STRINGS 2014

Scattering Equations
(Massless Particles)

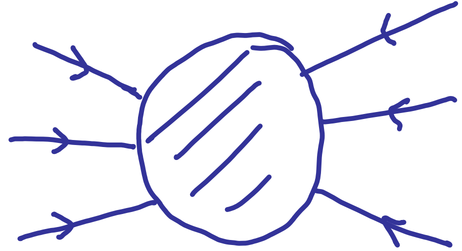
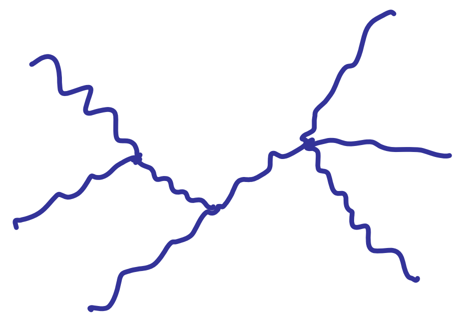
Based on:
w. Song He & Ellis Yuan
1309.0885, 1307.2199
w. Humberto Gomez
to appear.

Freddy Cachazo
Perimeter Institute

In this talk I concentrate on a new formula for a very OLD object.

In this talk I concentrate on a new formula for a very OLD object.

The S-matrix of massless particles (scalars, gluons, gravitons)

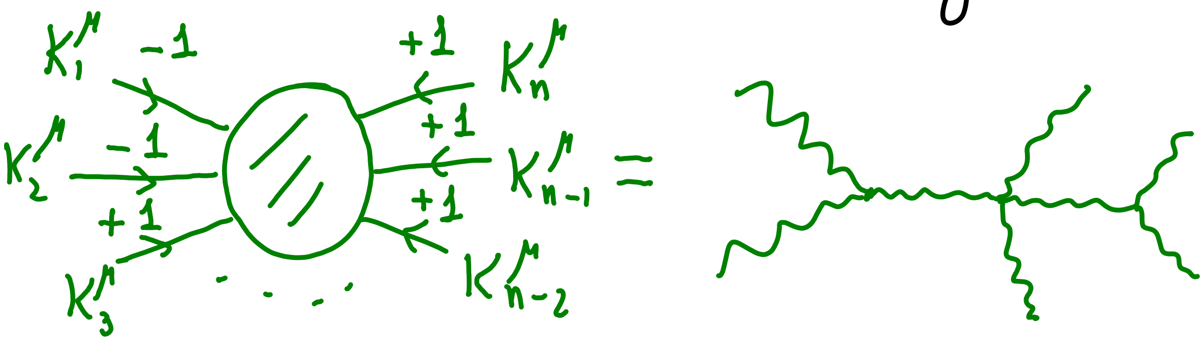
$\mathcal{M}_n =$  $=$  $+ \ldots$

Why look for a new formula?

First indications:

* 1980's Parke-Taylor

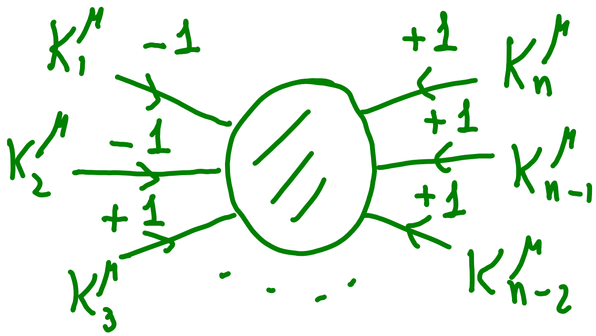
$\{k_a^\mu, \epsilon_a^\mu, A_a\}$ color SU(N) index



+ {Hundreds more even for small n}

$$K_a^\mu \sim \left(1+z_a\bar{z}_a,\ z_a+\bar{z}_a,\ -i(z_a-\bar{z}_a),\ 1-z_a\bar{z}_a\right)$$

$\epsilon_a^\mu \sim$ Similar formula which depends on $h=\pm 1$ and a reference (ω or $\bar{\omega}$).



$$= (z_1-z_2)^4 \left(\frac{\mathrm{Tr}\left(T^{A_1} T^{A_2} \cdots T^{A_n}\right)}{(z_1-z_2)(z_2-z_3)\cdots(z_n-z_1)} + \text{perm} \right)$$

$$K_a^\mu \sim \left(1+z_a\bar{z}_a,\ z_a+\bar{z}_a,\ -i(z_a-\bar{z}_a),\ 1-z_a\bar{z}_a\right)$$

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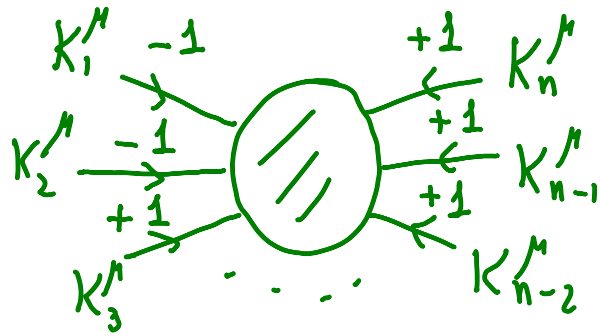


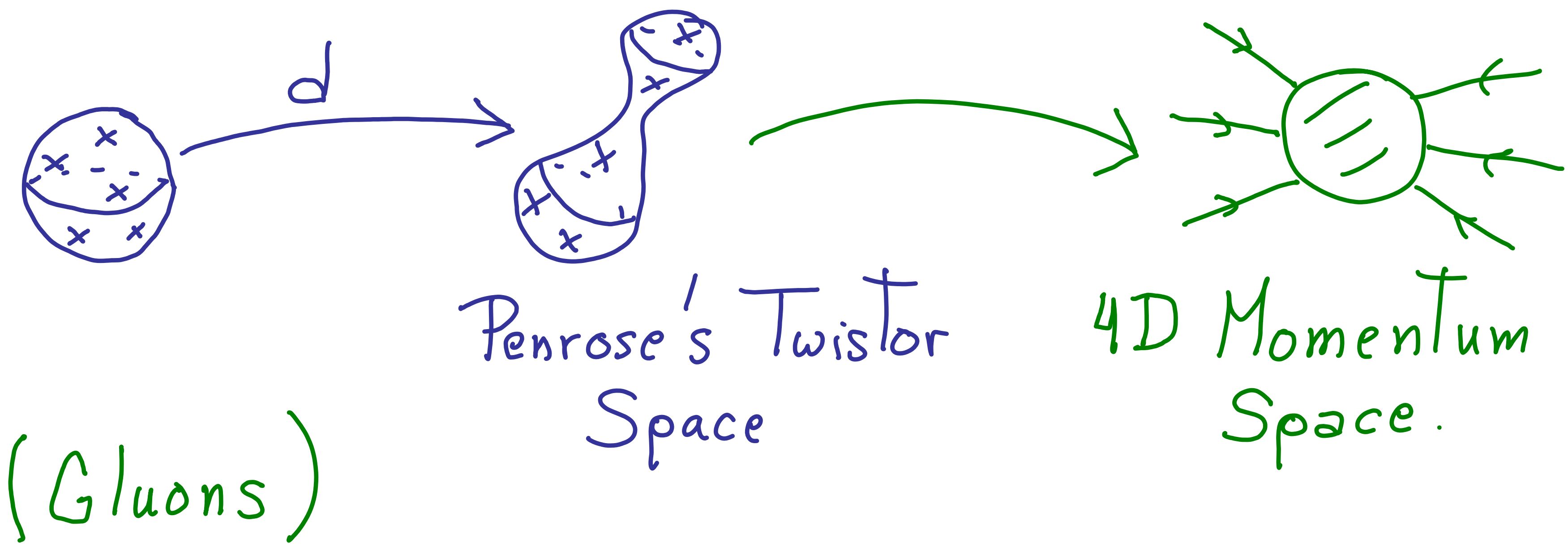
$$= (z_1-z_2)^4 \boxed{\left(\frac{\mathrm{Tr}\left(T^{A_1}T^{A_2}\cdots T^{A_n}\right)}{(z_1-z_2)(z_2-z_3)\cdots(z_n-z_1)} + \text{perm}\right)}$$

Does this miraculous simplification occur for other helicity configurations?

* Nair 90's

  Parke-Taylor = Correlation function on $\mathbb{CP}^1$.

* Witten 2003



Natural Questions:

* Is there a similar formula for gravitons (and for scalars)?
* Is there a formulation in any dimension or is it special to 4D where the magic variables $(z, \bar{z})$ exist?
* Can it work at tree and loops?

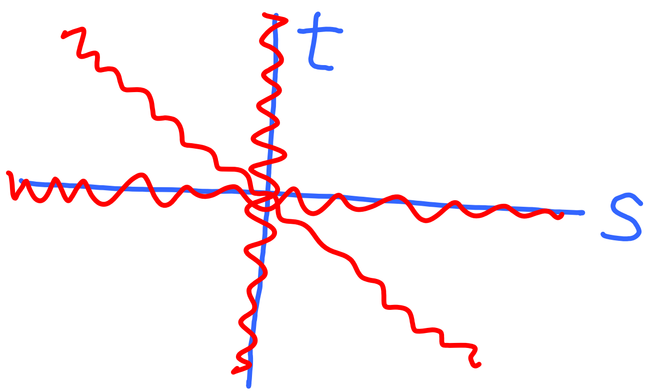
How can we search for such an object?

* Space of Kinematic invariants

$s_{ab} \equiv (k_a + k_b)^2 \qquad [\, k_a \cdot k_a = 0 \quad \forall\, a \,]$

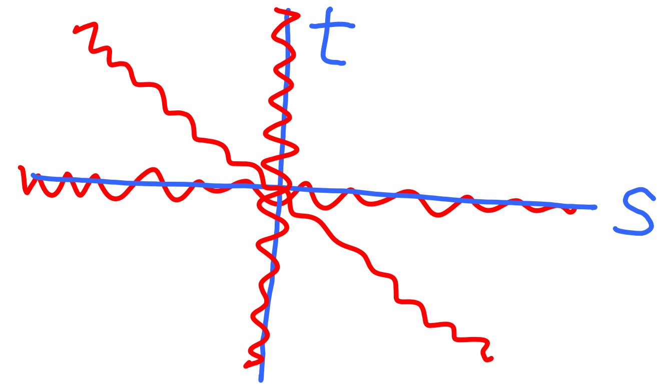
Consider $n = 4$ (Four particle scattering)

$s \equiv s_{12} \quad t = s_{14} \quad u = s_{13} \qquad s + t + u = 0$

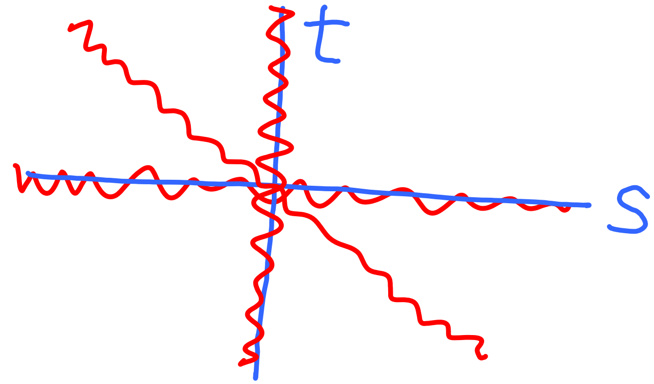


Singularities of $\mathcal{M}_4$ @ $s = 0$ or $t = 0$ or $u = 0$.

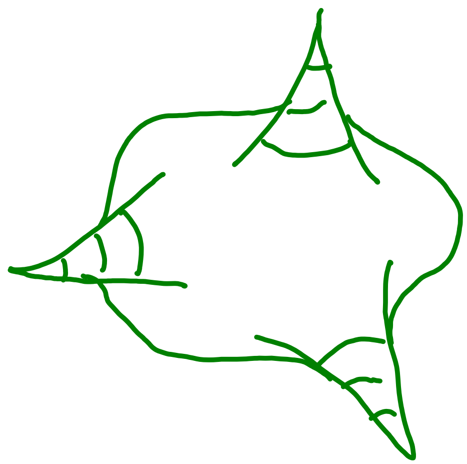
This space clearly does not know much PHYSICS!



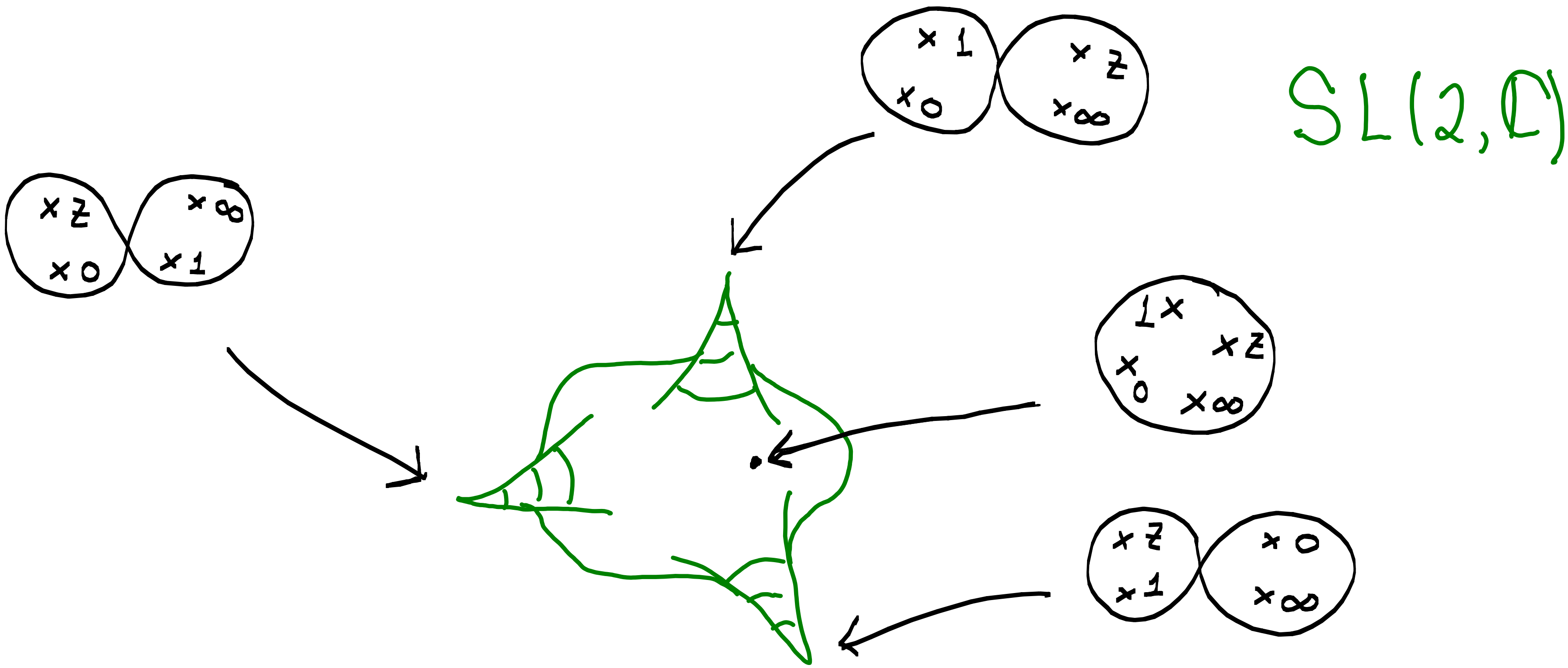
This space clearly does not know much PHYSICS!



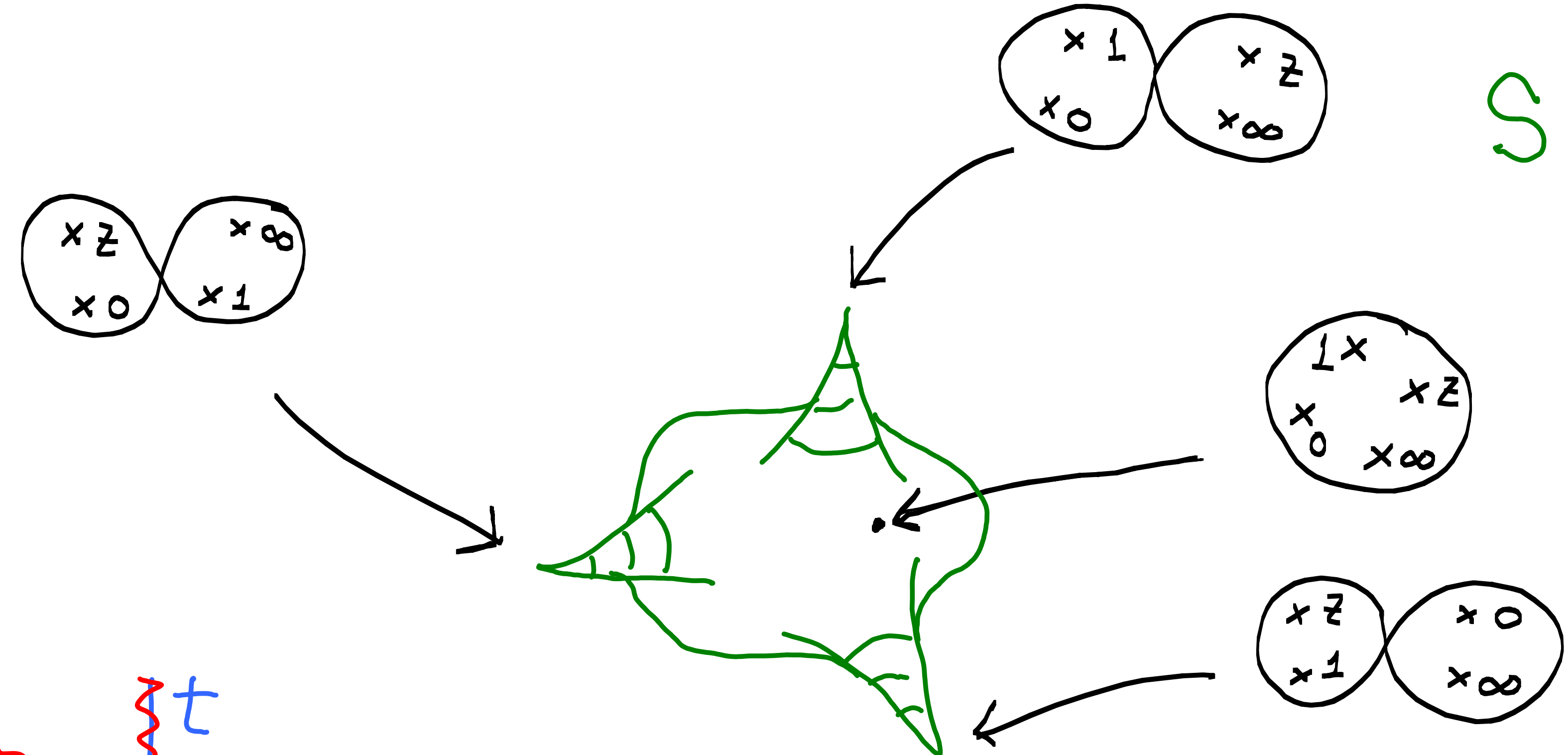
Riemann Spheres

$\mathcal{M}_{0,4} =$ 

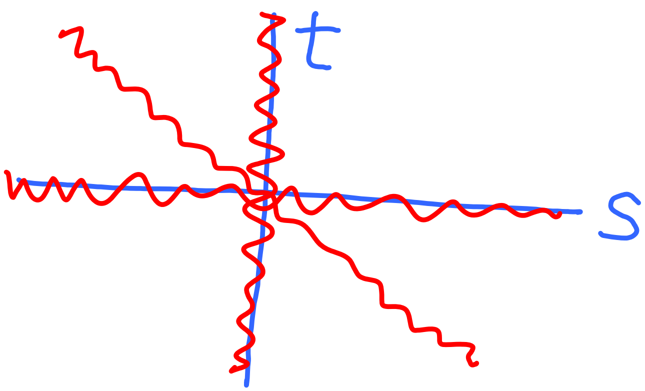
It has 3 singular points!

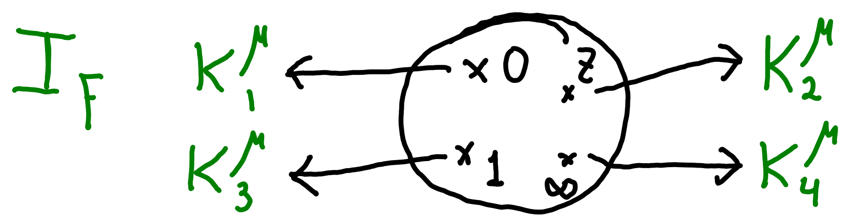


$SL(2,\mathbb{C})$

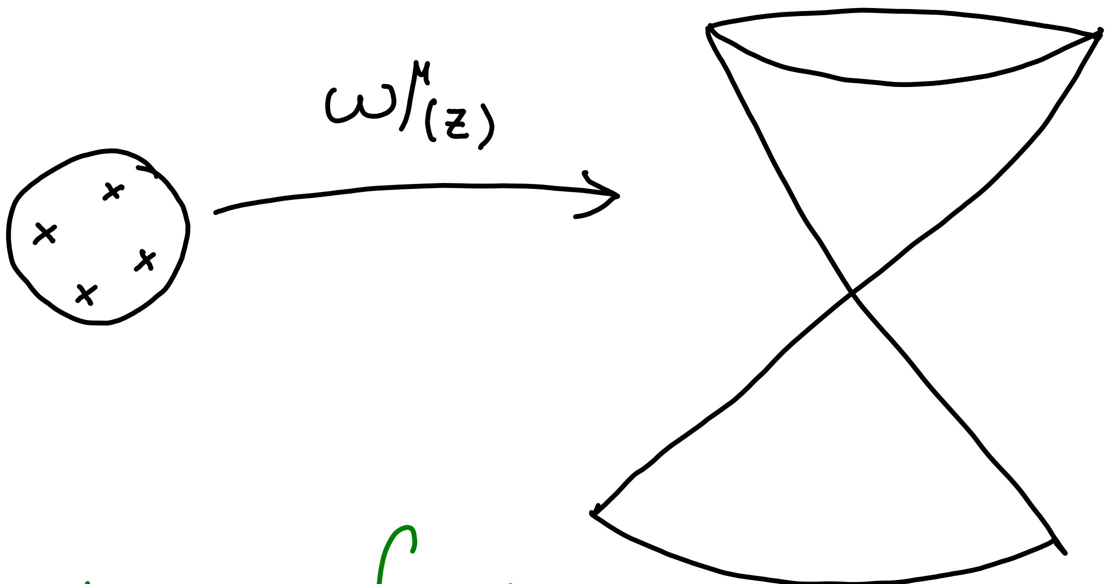


$SL(2,\mathbb{C})$





Riemann Knows a lot of Physics!



Null cone in D dimensional momentum space

Want $$\frac{1}{2\pi i}\oint_{|z-\sigma_a|=\epsilon} \omega^\mu(z)\,dz = K_a^\mu \qquad \& \qquad \omega^\mu(z)\,\omega_\mu(z)\,dz^2 = 0$$

Scattering Equations

① $$\omega^\mu(z) = \sum_{a=1}^{n} \frac{k_a^\mu}{z - \sigma_a}$$ ✓

Obs:

$\omega^\mu(z)$'s residue at $z = \infty$ is $\sum_{a=1}^{n} k_a^\mu = 0$!

② $$\omega^\mu(z)\,\omega_\mu(z) = \sum_{a,b} \frac{k_a \cdot k_b}{(z-\sigma_a)(z-\sigma_b)} = 0$$

$\Leftrightarrow$ All its residues are 0. $$\sum_{\substack{b=1 \\ b \neq a}}^{n} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b} = 0$$

Scattering Equations

* Link n points on $\mathbb{CP}^1$ to the space of kinematic invariants.

* Map singularities of $\mathcal{M}_{0,n} \Longleftrightarrow$ Physical Factorization

$$\boxed{\sum_{\substack{b=1 \\ b \neq a}}^{n} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b} = 0}$$

History

- 1972 Fairlie and Roberts (Unpublished)
  [Dual Models]
- 1988 Gross and Mende
  [High Energy Scattering of Strings]
- 2004 Witten [Parity in the Twistor String]
- 2012 Makeenko & Olesen [Wilson Loops]
- 2012 F.C. [BCJ from the Witten-RSV formula].

Proposal

[w. Song He & Ellis Yuan]

$$\mathcal{M}_n^{\text{Gravitons}} = \int \prod_{a=1}^{n} d\sigma_a \prod_{a=1}^{n} \delta\left(\sum_{\substack{b=1 \\ b\neq a}}^{n} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b} \right) E\left(\epsilon^{\mu\nu}, k^{\mu}, \sigma\right)$$

Formula for the complete Tree level S-matrix of Gravitons in D dimensions.

Proposal

$$\mathcal{M}_n^{\text{Gravitons}} = \int \prod_{a=1}^{n} d\sigma_a \prod_{a=1}^{n} \delta\left(\sum_{\substack{b=1 \\ b\neq a}}^{n} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b}\right) E(\epsilon^{\mu\nu}, k^{\mu}, \sigma)$$

Hints:
- * Permutation invariance.
- * Gauge invariance.
- * Weinberg's soft theorems.
- * Multilinear in $\epsilon_a^{\mu\nu}$.

Ψ-Matrix

$$
\Psi = \left[\begin{array}{cccc|cccc}
0 & \frac{K_1\cdot K_2}{\sigma_1-\sigma_2} & \cdots & \frac{K_1\cdot K_n}{\sigma_1-\sigma_n} & C_{11} & \frac{K_1\cdot \epsilon_2}{\sigma_1-\sigma_2} & \cdots & \frac{K_1\cdot \epsilon_n}{\sigma_1-\sigma_n} \\
\frac{K_2\cdot K_1}{\sigma_2-\sigma_1} & 0 & & & \frac{K_2\cdot \epsilon_1}{\sigma_2-\sigma_1} & \ddots & & \vdots \\
\vdots & & \ddots & & \vdots & & \ddots & \\
\frac{K_n\cdot K_1}{\sigma_n-\sigma_1} & & & 0 & & & & C_{nn} \\
\hline
-C_{11} & \frac{\epsilon_1\cdot K_2}{\sigma_1-\sigma_2} & \cdots & \frac{\epsilon_1\cdot K_n}{\sigma_1-\sigma_n} & 0 & \frac{\epsilon_1\cdot \epsilon_2}{\sigma_1-\sigma_2} & \cdots & \frac{\epsilon_1\cdot \epsilon_n}{\sigma_1-\sigma_n} \\
\frac{\epsilon_2\cdot K_1}{\sigma_2-\sigma_1} & \ddots & & \vdots & \frac{\epsilon_2\cdot \epsilon_1}{\sigma_1-\sigma_2} & \ddots & & \vdots \\
\vdots & & \ddots & & \vdots & & \ddots & \\
\vdots & & & -C_{nn} & \frac{\epsilon_n\cdot \epsilon_1}{\sigma_n-\sigma_1} & \cdots & & 0
\end{array}\right]
$$

$$
C_{aa} = \sum_{b\neq a} \frac{\epsilon_a\cdot K_b}{\sigma_a-\sigma_b}
$$

Ψ-Matrix

$$E = \det \Psi = \det \left[\begin{array}{cccc|cccc}
0 & \frac{K_1 \cdot K_2}{\sigma_1 - \sigma_2} & \cdots & \frac{K_1 \cdot K_n}{\sigma_1 - \sigma_n} & C_{11} & \frac{K_1 \cdot \epsilon_2}{\sigma_1 - \sigma_2} & \cdots & \frac{K_1 \cdot \epsilon_n}{\sigma_1 - \sigma_n} \\
\frac{K_2 \cdot K_1}{\sigma_2 - \sigma_1} & 0 & & & \frac{K_2 \cdot \epsilon_1}{\sigma_2 - \sigma_1} & \ddots & & \vdots \\
\vdots & & \ddots & & \vdots & & \ddots & \vdots \\
\frac{K_n \cdot K_1}{\sigma_n - \sigma_1} & & & 0 & & & & C_{nn} \\
\hline
-C_{11} & \frac{\epsilon_1 \cdot K_2}{\sigma_1 - \sigma_2} & \cdots & \frac{\epsilon_1 \cdot K_n}{\sigma_1 - \sigma_n} & 0 & \frac{\epsilon_1 \cdot \epsilon_2}{\sigma_1 - \sigma_2} & \cdots & \frac{\epsilon_1 \cdot \epsilon_n}{\sigma_1 - \sigma_n} \\
\frac{\epsilon_2 \cdot K_1}{\sigma_2 - \sigma_1} & \ddots & & \vdots & \frac{\epsilon_2 \cdot \epsilon_1}{\sigma_2 - \sigma_1} & & & \vdots \\
\vdots & & \ddots & \vdots & \vdots & & & \vdots \\
\vdots & & & -C_{nn} & \frac{\epsilon_n \cdot \epsilon_1}{\sigma_n - \sigma_1} & \cdots & \cdots & 0
\end{array}\right]$$

$$C_{aa} = \sum_{b \neq a} \frac{\epsilon_a \cdot K_b}{\sigma_a - \sigma_b}$$

Checks: Gauge Invariance $\quad \epsilon_1^\mu \to k_1^\mu$

$$E = \det \Psi' = \det \left[\begin{array}{cccc|cccc}
0 & \frac{k_1\cdot k_2}{\sigma_1-\sigma_2} & \cdots & \frac{k_1\cdot k_n}{\sigma_1-\sigma_n} & C_{11} \to 0 & \frac{k_1\cdot \epsilon_2}{\sigma_1-\sigma_2} & \cdots & \frac{k_1\cdot \epsilon_n}{\sigma_1-\sigma_n} \\
\frac{k_2\cdot k_1}{\sigma_2-\sigma_1} & 0 & & & \frac{k_2\cdot \epsilon_1}{\sigma_2-\sigma_1},\ \epsilon_1 \to k_1 & & & \vdots \\
\vdots & & \ddots & & \to k_1 & & & \vdots \\
\frac{k_n\cdot k_1}{\sigma_n-\sigma_1} & & & 0 & \to k_1 & & & C_{nn} \\
\hline
-C_{11} \to 0 & \frac{\epsilon_1\cdot k_2}{\sigma_1-\sigma_2} & \cdots & \frac{\epsilon_1\cdot k_n}{\sigma_1-\sigma_n} & 0 & \frac{\epsilon_1\cdot \epsilon_2}{\sigma_1-\sigma_2} & \cdots & \frac{\epsilon_1\cdot \epsilon_n}{\sigma_1-\sigma_n} \\
\frac{\epsilon_2\cdot k_1}{\sigma_2-\sigma_1} & & & & \frac{\epsilon_2\cdot \epsilon_1}{\sigma_1-\sigma_2},\ \epsilon_1 \to k_1 & & & \vdots \\
\vdots & & \ddots & & \vdots & & & \vdots \\
\vdots & & & -C_{nn} & \frac{\epsilon_n\cdot \epsilon_1}{\sigma_n-\sigma_1},\ \epsilon_1 \to k_1 & & \cdots & 0
\end{array} \right]$$

$$C_{11} = \sum_{b\neq 1} \frac{\epsilon_1\cdot k_b}{\sigma_1-\sigma_b} = 0, \qquad \epsilon_1 \to k_1$$

$$\Rightarrow E = 0$$

Proposal

$$\mathcal{M}_n^{\text{Gravitons}} = \int \prod_{a=1}^{n} d\sigma_a \prod_{a=1}^{n} \delta\left(\sum_{\substack{b=1 \\ b\neq a}}^{n} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b}\right) \det \Psi_{(\epsilon, k, \sigma)}$$

Proposal

$$\mathcal{M}_n^{\text{Gravitons}} = \int \prod_{a=1}^{n} d\sigma_a \prod_{a=1} \delta\left(\sum_{\substack{b=1 \\ b\neq a}}^{n} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b}\right) \det \Psi_{(\epsilon, k, \sigma)}$$

How about Gluons? Hint: $\Psi = -\Psi^T$

$$\Rightarrow \det \Psi = (\mathrm{Pf}\, \Psi)^2$$

$\mathrm{Pf} \longrightarrow$ Pfaffian

Proposal

$$\mathcal{M}_n^{\text{Gravitons}} = \int \prod_{a=1}^{n} d\sigma_a \prod_{a=1}^{n} \delta\left(\sum_{\substack{b=1 \\ b\neq a}}^{n} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b} \right) \left(\mathrm{Pf}\, \Psi_{(\epsilon, k, \sigma)} \right)^2$$

$$\mathcal{M}_n^{\text{Gluons}} = \int \prod_{a=1}^{n} d\sigma_a \prod_{a=1}^{n} \delta\left(\sum_{\substack{b=1 \\ b\neq a}}^{n} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b} \right) \mathrm{Pf}\, \Psi_{(\epsilon, k, \sigma)} \times \mathcal{C}$$

Something that contains color info.

Proposal

$$\mathcal{M}_n^{\text{Gravitons}} = \int \prod_{a=1}^{n} d\sigma_a \prod_{a=1}^{n} \delta\left(\sum_{\substack{b=1 \\ b\neq a}}^{n} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b}\right) \left(\mathrm{P}_F \Psi_{(\epsilon, k, \sigma)}\right)^2$$

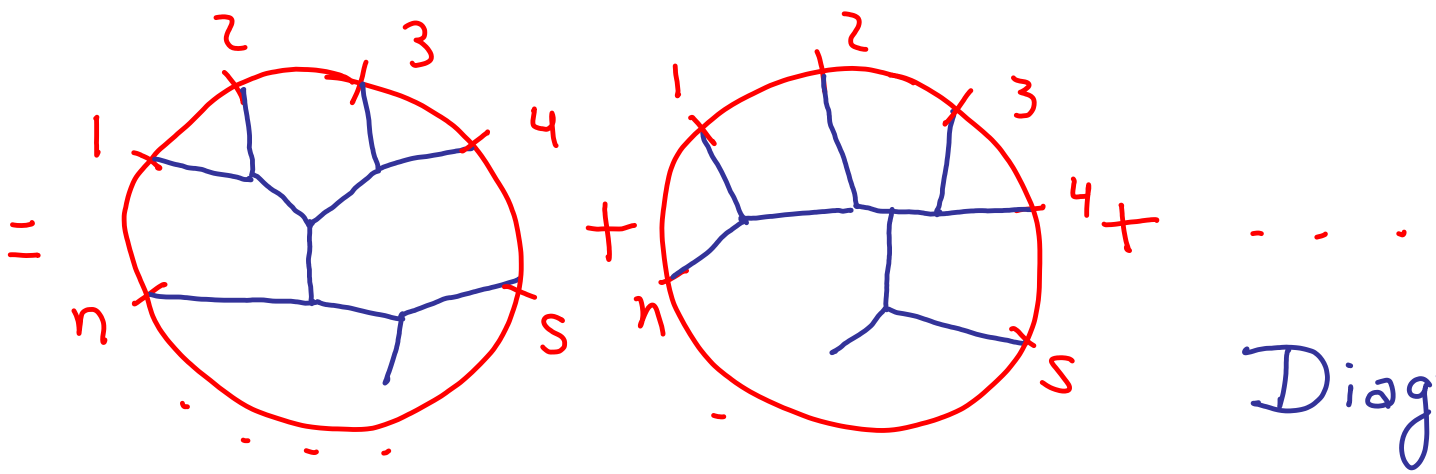
$$\mathcal{M}_n^{\text{Gluons}} = \int \prod_{a=1}^{n} d\sigma_a \prod_{a=1}^{n} \delta\left(\sum_{\substack{b=1 \\ b\neq a}}^{n} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b}\right) \mathrm{P}_F \Psi_{(\epsilon, k, \sigma)} \times \mathcal{C}$$

$$\mathcal{C} = \left(\frac{\mathrm{Tr}\left(T^{A_1} T^{A_2} \cdots T^{A_n}\right)}{(\sigma_1 - \sigma_2)(\sigma_2 - \sigma_3) \cdots (\sigma_n - \sigma_1)} + \text{perm} \right)$$

$$\mathcal{M}_n^{\text{Scalars}} = \int \prod_{a=1}^{n} d\sigma_a \prod_{a=1}^{n} \delta\left(\sum_{\substack{b=1 \\ b\neq a}}^{n} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b}\right) \mathcal{C}^2$$

$$\mathcal{E}_{\text{Grav.}} = (\mathrm{Pf}\,\Psi)^2 \longrightarrow \mathcal{E}_{\text{YM}} = \mathrm{Pf}\,\Psi \times \mathcal{C} \longrightarrow \mathcal{E}_{\text{Scalar}} = \mathcal{C}^2$$

$$\mathcal{M}_n^{\text{Scalars}} = \int \prod_{a=1}^{n} d\sigma_a \prod_{a=1}^{n} \delta\left(\sum_{\substack{b=1 \\ b\neq a}}^{n} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b}\right) \left(\frac{1}{(\sigma_1-\sigma_2)(\sigma_2-\sigma_3)\cdots(\sigma_n-\sigma_1)}\right)^2$$



Diagrams in ϕ^3.

[If color factors are removed, then one gets $\lambda\phi^3$]
Dolan & Goddard

Further Developments (Partial list)

- Monteiro & O'Connell [Kinematic Algebras]
- Mason & Skinner [Ambitwistor Strings]
- Dolan & Goddard [Proof in YM, Polynomial Form]
- Berkovits, Gomez, Yuan [Infinite tension Limits]
- Casali, Mason, Skinner [Loop amplitudes in Ambitwistor Strings]
- Geyer, Lipstein, Mason [Ambitwistors in 4D]
- . . .

Hyperelliptic Scattering Equations

[w. Humberto Gomez]

Hyperelliptic Scattering Equations

At genus 0 :

1. Construct a Lorentz vector of meromorphic differentials $\Omega^\mu = \omega^\mu(z)\,dz$ with simple poles at $z=\sigma_a$ and residue k_a^μ.
2. Impose that the Lorentz scalar quadratic differential $\Omega^\mu \Omega_\mu$ vanishes.

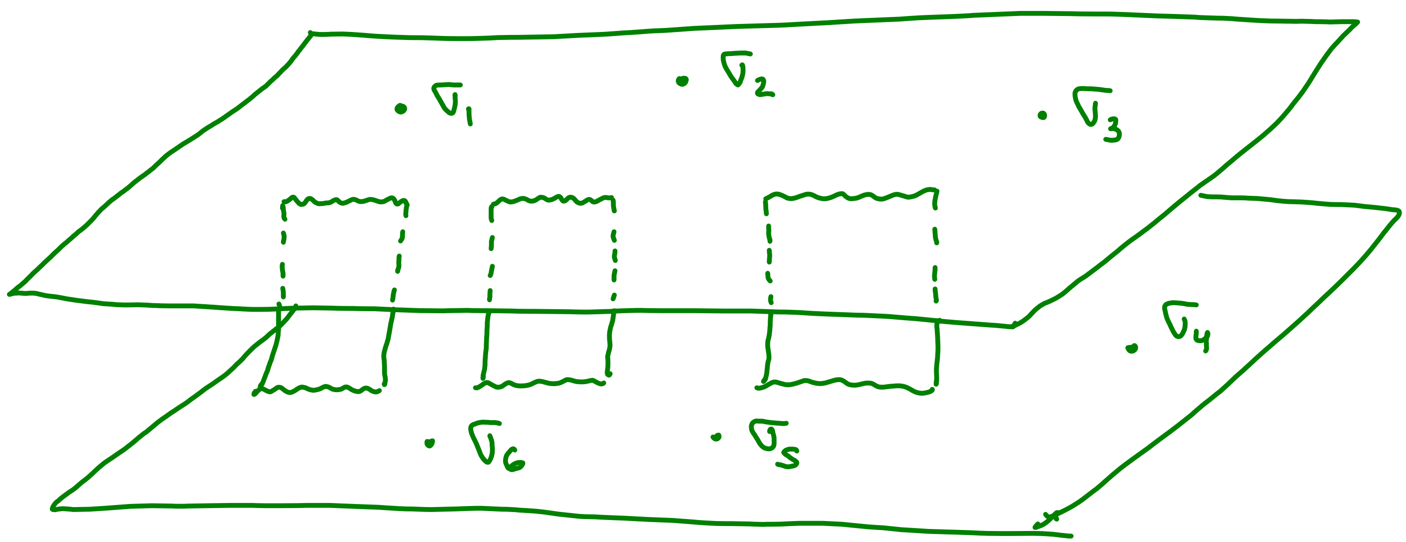
Hyperelliptic Scattering Equations

At genus g :

1. Construct a Lorentz vector of meromorphic differentials $\Omega^\mu = \omega^\mu(z)\, dz$ with simple poles at $z = \sigma_a$ and residue k_a^μ.
2. Impose that the Lorentz scalar quadratic differential $\Omega^\mu \Omega_\mu$ vanishes.

Hyperelliptic Construction

1. $$y^2 = z(z^2-1)\prod_{i=1}^{2g-1}(z-\lambda_i)$$



Hyperelliptic Construction

1. $y^2 = z(z^2-1)\prod_{i=1}^{2g-1}(z-\lambda_i)$

2. $\tau_{P,Q}$ = Meromorphic differential with simple poles at P and Q with residues +1 and −1. (Unique if $\oint_{A_m} \tau = 0$).

3. Consider the combination

$$\sum_{a=1}^{n-1} k_a^\mu \, \tau_{\sigma_a,\sigma_n} \longrightarrow \text{Simple pole at } \sigma_a \text{ with residue } k_a^\mu, \quad a \in \{1, 2, \ldots, n-1\}$$

$\downarrow$

Simple pole at σ_n with residue $-k_1^\mu - k_2^\mu - \ldots - k_{n-1}^\mu = k_n^\mu$

3. Consider the combination

$$\Omega^{\mu}_{\lrcorner} - \sum_{a=1}^{n-1} K_a^{\mu}\, \tau_{\sigma_a, \sigma_n} = \text{Holomorphic}$$

$$= \sum_{i=1}^{g} \ell_i^{\mu}\, z^{i-1} \frac{dz}{y(z)}$$

3. Consider the combination

$$\Omega^{\mu} = \sum_{a=1}^{n-1} K_a^{\mu}\, \tau_{\sigma_a,\sigma_n} + \sum_{i=1}^{g} \ell_i^{\mu}\, z^{i-1} \frac{dz}{y(z)}$$

4. Impose $\Omega^{\mu}\Omega_{\mu} = 0$.

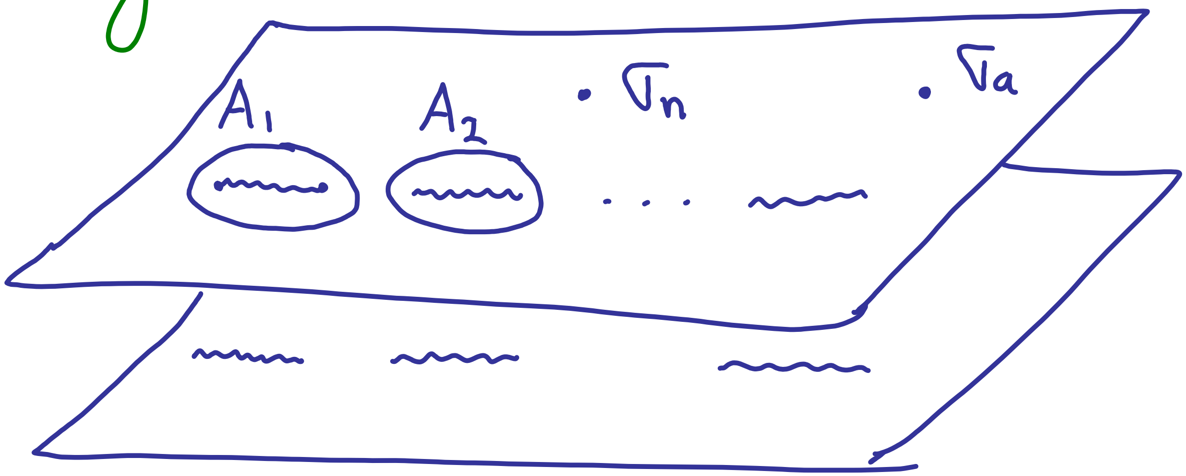
3. Consider the combination

$$\Omega^{\mu} = \sum_{a=1}^{n-1} K_a^{\mu} \tau_{\sigma_a, \sigma_n} + \sum_{i=1}^{g} l_i^{\mu} z^{i-1} \frac{dz}{y(z)}$$

4. Impose $\Omega^{\mu} \Omega_{\mu} = 0$.

How can we construct them?

Luckily, this problem is familiar in Physics (SW, MM, etc.)



$$\tau_{\sigma_a,\sigma_n} = d\log\left(\frac{P_a(z) - \sqrt{P_a^2(z) - b_a(z-\sigma_a)\,Q(z)}}{P_n(z) - \sqrt{P_n^2(z) - b_n(z-\sigma_n)\,Q(z)}}\right)$$

Need $P_a^2(z) - b_a(z-\sigma_a)\,Q(z) = y^2(z)\, H_a^2(z)$

$y^2 = z(z^2-1)\prod_{i=1}^{g}(z-\lambda_i)$, $Q(z)$ same polynomial $\forall\, a$.

These equations together with

$\Omega^\mu \Omega_\mu = 0$ are the

Hyperelliptic Scattering Equations.

Example I: Genus 0

$y^2 = z^2 - 1$ $\qquad P_a^2(z) - b_a(z - \sigma_a)\, Q(z) = (z^2 - 1)\, H_a^2(z)$

Minimal Solution: $P_a(z) = z - p_a$, $Q(z) = 1$, $H_a(z) = 1$.

$$\Rightarrow \quad (z - p_a)^2 - b_a(z - \sigma_a) = (z^2 - 1)$$

$$\Rightarrow \quad \boxed{p_a = \sigma_a - y(\sigma_a)}$$

Form : $$\Omega^{\mu} = \sum_{a=1}^{n} K_a^{\mu} \, d\log\Big(z - \sigma_a - \big(y(z) - y(\sigma_a)\big)\Big)$$

S.E : $$\sum_{c=1,\, c\neq a}^{n-1} K_a \cdot K_c \, \frac{d}{dz} \log\left(\frac{z - \sigma_c - \big(y(z) - y(\sigma_c)\big)}{z - \sigma_n - \big(y(z) - y(\sigma_n)\big)}\right)\Bigg|_{z=\sigma_a} = 0$$

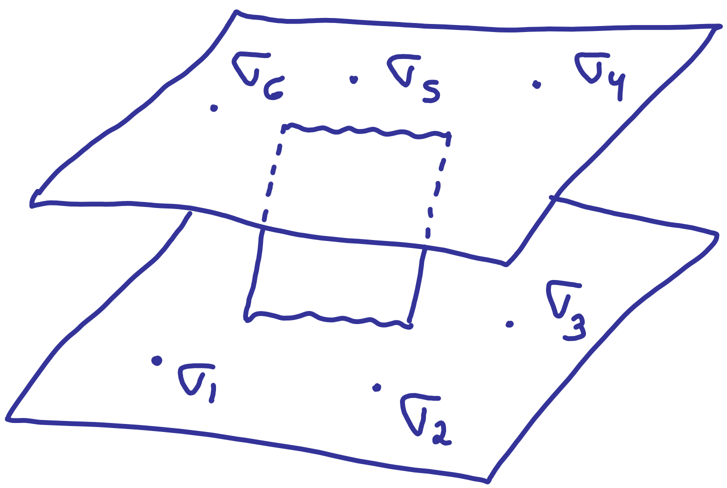
Form : $\Omega^\mu = \sum_{a=1}^{n} K_a^\mu \, d\log\left(z-\sigma_a-(y(z)-y(\sigma_a))\right)$

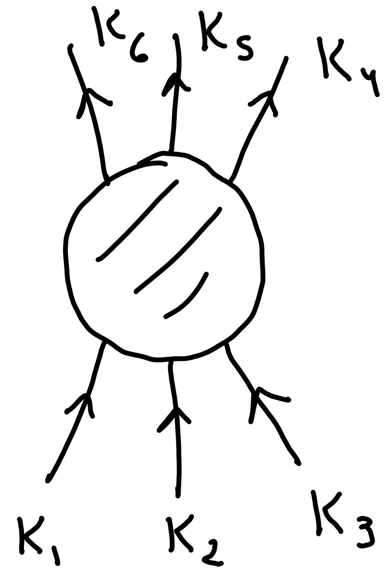
S.E : $\displaystyle \sum_{c=1,c\neq a}^{n-1} K_a\cdot K_c \frac{d}{dz}\log\left(\frac{z-\sigma_c-(y(z)-y(\sigma_c))}{z-\sigma_n-(y(z)-y(\sigma_n))}\right)\Bigg|_{z=\sigma_a}=0$

Back to Original Presentation : $z=\frac{1}{2i}\left(\tilde{z}-\frac{1}{\tilde{z}}\right)$ $\quad \sigma_a=\frac{1}{2i}\left(\tilde{\sigma}_a-\frac{1}{\tilde{\sigma}_a}\right)$

$$\Rightarrow \sum_{c=1}^{n} \frac{K_a\cdot K_c}{\tilde{\sigma}_a-\tilde{\sigma}_c}=0 .$$

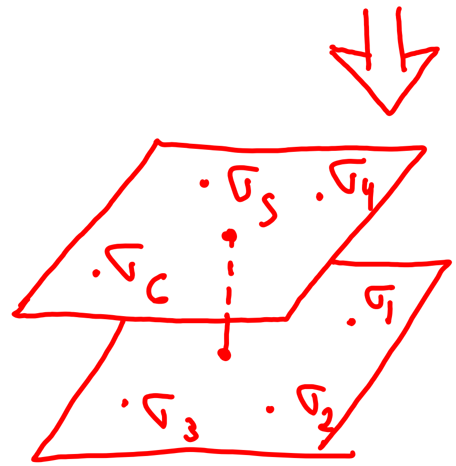
Crossing & Factorization

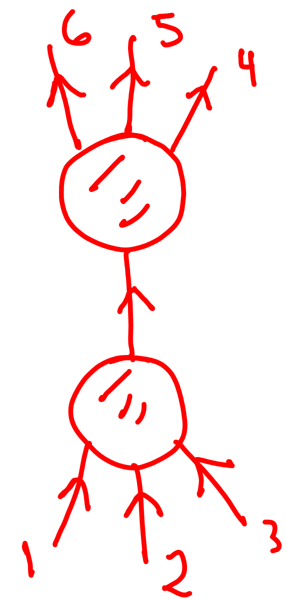




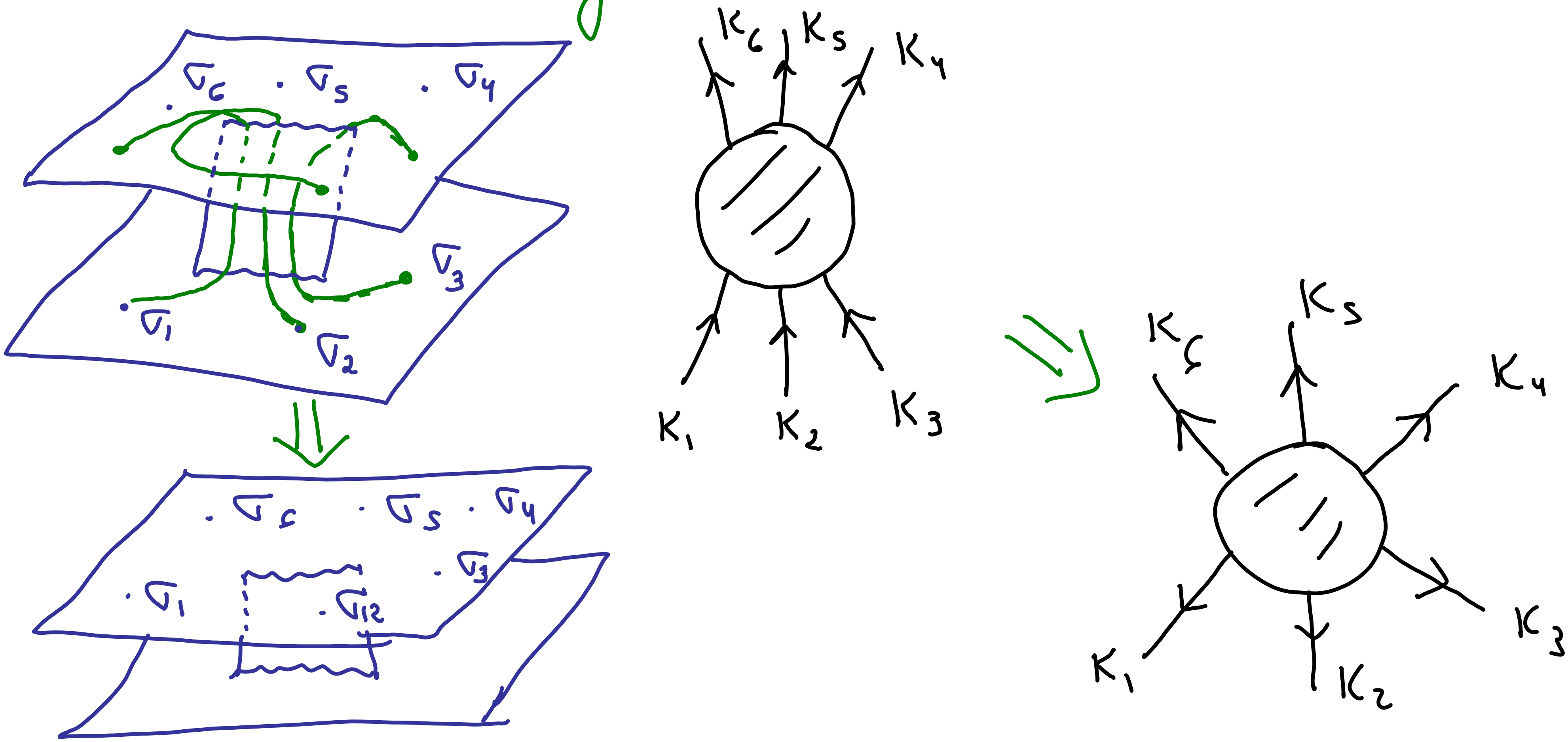
$$y^2 = z^2 - \Lambda^2$$

$$\Lambda^2 \sim (K_1 + K_2 + K_3)^2$$





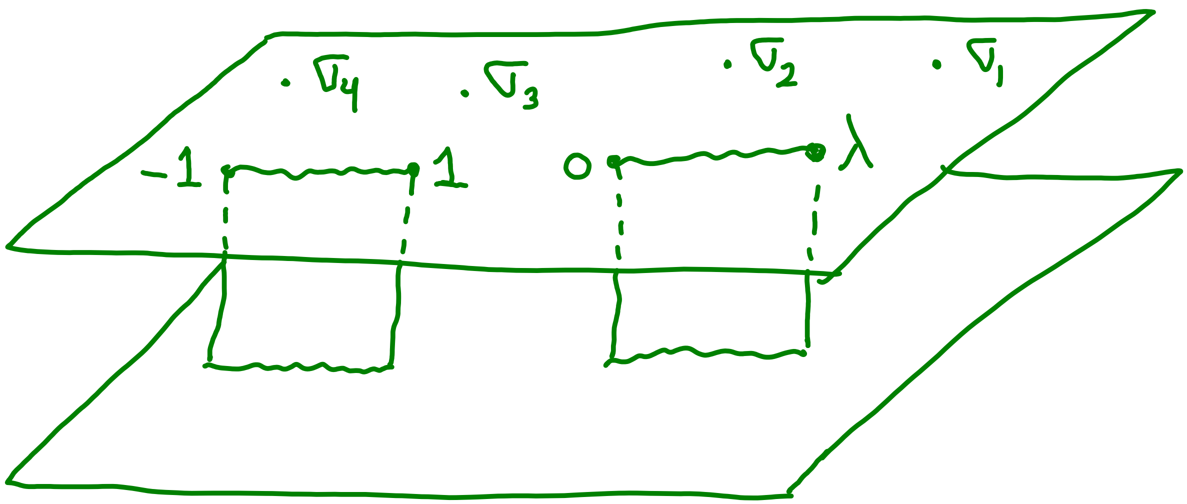
Crossing & Factorization



Example II : $g = 1$

Curve: $y^2 = z(z^2-1)(z-\lambda)$

Form: $\Omega^\mu = \sum_{a=1}^{n} K_a^\mu \, d\log\left(P_{a(z)} - y_{(z)} H_{a(z)}\right) + \ell^\mu \frac{dz}{y_{(z)}}$



Example II : $g = 1$

Curve : $y^2 = z(z^2-1)(z-\lambda)$

Embedding : $P_a^2(z) - b_a(z-\sigma_a)Q(z) = y^2(z)\, H_a^2(z)$

S.E : $\begin{cases} \displaystyle\sum_{a=1}^{n} K_a \cdot K_c \frac{d}{dz}\log\left(P_a(z) - y(z) H_a(z)\right)\Big|_{z=\sigma_a} + \frac{\ell \cdot K_c}{y(\sigma_c)} = 0 \\ \displaystyle \ell^2 = \lambda\left(\sum_{a=1}^{n} K_a \cdot \ell \frac{H_a(0)}{P_a(0)} - \frac{\lambda}{2}\left(\sum_{a=1}^{n} K_a^{\mu} \frac{H_a(0)}{P_a(0)}\right)^2\right) \end{cases}$

Observations

1. These are all polynomial equations !

2. $\ell^2 \sim \lambda$ $\qquad y^2 = z(z^2-1)(z-\lambda)$ $\Rightarrow \ell^\mu \sim$ Loop momentum ?

3. Any integral of the form

$$\int d\lambda \int \prod_a \frac{d\sigma_a}{y(\sigma_a)} \, \delta(\ell^2 - \lambda(\ldots)) \prod \delta\left(\sum k_a \cdot k_c \ldots\right) \tilde{I}(\ldots)$$

is a rational function of $k_a \cdot k_b, \ell^2$ & $\ell \cdot k_a$!

[Loop Integrands]

Open Questions

Q1: What's the number of solutions?

Q2: Is there a systematic way of constructing integrands?

Q3: For $g = 0, 1, 2$ this is complete. How about non-hyperelliptic RS $g \geq 3$.

Q4: Can general perturbative QFT S-matrices be formulated in this "pre-String" form?

Q5: How can the pre-String form be deformed into a full String theory?

Challenge: Solve the HSE for $n=4$.

$$P_a^2(z) - b_a(z-\sigma_a)\,Q(z) = z(z^2-1)(z-\lambda)\,H_a^2(z)$$

$$P_a(z) = z^3 + p_{a,1} z^2 + p_{a,2} z + p_{a,3} \qquad Q(z) = z^4 + q_1 z^3 + \dots q_4 \qquad H_a(z) = z - h_a$$

$a = 1, 2, 3, 4$

Assume that $\{\sigma_a\}$ & λ are given data.