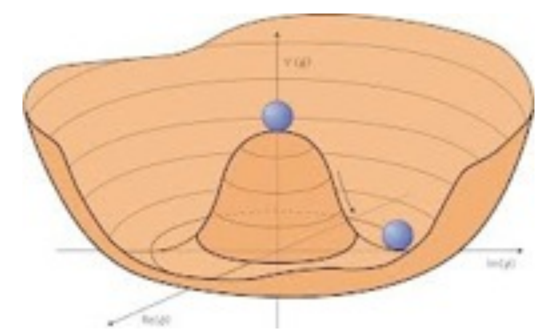
[See for instance: Talk by S. Kachru, Taormina–Wendland, Harvey–Murthy '13]

Why should we care?

String Theory ⟷ **Moonshine**

- Symmetries are important in physics!





- BPS states are important in string theory!

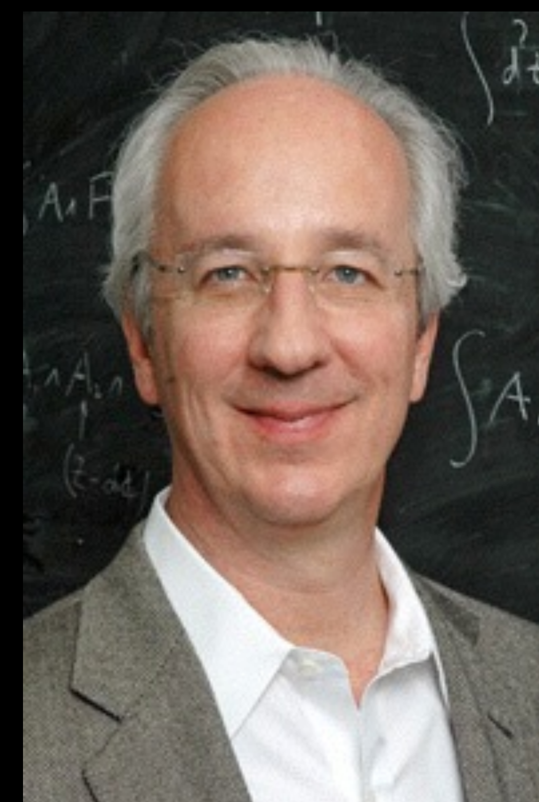
See 80% of the talks in the String-Math conferences for instance.

- K3 compactification is ubiquitous in string theory!

From black hole states counting to string duality, we need K3.

- An elegant and important mathematical problem involving number theory, group theory, algebra and geometry.

I. Umbral Moonshine



based on work with **John Duncan** and **Jeff Harvey**
arXiv: 1204.2779, 1307.5793

What are these as mathematical objects?

$$H(\tau) = 2\,q^{-1/8}(-1 + \underline{45}\,q + \underline{231}\,q^2 + \underline{770}\,q^3 + \dots)$$

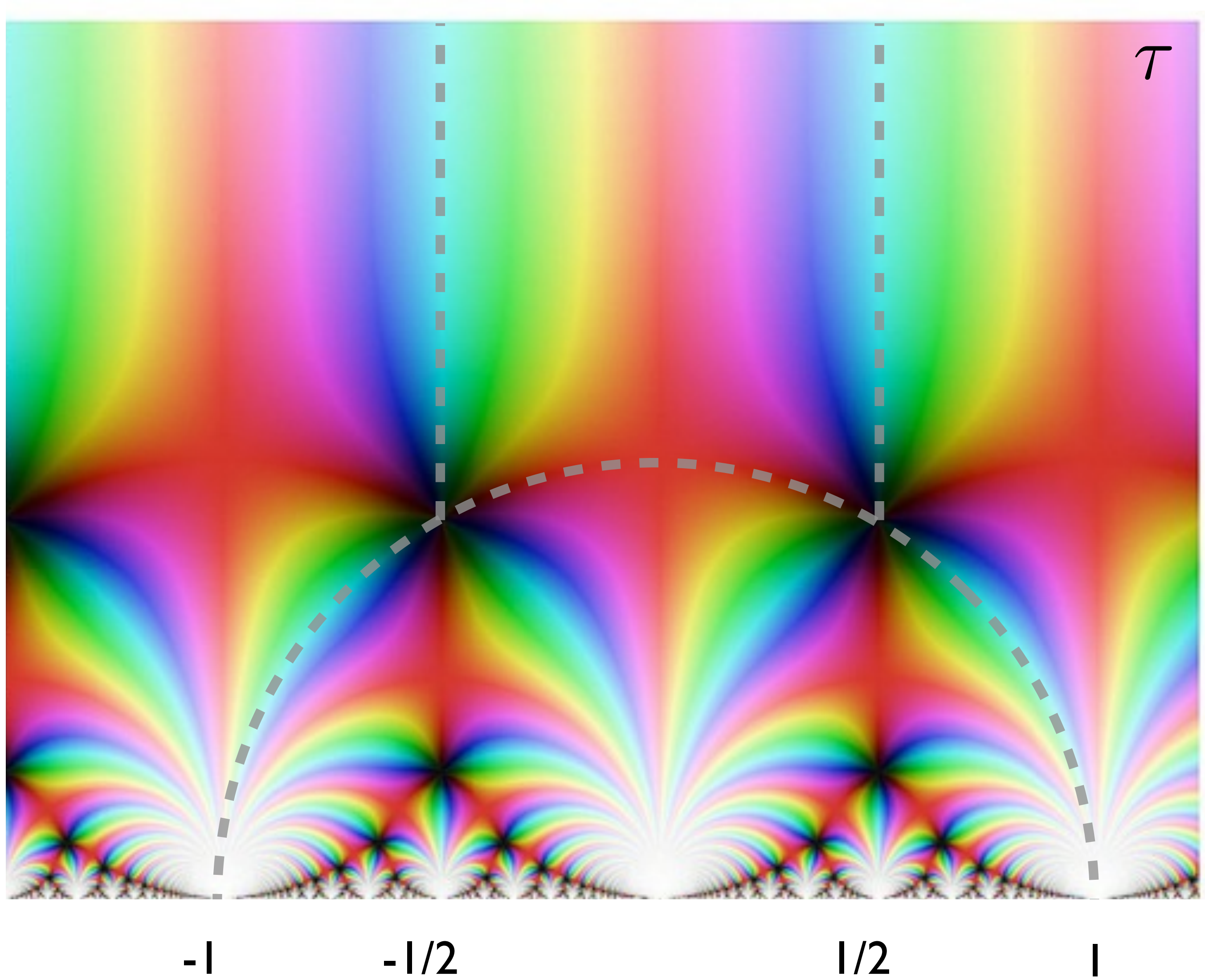
dimensions of irreps of ***M_{24}***,
an interesting finite group with ~10^8 elements

- $H(\tau)$ is a **mock modular form.**
- M_{24} is a **sporadic group.**

Modular Forms

= Holomorphic functions on the upper-half plane that transforms "nicely" under *SL(2,Z)*.

Example: the J-function



$$J(\tau) = J(\tau+1) = J(-1/\tau)$$

(*:modularity)

$$= q^{-1} + \mathcal{O}(q)$$

(**: pole condition)

* & ** $\Rightarrow$ *J* is unique.

Mock Modular Form

: a variant of modular form that comes with a **non-holomorphic correction**, given by the **shadow** (or **umbra**) of the mmf.

$$f(\tau) + \int_{-\bar{\tau}}^{\infty} d\tau' (\tau + \tau')^{-w} \overline{s(-\bar{\tau}')} = \hat{f}(\tau, \bar{\tau})$$

[Ramanujan '20]
....
[Zwegers '02]

$f(\tau)$ ↓ holomorphic, mock modular

s ↓ **shadow**

$\hat{f}$ ↓ modular

controls the modularity

Mock modular forms appear in physics, from 2 origins!

Geometric: Mod. anomaly from the **non-compactness** of the relevant spaces.

eg. wall-crossing, the cigar theory, ...

[Vafa–Witten '94, Troost '10, Dabholkar–Murthy–Zagier '12, ...]

Algebraic: Mod. anomaly from characters of **super**-algebras.

eg. 2d superconformal algebra, affine Lie superalgebras

[Eguchi–Hikami '09, Kac–Wakimoto, Bringmann–Ono, ...]

Sporadic Groups

The only 26 finite simple groups that do not fall into infinite families.

Example 1. M_{24} = the biggest of the 5 "*Mathieu Groups*".

[Mathieu 1860]

Example 2. "*The **Monster***" = the largest sporadic groups.

$|M| \sim 10^{54}$ ~nr of atoms in the solar system.

[Fischer, Griess 1970-80]

Reminiscence: Monstrous Moonshine

$$H(\tau) = 2\,q^{-1/8}(-1 + \underline{45}\,q + \underline{231}\,q^2 + \underline{770}\,q^3 + \ldots)$$

dimensions of irreps of ***M₂₄***

$$J(\tau) = J(\tau+1) = J(-\tfrac{1}{\tau}) = q^{-1} + \mathcal{O}(q)$$

$$= q^{-1} + \underline{196884}\,q + \underline{21493760}\,q^2 + \ldots$$

$$\| \qquad\qquad\qquad \|$$

$$1{+}196883 \qquad 1{+}196883{+}21296876$$

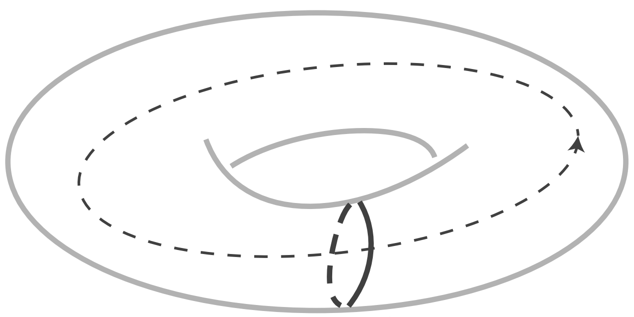
[McKay late 70's]

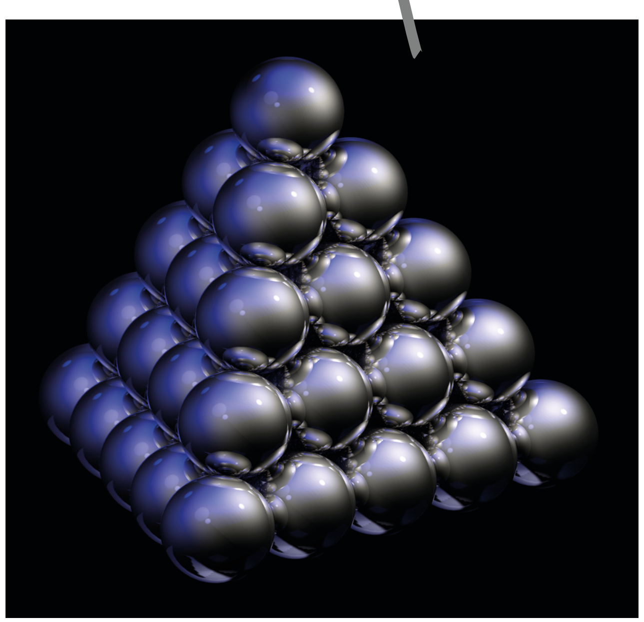
dimensions of the irreps of the Monster group

String Theory

explains Monstrous Moonshine.

Strings in the Leech lattice background:

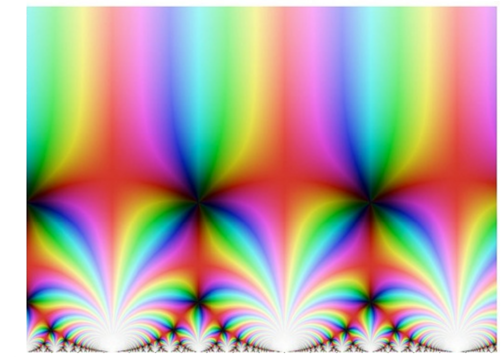




$\mathbb{R}^{24}/\Lambda_{Leech}$

Modular Symmetry

$$\mathrm{Tr}_{\mathcal{H}}(q^{L_0 - c/24}) = J(\tau)$$

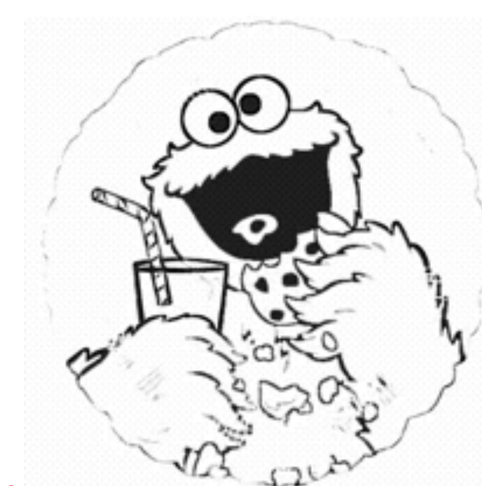


Monstrous Moonshine

Sporadic Symmetry

Λ_{Leech}

$Co\,I \xrightarrow{Z/2}$ *Monster*



[Conway–Norton '79, Frenkel–Lepowsky–Meurman, Borcherds 80's–90's]

Why these forms and groups?

$$H(\tau) = 2\,q^{-1/8}(-1 + \underline{45}\,q + \underline{231}\,q^2 + \underline{770}\,q^3 + \dots)$$

dimensions of irreps of ***M_{24}***,
an interesting finite group with ~10^8 elements

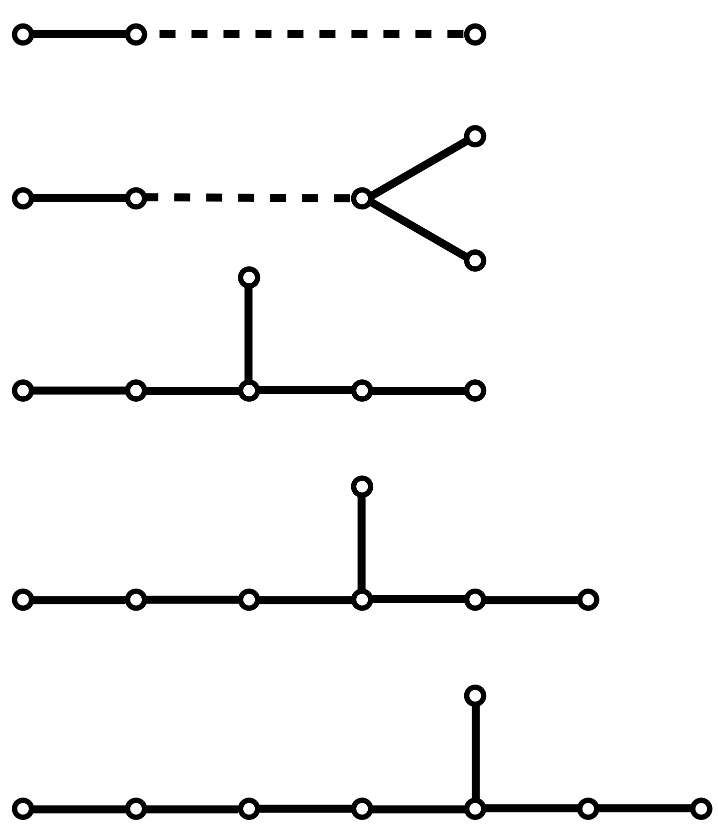
- Why $H(\tau)$ from all mock modular forms?
- Why M_{24} and not other sporadic groups?

M_{24} from Lattice Symmetries:

it is the symmetry of the Niemeier lattice N^X, $X=24\,A_1$.

The 23 Niemeier Lattices

ADE root systems



There are exactly 23 unions of ADE rt sys. X

$24A_1$, $12A_2$,.... , $6D_4$, $4D_6$, $2A_9 \oplus D_6$,…., $3E_8$, $D_{16} \oplus E_8$, D_{24}

with 1. total rank =24

2. the same *h(=Coxeter nr.)* for all components

↓ adding points to the root lattice

the 23 even, self-dual Niemeier Lattices N^X

↓ lattice symmetries

Finite Groups G^X

eg. : $G^X \simeq M_{24}$, $X=24\,A_1$. $G^X \simeq 2.M_{12}$, $X=12\,A_2$.

The shadow fixes the function!

$$H(\tau) = 2\,q^{-1/8}(-1 + 45\,q + 231\,q^2 + 770\,q^3 + \dots)$$ satisfies

- mock modular with shadow $S(\tau) = \underline{24\,\eta^3(\tau)}$

(*:modularity)

- $q^{1/8}H(\tau) = \mathcal{O}(1) \quad \text{as} \quad q \to 0$

(**: pole condition)

* & ** ⇒ ***H*** **is unique** (up to normalisation)!   [MC–Duncan ‘11]

Using the same math underlying the
ADE classification of N=2 minimal models.
[Cappelli–Itzykson–Zuber ’87]

X=24 A₁

Niemeier Shadows

$S^X(\tau)$

Using the same math underlying the
ADE classification of N=2 minimal models.

$X = 24A_1, 12A_2, \ldots, 6D_4, 4D_6, 2A_9 \oplus D_6, \ldots, 3E_8, D_{16} \oplus E_8, D_{24}.$

Niemeier Lattices N^X ⟶ **Mock Modular Forms H^X**

The vector of mmf $H^X = (H^X_r)\ ,\ r = 1,\ldots,h-1$ satisfies

- mock modular with shadow $S^X(\tau)$ (*:modularity)
- $q^{1/4h} H^X_r(\tau) = \mathcal{O}(1)\ ,$ for all $r = 1,\ldots,h-1$ (**: pole condition)

* & **⇒ ***H^X*** **is unique** (up to normalisation).

$$H^X = (H^X_r) \longleftrightarrow \psi^X(\tau,z)$$

Umbral Moonshine:
2nd example

Corresponding to the "next simplest" Niemeier lattice N^X, $X=12\,A_2$

$$H_1^X = 2\,q^{-1/12}\,(-1 + \underline{16}\,q + \underline{55}\,q^2 + \underline{144}\,q^3 + \ldots)$$

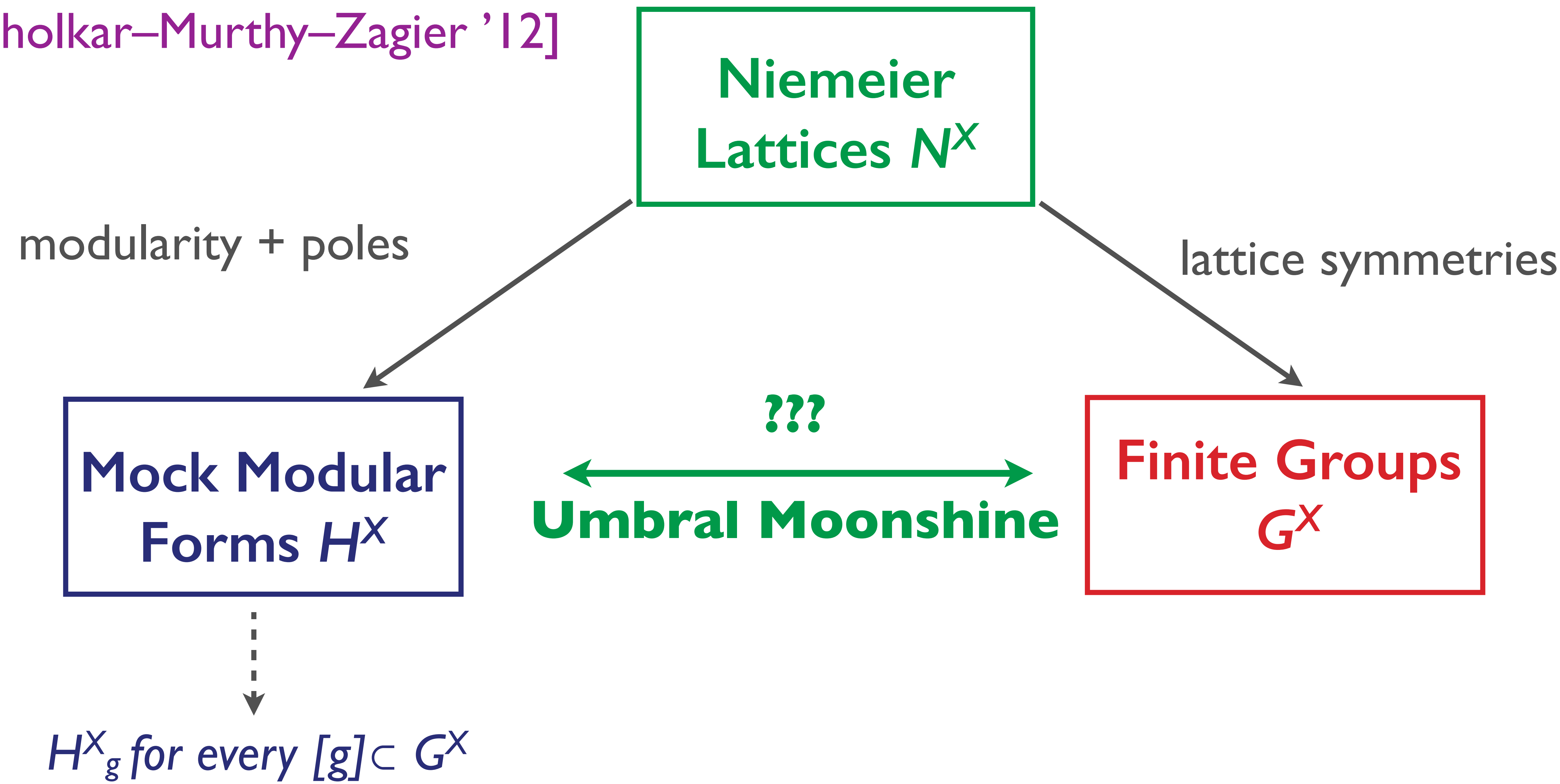
$$H_2^X = 2\,q^{8/12}\,(\underline{10} + \underline{44}\,q + \underline{110}\,q^2 + \ldots)$$

They are dimensions of irreps of the
Mathieu sporadic group $G^X=2.M_{12}$!

Niemeier Groups and Forms

For each of the 23 Niemeier lattice N^X, we associate a (unique) mock modular form H^X and a finite group G^X.

[Inspired by Cappelli–Itzykson–Zuber '87, Dabholkar–Murthy–Zagier '12]



Umbral Moonshine

A New Type of Moonshine

Mock Modular Forms H^X ⟷ **Umbral Moonshine** ⟷ **Finite Groups G^X**

23 cases with a uniform construction.

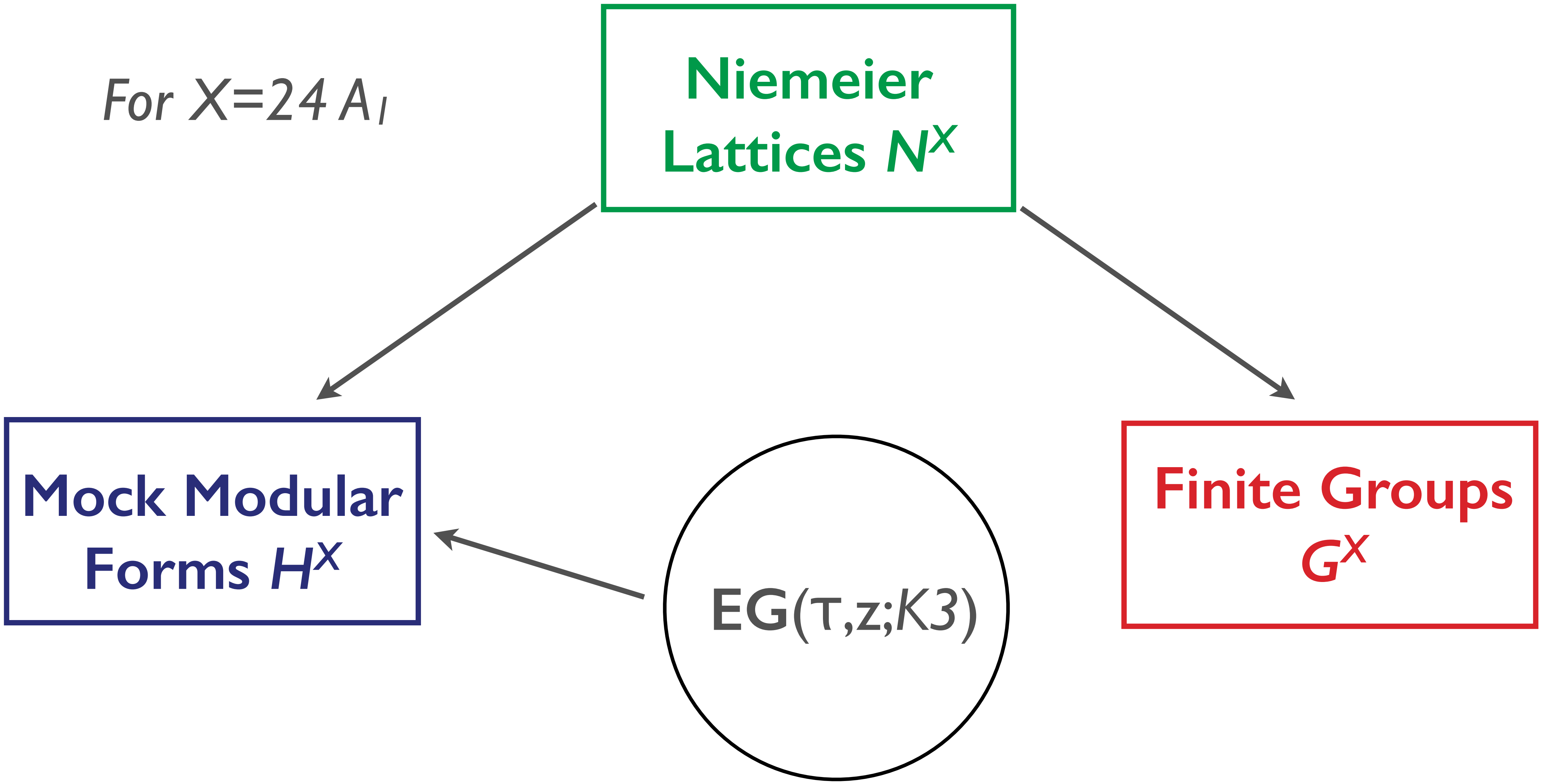
Umbral Moonshine Conjecture

G^X–*representation* $K^X_{r,n}$ that gives the coefficients of the special mmf H^X_g:

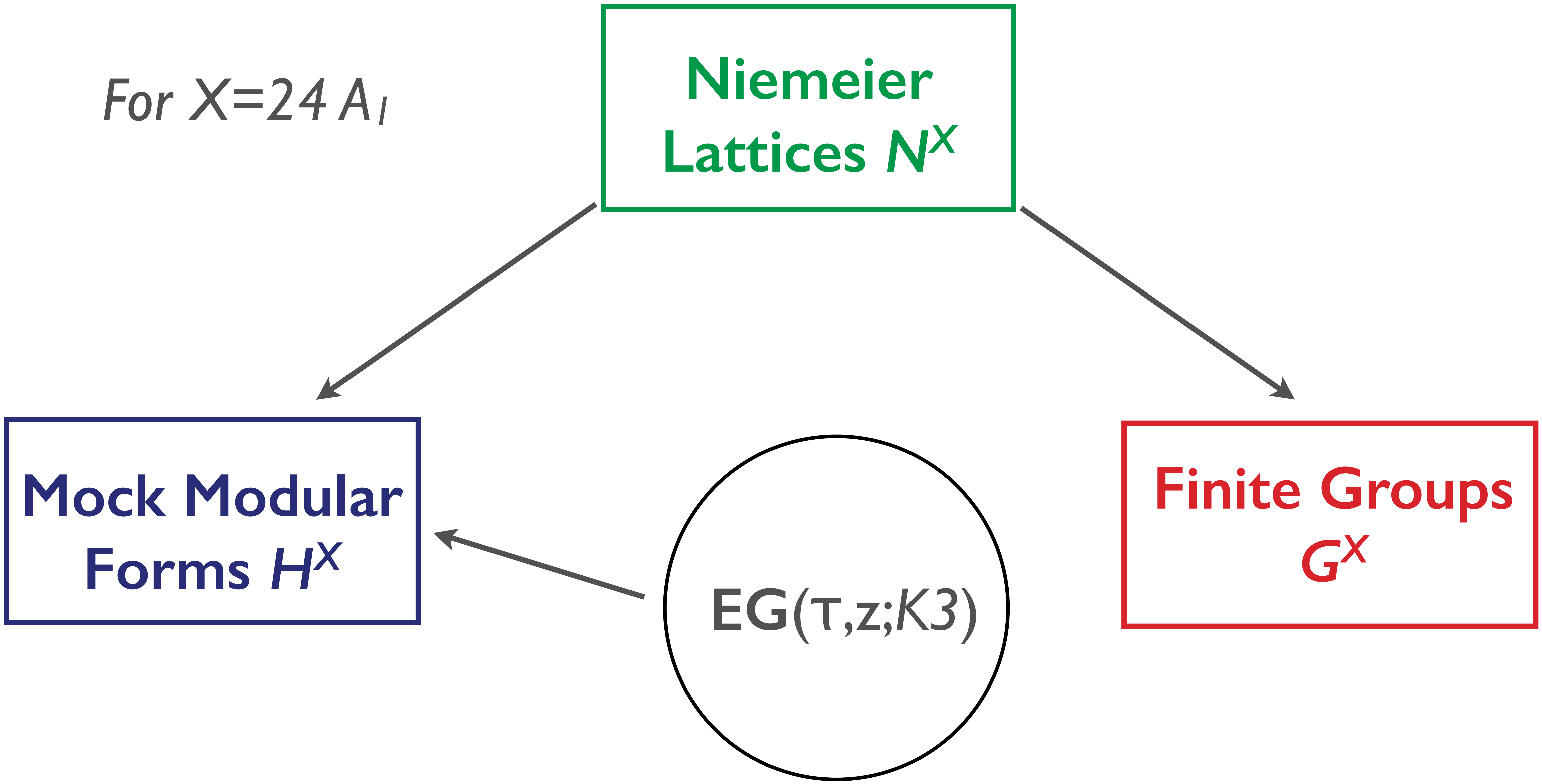
$$H^X_{g,r}(\tau) = q^{-r^2/4h} \sum_n q^n \left(\mathrm{Tr}_{K^X_{r,n}} g\right)$$

String Theory on $K3$ and M_{24}

For $X=24\,A_1$



String Theory on *K3* and Umbral Moonshine?



Q: What about the other 22 cases ($X\neq 24A_1$) of umbral moonshine?
What is the physical and geometric context of umbral moonshine in general?

II. Umbral Moonshine and K3 Compactifications

based on work with **Sarah Harrison**
arXiv: 1406.0619

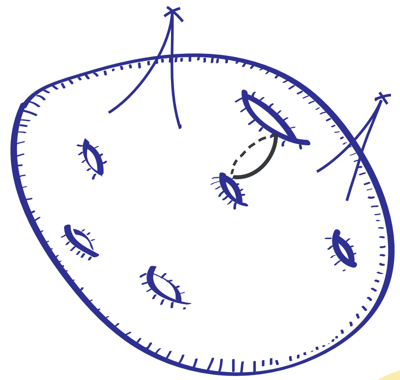
What is the **physical and geometric meaning** of this construction?

Niemeier Lattices N^X ⟶ **Mock Modular Forms H^X**

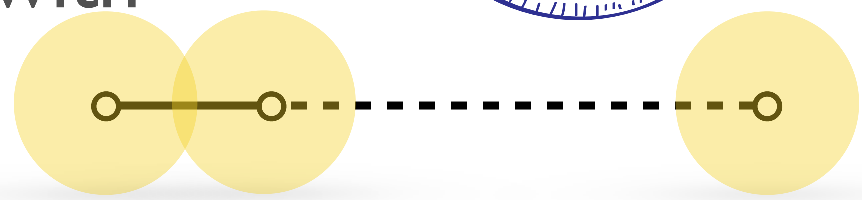
Using the same math underlying the
ADE classification of N=2 minimal models.

- The classification can be understood through the ADE *du Val surface singularities*. [E. Martinec, Vafa-Warner '89]

- They are singularities a K3 can develop. Locally they look like C^2/Γ.



- Their resolution gives rise to genus 0 curves with the corresponding ADE intersection matrix.



ADE root system *X* ↔ <u>ADE singularities/rational curves</u> *X* ?

What is the **physical and geometric meaning** of this construction?

Niemeier Lattices N^X	lattice symmetries →	Finite Groups G^X

Niemeier lattices provide a framework to study K3 geometry. [Nikulin 2011]

Depending on the shape of the *K3*, every *K3* surface *M* can be associated with (marked by) at least one Niemeier lattice N^X, and such that

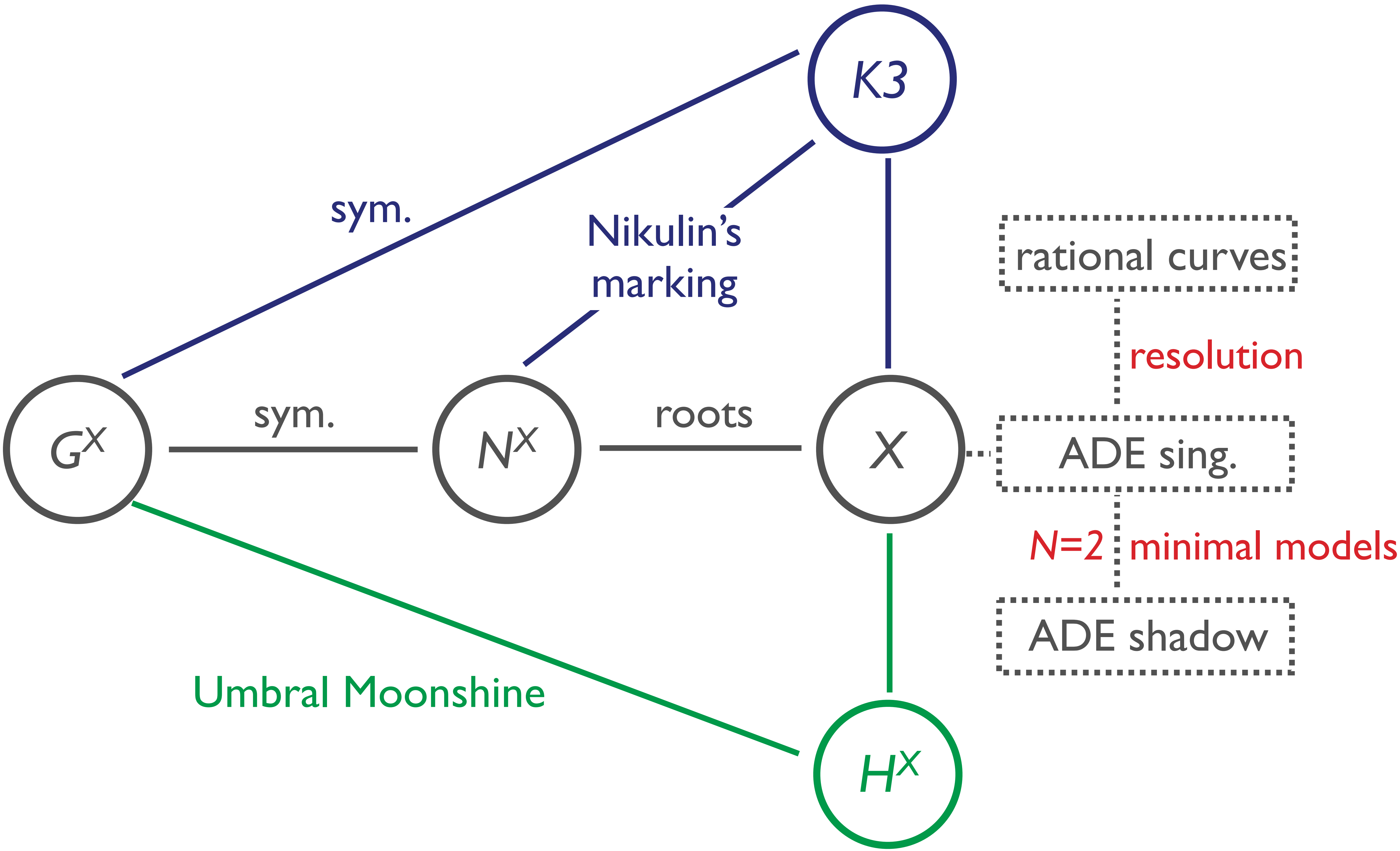
- Rational curves $\subset X$
- $Sym(M) \subset G^X$

eg.: $M_1 \simeq T^4/Z_2$, $16\,A_1 \subset 24\,A_1 = X_1$, $Sym(M_1) \subset G^{X_1} = M_{24}$

[cf. Taormina–Wendland '13]

$M_2 \simeq T^4/Z_3$, $9\,A_2 \subset 12\,A_2 = X_2$, $Sym(M_2) \subset G^{X_2} = 2.M_{12}$

A Geometric and a Moonshine Story



Q: Are the 2 stories related? What is the role of EG?

Elliptic Genus of *K3* Singularities

EG(τ,*z*;*X*) for $X=A_{h-1}, D_{1+h/2}, E_6, E_7, E_8$ can be computed thanks to the proposal in [Ooguri-Vafa '95] and the recent progress in computing EG of non-compact theories [Troost '10, …..].

They are all mock modular due to their non-compactness.

$$\mathbf{EG}(\tau, z; K3) = 24 \text{ massless multiplets} + \infty\text{−tower of massive multiplets}$$

$$\|$$

$$\mathbf{EG}(\tau, z; A_1)$$

$$= \frac{\theta_1^2(\tau, z)}{\eta^3(\tau)} \Big(24\, \mu(\tau, z) + 2\, q^{-1/8}(-1 + 45\, q + 231\, q^2 + 770\, q^3 + \dots) \Big)$$

Appell–Lerch sum, mock modular

Elliptic Genus of *ADE* Singularities

$\mathbf{EG}(\tau, z; A_1)$

$\mathbf{EG}(\tau, z; K3)$

$= 24$ massless multiplets $+ \infty-$tower of massive multiplets

$$= \frac{\theta_1^2(\tau,z)}{\eta^3(\tau)}\Big(24\,\mu(\tau,z) + 2\,q^{-1/8}(-1 + 45\,q + 231\,q^2 + 770\,q^3 + \ldots)\Big)$$

(algebraic)

$$= \mathbf{EG}(\tau, z; X) + \sum_{a,b\in\mathbb{Z}/h\mathbb{Z}} q^{a^2} y^{2a} \psi^X(\tau, \tfrac{z+a\tau+b}{h})$$

(geometric)

for $X=24A_1$

contribution from the singularities

contribution from the umbral mock mod. form

K3 Elliptic Genus and Umbral Moonshine

$$H^X = (H^X_r) \longleftrightarrow \psi^X(\tau, z)$$

$$\mathbf{EG}(\tau, z; K3)$$

$$= \mathbf{EG}(\tau, z; X) + \sum_{a,b \in \mathbb{Z}/h\mathbb{Z}} q^{a^2} y^{2a} \psi^X(\tau, \tfrac{z+a\tau+b}{h})$$

contribution from the singularities

contribution from the umbral mock mod. form

for **all** 23 Niemeier lattices with
X=24A₁, ...

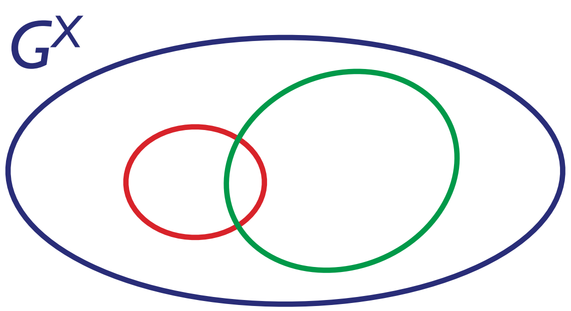
23 ways to split the same EG into 2 parts, depending on the moduli/marking of the *K3*.

All 23 cases of UM are related to *K3*?!

Twined *K3* Elliptic Genus and Umbral Moonshine

For any g generating *any symmetry of any K3 surface*, we can define

$$\mathbf{EG}_g(\tau,z) = \mathrm{Tr}_{\mathcal{H}_{RR}}\left(g\,(-1)^{J_0+\bar{J}_0} q^{L_0-c/24}\bar{q}^{\bar{L}_0-c/24} y^{J_0}\right)$$



$$= \mathbf{EG}_g(\tau,z;X) + \sum_{a,b\in\mathbb{Z}/h\mathbb{Z}} q^{a^2} y^{2a} \psi_g^X\left(\tau, \frac{z+a\tau+b}{h}\right)$$

K3 symmetries

contribution from the singularities

contribution from the umbral mock mod. form

$H_g^X(\tau) \longleftrightarrow \psi_g^X(\tau,z)$

It holds for **all** such elements g and for **all** 23 Niemeier lattices X.
All geometric symmetries of all K3 sigma models are captured by UM

All 23 cases of UM are related to *K3!*

What Have We Learned?

- It's a moonshine story.

- It's a lattice story.

- It's a *K3* story, in which the rational curves play an important role.

- A lot to explore in *K3* compactifications. We believe this will lead to an explanation of the umbral moonshine and the discovery of many novel features of *K3* compactifications, BPS algebras, and beyond!

Thank You!