A Proof of the Covariant Entropy Bound

Joint work with H. Casini, Z. Fisher, and J. Maldacena,
arXiv:1404.5635 and 1406.4545

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The World as a Hologram

- The Covariant Entropy Bound is a relation between information and geometry. RB 1999
- Motivated by holographic principle
  Bekenstein 1972; Hawking 1974
  ’t Hooft 1993; Susskind 1995; Susskind and Fischler 1998
- Conjectured to hold in arbitrary spacetimes, including cosmology.
- The entropy on a light-sheet is bounded by the difference between its initial and final area in Planck units.
- If correct, origin must lie in quantum gravity.

A Proof of the Covariant Entropy Bound

- In this talk I will present a proof, in the regime where gravity is weak ($G\hbar \to 0$).
- Though this regime is limited, the proof is interesting.
- No need to assume any relation between the entropy and energy of quantum states, beyond what quantum field theory already supplies.
- This suggests that quantum gravity determines not only classical gravity, but also nongravitational physics, as a unified theory should.

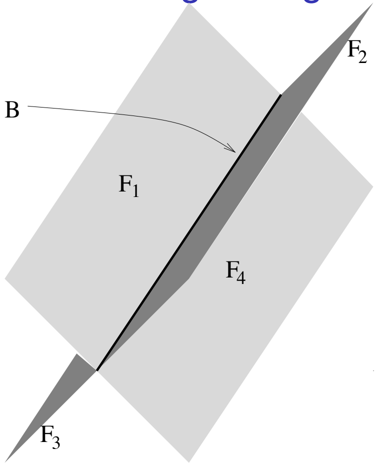
Covariant Entropy Bound

Entropy ΔS

Modular Energy ΔK

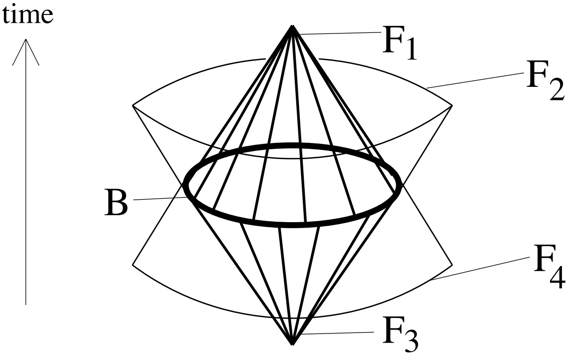
Area Loss ΔA

Surface-orthogonal light-rays



- Any 2D spatial surface *B* bounds four (2+1D) null hypersurfaces
- Each is generated by a congruence of null geodesics ("light-rays") $\perp B$

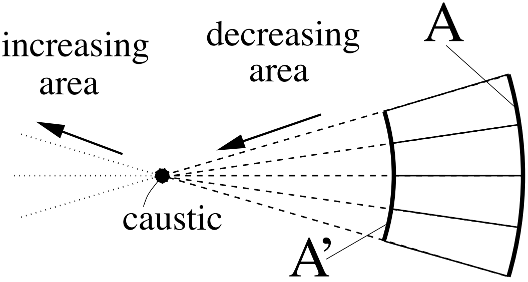
Light-sheets



Out of the 4 orthogonal directions, usually at least 2 will initially be nonexpanding.

The corresponding null hypersurfaces are called light-sheets.

The Nonexpansion Condition



$$\theta = \frac{d\mathcal{A}/d\lambda}{\mathcal{A}}$$

Demand

$\theta \leq 0 \quad \leftrightarrow \quad$ nonexpansion

everywhere on the light-sheet.

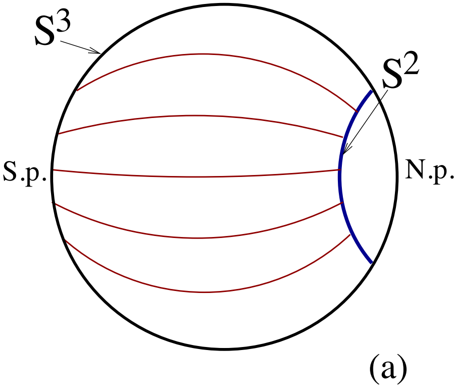
Covariant Entropy Bound

In an arbitrary spacetime, choose an arbitrary two-dimensional surface B of area A. Pick any light-sheet of B.

Then $S \leq A/4G\hbar$, where S is the entropy on the light-sheet.

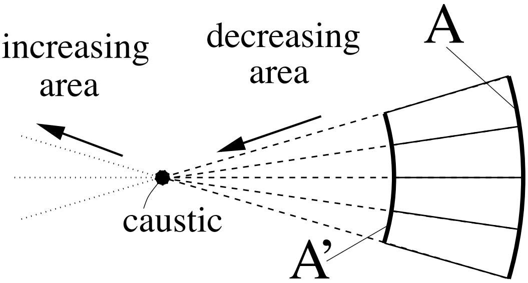
RB 1999

Example: Closed Universe



- S(volume of most of $\mathbf{S}^3$) $\gg A(\mathbf{S}^2)$
- The light-sheets are directed towards the "small" interior, avoiding an obvious contradiction.

Generalized Covariant Entropy Bound

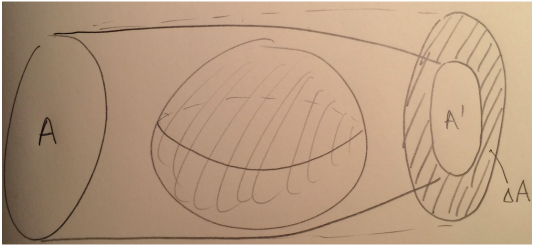


If the light-sheet is terminated at finite cross-sectional area A', then the covariant bound can be strengthened:

$$S \leq \frac{A - A'}{4G\hbar}$$

Flanagan, Marolf & Wald, 1999

Generalized Covariant Entropy Bound



$$S \leq \frac{\Delta A}{4G\hbar}$$

For a given matter system, the tightest bound is obtained by choosing a nearby surface with initially vanishing expansion.

Bending of light implies

$$A - A' \equiv \Delta A \propto G\hbar \; .$$

Hence, the bound remains nontrivial in the weak-gravity regime ($G\hbar \to 0$). RB 2003

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- In cosmology, and for well-isolated systems: usual, “intuitive” entropy. But more generally?
- Quantum systems are not sharply localized. Under what conditions can we consider a matter system to “fit” on L?
- The vacuum, restricted to L, contributes a divergent entropy. What is the justification for ignoring this piece?

In the $G\hbar \to 0$ limit, a sharp definition of S is possible.

Vacuum-subtracted Entropy

Consider an arbitrary state ρ_{global}. In the absence of gravity, $G = 0$, the geometry is independent of the state. We can restrict both ρ_{global} and the vacuum $|0\rangle$ to a subregion V:

$$\begin{aligned} \rho &\equiv \text{tr}_{-V}\, \rho_{\text{global}} \\ \rho_0 &\equiv \text{tr}_{-V}\, |0\rangle\langle 0| \end{aligned}$$

The von Neumann entropy of each reduced state diverges like A/ϵ^2, where A is the boundary area of V, and ϵ is a cutoff. However, the difference is finite as $\epsilon \to 0$:

$$\Delta S \equiv S(\rho) - S(\rho_0)\ .$$

Marolf, Minic & Ross 2003, Casini 2008

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Relative Entropy

Given any two states, the (asymmetric!) *relative entropy*

$$S(\rho|\rho_0) = -\text{tr}\,\rho\log\rho_0 - S(\rho)$$

satisfies positivity and monotonicity: under restriction of ρ and ρ_0 to a subalgebra (e.g., a subset of V), the relative entropy cannot increase.

Lindblad 1975

Modular Hamiltonian

Definition: Let ρ_0 be the vacuum state, restricted to some region V. Then the *modular Hamiltonian*, K, is defined up to a constant by

$$\rho_0 \equiv \frac{e^{-K}}{\mathrm{tr}\, e^{-K}} \ .$$

The *modular energy* is defined as

$$\Delta K \equiv \mathrm{tr}\, K\rho - \mathrm{tr}\, K\rho_0$$

A Central Result

Positivity of the relative entropy implies immediately that

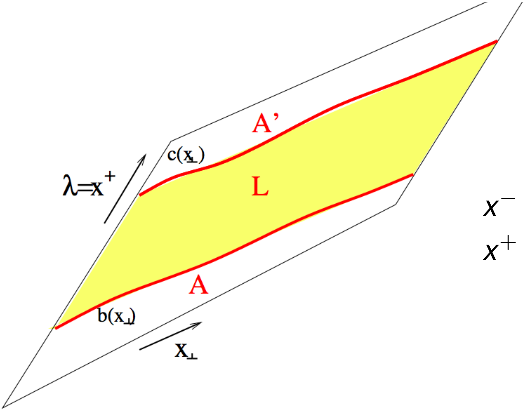
$$\Delta S \leq \Delta K \; .$$

To complete the proof, we must compute ΔK and show that

$$\Delta K \leq \frac{\Delta A}{4G\hbar} \; .$$

Light-sheet Modular Hamiltonian

In finite spatial volumes, the modular Hamiltonian K is nonlocal.
But we consider a portion of a null plane in Minkowski:



$$x^- \equiv t - x = 0\,;$$
$$x^+ \equiv t + x\,; 0 < x^+ < 1\,.$$

In this case, K simplifies dramatically.

Free Case

- The vacuum on the null plane factorizes over its null generators.
- The vacuum on each generator is invariant under a special conformal symmetry. Wall (2011)

Thus, we may obtain the modular Hamiltonian by application of an inversion, $x^+ \to 1/x^+$, to the (known) Rindler Hamiltonian on $x^+ \in (1, \infty)$. We find

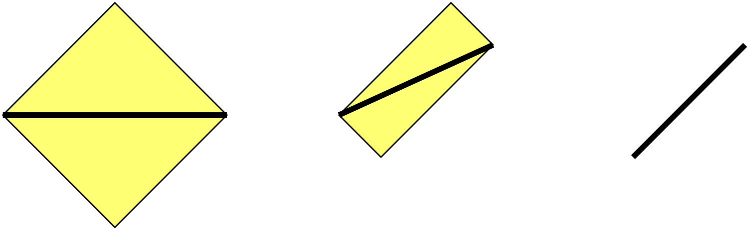
$$K = \frac{2\pi}{\hbar} \int d^2x^\perp \int_0^1 dx^+ \, g(x^+) \; T_{++}$$

with

$$g(x^+) = x^+(1 - x^+) \, .$$

Interacting Case

In this case, it is not possible to define ΔS and K directly on the light-sheet. Instead, consider the null limit of a spatial slab:



(a) (b) (c)

Interacting Case

We cannot compute ΔK on the spatial slab.

However, it is possible to constrain the form of ΔS by analytically continuing the Rényi entropies,

$$S_n = (1-n)^{-1} \log \mathrm{tr} \rho^n \ ,$$

to $n = 1$.

Interacting Case

The Renyi entropies can be computed using the replica trick,

Calabrese and Cardy (2009)

as the expectation value of a pair of defect operators inserted at the boundaries of the slab. In the null limit, this becomes a null OPE to which only operators of twist d-2 contribute. The only such operator in the interacting case is the stress tensor, and it can contribute only in one copy of the field theory.

This implies

$$\Delta S = \frac{2\pi}{\hbar} \int d^2 x^{\perp} \int_0^1 dx^+ \; g(x^+) \; T_{++} \; .$$

Interacting Case

Because ΔS is the expectation value of a linear operator, it follows that

$$\Delta S = \Delta K$$

for all states.

Blanco, Casini, Hung, and Myers 2013

This is possible because the operator algebra is infinite-dimensional; yet any given operator is eliminated from the algebra in the null limit.

Interacting Case

We thus have

$$\Delta K = \frac{2\pi}{\hbar} \int d^2 x^\perp \int_0^1 dx^+ \; g(x^+) \; T_{++} \; .$$

Known properties of the modular Hamiltonian of a region and its complement further constrain the form of $g(x^+)$:

$g(0) = 0$, $g'(0) = 1$, $g(x^+) = g(1 - x^+)$, and $|g'| \leq 1$.

I will now show that these properties imply

$$\Delta K \leq \Delta A / 4G\hbar \; ,$$

which completes the proof.

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Area Loss in the Weak Gravity Limit

Integrating the Raychaudhuri equation twice, one finds

$$\Delta A = -\int_0^1 dx^+ \theta(x^+) = -\theta_0 + 8\pi G \int_0^1 dx^+ (1 - x^+) T_{++} \ .$$

at leading order in G.

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Since $\theta_0 \leq 0$ and $g(x^+) \leq (1 + x_+)$, we have $\Delta K \leq \Delta A / 4G\hbar$

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if we assume the Null Energy Condition, $T_{++} \geq 0$.

Violations of the Null Energy Condition

- It is easy to find quantum states for which $T_{++} < 0$.
- Explicit examples can be found for which $\Delta S > \Delta A/4G\hbar$, if $\theta_0 = 0$.
- Perhaps the Covariant Entropy Bound must be modified if the NEC is violated?
- E.g., evaporating black holes

Lowe 1999

Strominger and Thompson 2003

- Surprisingly, we can prove $S \leq (A - A')/4$ without assuming the NEC.

Negative Energy Constrains θ_0

- If the null energy condition holds, $\theta_0 = 0$ is the "toughest" choice for testing the Entropy Bound.
- However, if the NEC is violated, then $\theta_0 = 0$ does not guarantee that the nonexpansion condition holds everywhere.
- To have a valid light-sheet, we must require that

$$0 \geq \theta(x^+) = \theta_0 + 8\pi G \int_{x^+}^{1} d\hat{x}^+ \, T_{++}(\hat{x}^+) \; ,$$

  holds for all $x^+ \in [0, 1]$.
- This can be accomplished in any state.
- But the light-sheet may have to contract initially:

$$\theta_0 \sim O(G\hbar) < 0 \; .$$

Proof of $\Delta K \leq \Delta A / 4G\hbar$

Let $F(x^+) = x^+ + g(x^+)$. The properties of g imply $F' \geq 0$, $F(0) = 0$, $F(1) = 1$.

By nonexpansion, we have $0 \geq \int_0^1 F' \, \theta \, dx^+$, and thus

$$\theta_0 \leq 8\pi G \int dx^+ [1 - F(x^+)] T_{++} \ . \qquad (1)$$

For the area loss, we found

$$\Delta A = -\int_0^1 dx^+ \theta(x^+) = -\theta_0 + 8\pi G \int_0^1 dx^+ (1 - x^+) T_{++} \ . \qquad (2)$$

Combining both equations, we obtain

$$\frac{\Delta A}{4G\hbar} \geq \frac{2\pi}{\hbar} \int_0^1 dx^+ \, g(x^+) \, T_{++} = \Delta K \ . \qquad (3)$$

Monotonicity

- In all cases where we can compute g explicitly, we find that it is concave:
$$g'' \leq 0$$
- This property implies the stronger result of monotonicity:
- As the size of the null interval is increased, $\Delta S - \Delta A/4G\hbar$ is nondecreasing.
- No general proof yet.

Covariant Bound vs. Generalized Second Law

- The Covariant Entropy Bound applies to any null hypersurface with $\theta \leq 0$ everywhere.
- It constrains the vacuum subtracted entropy on a finite null slab.
- The GSL applies only to causal horizons, but does not require $\theta \leq 0$.
- It constrains the entropy difference between two nested semi-infinite null regions.
- Limited proofs exist for both, but is there a more direct relation?