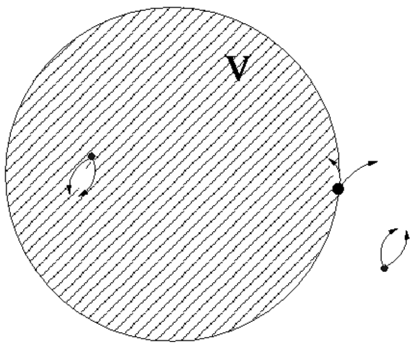
Entropy inequalities and quantum field theory

Horacio Casini

Instituto Balseiro, Centro Atomico Bariloche, Argentina

Entanglement entropy

$$S(V) = -tr(\rho_V \log \rho_V)$$



$$S(V) = \frac{g_{d-2}[\partial V]}{\epsilon^{d-2}} + \cdots + \frac{g_1[\partial V]}{\epsilon} + g_0[\partial V]\log(\epsilon) + S_0(V)$$

Area law: $S \sim \frac{R^{d-2}}{\epsilon^{d-2}}$

«There are also finite quantities lurking within this circle of ideas which seem likely to be of physical significance. One class of such quantities is the difference between geometric entropy for the same region but with respect to different states (for example, thermal or solitonic states). Another class is the geometric entropy for the ground state in its dependence on the shape of the bounding surface.»

C.Callan, F. Wilczek, hep-th/9401072

Entropy inequalities

Two states, one region

Positivity and monotonicity of relative entropy

$$S(\rho_V|\rho_V^0) = \mathrm{tr}\left(\rho_V \log \rho_V - \rho_V \log \rho_V^0\right)$$

↓

Entropy bounds

- Bekenstein bound
- Bousso bound (weak gravity).
  R. Bousso, H.C., Z. Fisher, J. Maldacena (2014)
- Generalized second law (weak gravity)
  A. Wall (2011)

One state two regions

Strong subadditivity

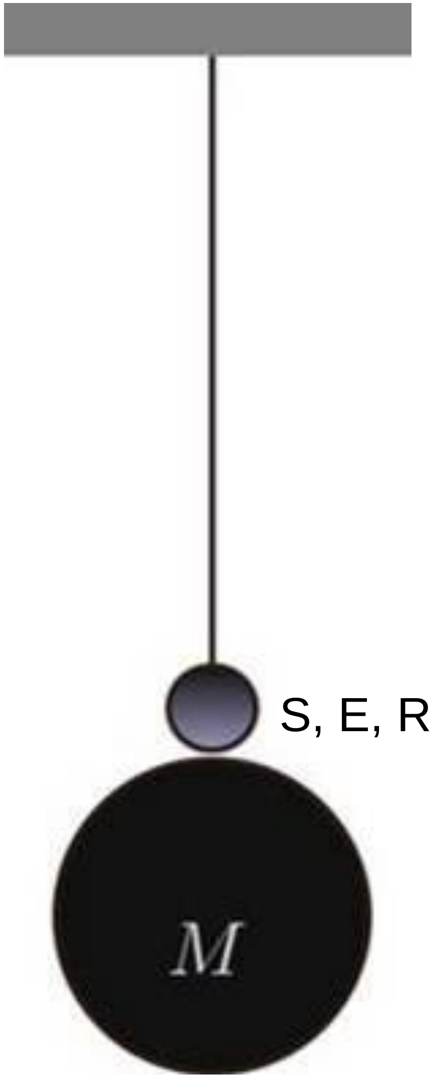
$$S(A) + S(B) \geq S(A \cap B) + S(A \cup B)$$

↓

C-theorems in d=2 and d=3

H.C, M. Huerta (2004,2012)

Bekenstein (1981)



Inicial entropy: $S + \frac{A_i}{4G}$

Final entropy: $\frac{A_f}{4G}$

Einstein equations + generalized second law:

$$S \leq \frac{2\pi RE}{\hbar c}$$

Bekenstein universal bound on entropy

It is independent of G

Does black hole thermodynamics tell something new about flat space physics?

$$S \leq \frac{2\pi R E}{\hbar c}$$

Some puzzles:

What is the meaning of R? Does it imply boundary conditions?

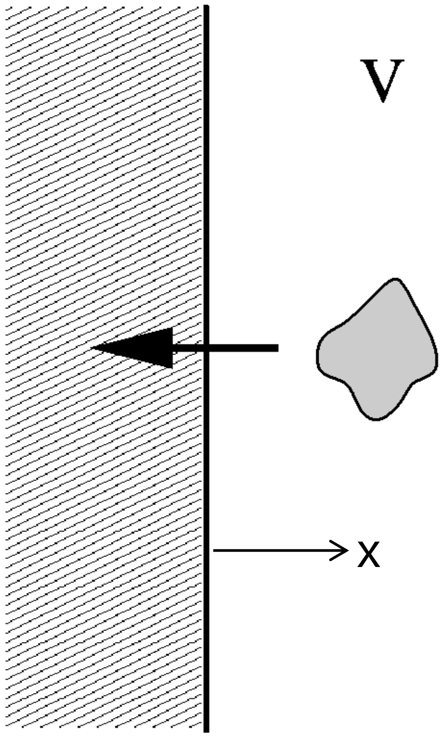
Was not the localized entropy divergent?

Species problem: What if we increase the number of particle species? (S increases, E is fixed)

The local energy can be negative while entropy is positive…

Quantum Bekenstein bound

Marolf, Minic, Ross 2004,
Sorkin 2002,
H.C. 2008



The near horizon limit of a large BH

Entanglement entropy in vacuum! Hence, the left hand side of the inequality is

$$\Delta S(V) = S(V) - S_0(V)$$

The right hand side is more precisely

$$2\pi R E \rightarrow 2\pi \int_V x \, \langle \rho_E \rangle$$

Then the bound reads

$$2\pi \int_V x \, \langle \rho_E \rangle \geq S(V) - S_0(V)$$

Quantum Bekenstein bound

$$2\pi \int_V x \langle \rho_E \rangle \geq S(V) - S_0(V)$$

This clarifies all the puzzles:

What is the meaning of R? The product ER is well defined

Does it imply boundary conditions? NO

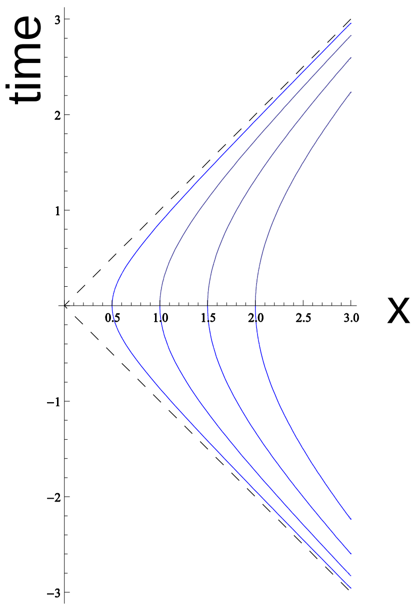
Was not the localized entropy divergent? The difference is not!

What if we increase the number of particle species? The entropy difference saturates

The local energy can be negative while entropy is positive…
The entropy difference can be negative

Proof of quantum Bekenstein bound

Preestablished relation between energy and entropy: Vacuum state in half space determined by the energy density operator for all quantum field theories.



$$\rho_V^0 \sim e^{-2\pi \int_V \, x \, \rho_E}$$

Bisognano Wichmann (1975). Unruh (1976): $T = \frac{a}{2\pi} = \frac{1}{2\pi x}$

Relative entropy between two states is positive

$$S(\rho_V|\rho_V^0) = \mathrm{tr}\left(\rho_V \log \rho_V - \rho_V \log \rho_V^0\right)$$

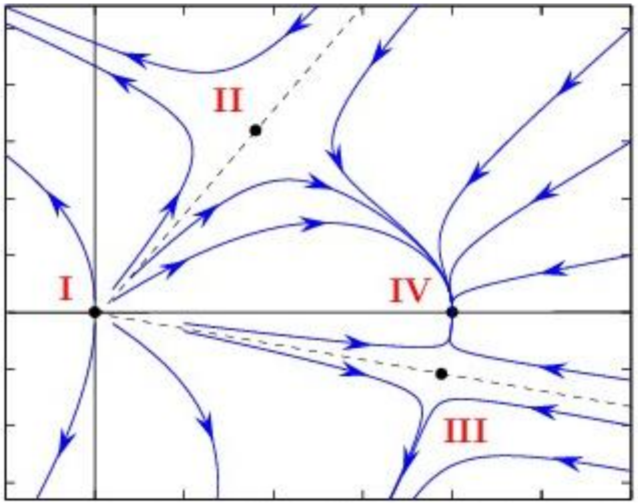
$$= 2\pi \int_V x \langle \rho_E \rangle - (S(V) - S_0(V)) \geq 0$$

Conclusion: quantum Bekenstein bound holds. It is saved by quantum effects. It is consistent with black hole thermodynamics, but follows already from the combination of special relativity and quantum mechanics. Then it is not a new constraint coming from black hole physics.

(entropic) c-theorems in 1+1 and 2+1 dimensions

Teorema C

There is a dimensionless function C on the space of theories which decreases along the renormalization group trajectories from the UV fixed point to the IR fixed point and has finite values at the fixed points.



↓

General constraint for the renormalization group. Ordering of the fixed points

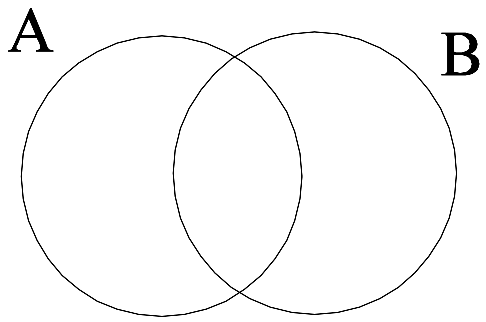
Proofs not using entanglement entropy:
d=1+1: A.B.Zamolodchikov (1986)
d=3+1: Z. Komargodski,
A Schwimmer (2011)

Conjectured in d=2+1:
Holographic C theorems.
R.C.Myers and A.Sinha (2010).
F theorem, D.Jafferis, I.Klebanov,
S.Pufu, B.Safdi (2011).

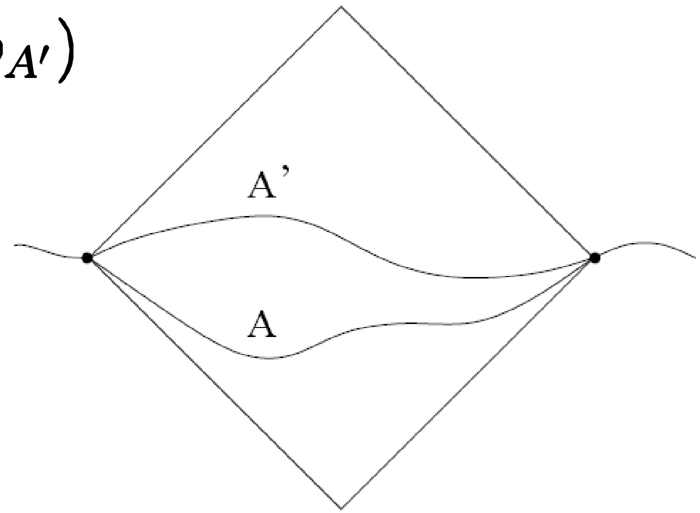
Strong subadditivity + Causality + Lorentz invariace

Strong subadditivity

$$S(A) + S(B) \geq S(A \cap B) + S(A \cup B)$$



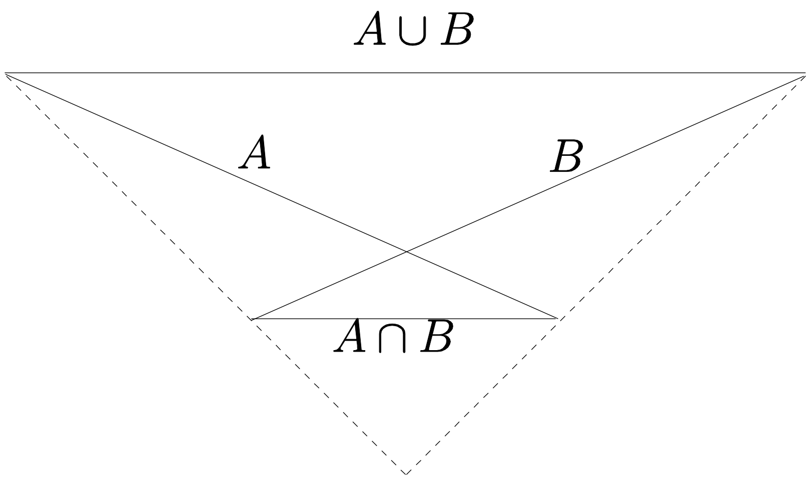
Causality $S(A) = S(A') \quad (\rho_A = \rho_{A'})$



S is a function of causal regions, or «diamonds»

1+1 dimensions

H.C., M. Huerta, 2004



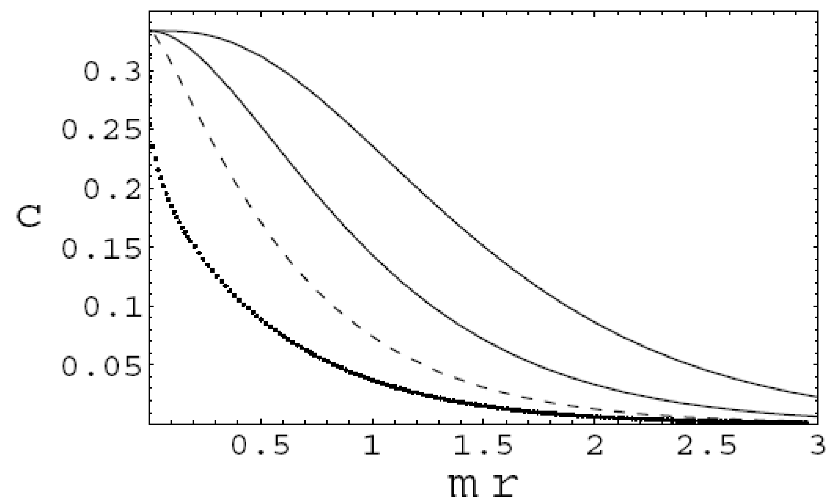
$$C'(r) \leq 0$$

$$C(r) = r S'(r)$$

C(r) is dimensionless, well defined, and decresing.
At the fixed point (scale invariant theory):

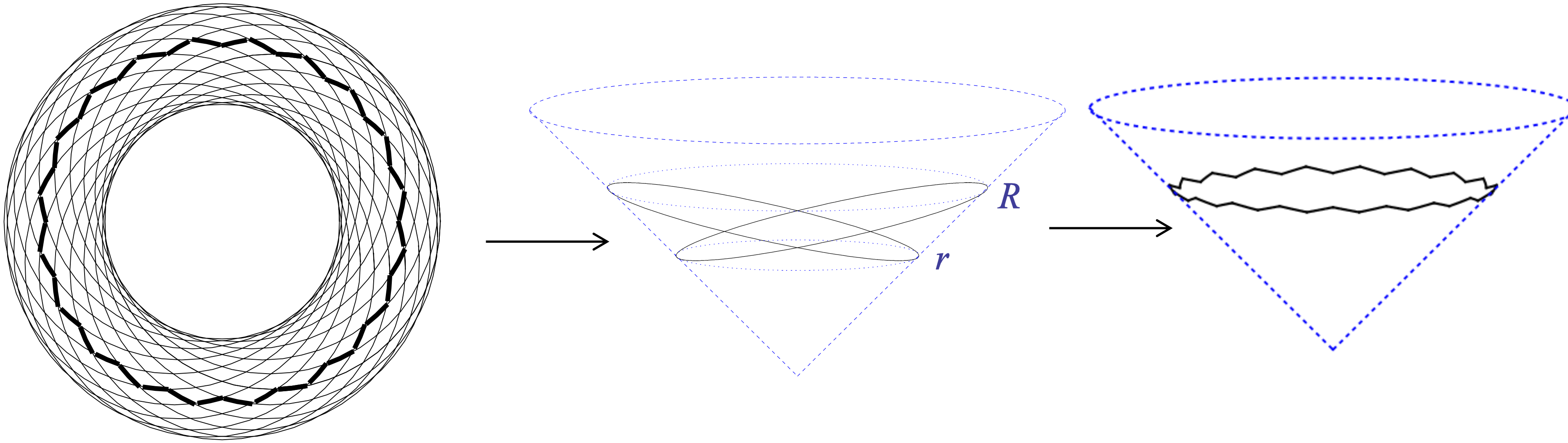
$$S(r) = \frac{c}{3} \log(r/\epsilon) + c_0 \longrightarrow$$ C(r)=c/3

The c-charge is proportional to Virasoro central charge at fixed points in 1+1. This is the same result as Zamolodchikov's
but the function C is very different outside the fixed points



2+1 dimensions

H.C., M. Huerta, 2012



Many circles to obtain circles as a limit. Circles at null cone to avoid divergent logarithmic terms due to corners

$$c_0(r) = r S'(r) - S(r) \Rightarrow c_0(r)' \leq 0$$

Previously conjectured by H. Liu, M. Mezei, 2012

At fixed points $S(R) = c_1 R - c_0$

$c_0(r) = c_0$ Is the constant term in the entropy of a circle = free energy F of the conformal theory on a 3-sphere.

There is a c-theorem in 2+1 dimension for relativistic theories (also called F-theorem). No proof has been found yet for d=3 that does not use entanglement entropy.

Is C a measure of «number of field degrees of freedom»?

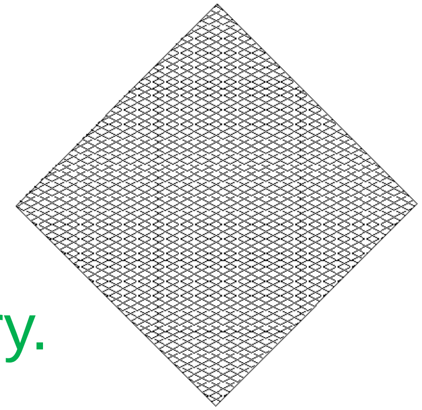
C is not an anomaly in d=3. It is a small universal term in a divergent entanglement entropy. It is very different from a «number of field degrees of freedom»: Topological theories with no local degree of freedom can have a large C (topological entanglement entropy)! C does measure some form of entanglement that is lost under renormalization, but what kind of entanglement?

Is there some loss of information interpretation?

Even if the theorem applies to an entropic quantity, there is no known interpretation in terms of some loss of information. Understanding this could tell us whether there is a version of the theorem that extends beyond relativistic theories.

More inequalities seem to be needed for an entropic c-theorem in higher dimensions.

Entanglement entropy is a funcion of the global state and the algebras of operators associated to causal regions. It fits naturally within the algebraic approach to QFT as a kind of «statistical correlator» which exists for any theory.



Renyi twisting operators are surface operators also attached to the algebras.

Does entanglement entropy of vacuum uniquely determine the theory?

If yes, likely there are infinitely many other inequalities beyond strong subadditivity to reconstruct the Hilbert space.