# Unitarity and positivity constraints for CFT at large N

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# Plan

- Basics of CFT in $d > 2$
- Four point functions in $\mathcal{N} = 4$ SYM with $SU(N)$ gauge group and crossing symmetry
- Large $N$ expansion:
  - construct solutions satisfying crossing
  - consequences for the spectrum of the theory
  - connection with causality constraints for effective field theories
- Conclusions

# Conformal field theories

- Conformal field theories in $d > 2$ dimensions are described by:
  - dimensions of primary operators $\phi_i$ $\Delta_i \to \langle \phi_i(x_1)\phi_j(x_2)\rangle$
  - OPE coefficients $c_{ijk} \to \langle \phi_i(x_1)\phi_j(x_2)\phi_k(x_3)\rangle$
- Four point functions of identical scalar primary operators is fixed by conformal symmetry to be:

$$\boxed{\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{\mathcal{G}(u,v)}{x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi}}}$$

where the cross ratios $u$ and $v$ are

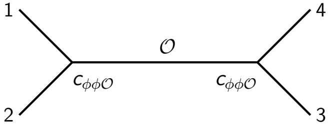
$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$$

# Conformal block decomposition

- OPE: $\phi \times \phi = 1 + \mathcal{O} + \textit{descendants}$
- conformal block decomposition:

$$\mathcal{G}(u,v) = \underbrace{1}_{\substack{\text{identity}\\ \text{operator}}} + \sum_{\mathcal{O}} \underbrace{c^2_{\phi\phi\mathcal{O}}}_{\substack{\text{OPE coeff}\\ \text{unitarity: } c^2 \geq 0}} \underbrace{g_{\mathcal{O}}(u,v)}_{\substack{\text{conformal blocks}\\ \text{primary + descendants}}}$$

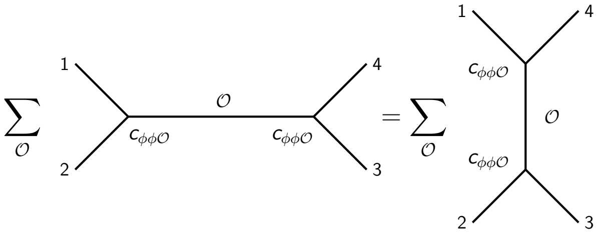
$$\mathcal{G}(u,v) = \sum_{\mathcal{O}}$$



# Crossing symmetry

- Crossing symmetry or associativity of the OPE:

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle$$




$$\boxed{\frac{\mathcal{G}(u,v)}{x_{12}^{2\Delta_\phi} x_{34}^{\Delta_\phi}} = \frac{\mathcal{G}(v,u)}{x_{23}^{2\Delta_\phi} x_{14}^{2\Delta_\phi}} \;\rightarrow\; v^{\Delta_\phi}\mathcal{G}(u,v) = u^{\Delta_\phi}\mathcal{G}(v,u)}$$

## $\mathcal{N} = 4$ SYM

- In this talk I will consider 4 dimensional $\mathcal{N} = 4$ SYM with gauge group $SU(N)$
- The four point function of $\frac{1}{2}$-BPS operators $\mathcal{O}_{\mathbf{20'}}$ of protected dimension 2, transforming under the $\mathbf{20'}$ of $SU(4)_R$ can be written as

$$\boxed{\langle \mathcal{O}_{\mathbf{20'}}(x_1)\mathcal{O}_{\mathbf{20'}}(x_2)\mathcal{O}_{\mathbf{20'}}(x_3)\mathcal{O}_{\mathbf{20'}}(x_4)\rangle = \sum_{\mathcal{R}} \frac{\mathcal{G}^{\mathcal{R}}(u,v)}{x_{12}^4 x_{34}^4}}$$

- $\mathcal{R}$ denotes the representations appearing in the tensor product $\mathbf{20'} \times \mathbf{20'}$
- Superconformal Ward identities imply relations among $\mathcal{G}^{\mathcal{R}}$ allowing the entire 4 point function to be expressed only in terms of one non trivial function $\mathcal{G}(u,v)$. Dolan, Osborn- Beem, Rastelli, van Rees

## Intermediate operators: $\mathcal{O} = \{\mathcal{O}_L, \mathcal{O}_S\}$

LONG MULTIPLET:

- acquire anomal. corrections

$$\Delta_L = \Delta(g_{YM}, N)$$
$$c^2_{\mathcal{O}_{20'}\mathcal{O}_{20'}L} = a(g_{YM}, N)$$

SHORT MULTIPLET:

- identity, $\frac{1}{2}$-BPS and $\frac{1}{4}$-BPS

$$\Delta_S = \Delta(N)$$
$$c^2_{\mathcal{O}_{20'}\mathcal{O}_{20'}S} = c^2_s(N)$$

- The contribution to the four point function can be split as $\mathcal{G}(u,v) = \mathcal{G}_L(u,v) + \mathcal{G}_S(u,v)$
- $\mathcal{G}_S(u,v)$ depends only on $N$ (what was 1 in the conformal case)
- $\mathcal{G}_L(u,v) = \underbrace{\sum_{\Delta,\ell}}_{\text{sum over superconf. primaries}} a_{\Delta,\ell} u^{\frac{\Delta-\ell}{2}} \underbrace{g_{\Delta+4,\ell}(u,v)}_{\text{superconformal blocks}}$

# Crossing equation

Crossing symmetry requires

$$v^2\mathcal{G}_L(u,v)-u^2\mathcal{G}_L(v,u)=u^2\mathcal{G}_S(v,u)-v^2\mathcal{G}_S(u,v)-4(u^2-v^2)-\frac{16}{N^2-1}(u-v)$$

- Solutions to this equation can be written as

$$\mathcal{G}_L(u,v)=\mathcal{G}_L^P(u,v)+\frac{\mathcal{A}(u,v)}{v^2}$$

- $\mathcal{G}_L^P(u,v)$ satisfies the crossing equation above by itself
- $\mathcal{A}(u,v)$ is crossing symmetric

$$\mathcal{A}(u,v)\underbrace{=}_{x_1\leftrightarrow x_3}\mathcal{A}(v,u)\underbrace{=}_{x_1\leftrightarrow x_4}v^2\mathcal{A}(\frac{u}{v},\frac{1}{v})$$

## Solution for $N = \infty$

- The contribution coming from the particular solution admits an expansion of the form

$$\mathcal{G}_L^P(u,v) = \mathcal{G}_L^{P,(0)}(u,v) + \frac{1}{N^2}\mathcal{G}_L^{P,(1)}(u,v)$$

- The solution $\mathcal{A}(u,v)$ starts at order $\frac{1}{N^2}$
- Intermediate operators:
  - double trace operators

$$\mathcal{O}_{n,\ell} = \mathcal{O}\Box^n \partial_{\mu_1} \cdots \partial_{\mu_\ell} \mathcal{O}$$

  - dimension $\Delta_{n,\ell} = 4 + 2n + \ell$ and spin $\ell$
  - $a_{n,\ell}^{(0)}$ are fixed by $\mathcal{G}_L^{P,(0)}(u,v)$

# Solutions at order $\frac{1}{N^2}$

- Intermediate operators: double trace
  - dimension $\Delta_{n,\ell} = 4 + 2n + \ell + \frac{1}{N^2}\gamma_{n,\ell}$
  - $a^{(0)}_{n,\ell} + \frac{1}{N^2} a^{(1)}_{n,\ell}$

Heemskerk, Penedones, Polchinski, Sully- Alday, AB, Lukowski

- For $\lambda = g^2_{YM} N$ large, the absence of twist two operators requires the presence of the supergravity solution

$$\frac{1}{N^2}\left(\mathcal{G}^{P,(1)}_{L}(u,v) + \mathcal{A}^{sugra}(u,v)\right) \to \gamma^{sugra}_{n,\ell} \text{ and } a^{sugra}_{n,\ell}$$

- Intermediate operators (at finite $\lambda$): single trace $\to \mathcal{A}^{ex}(u,v)$
  - dimension $\delta$ of order $N^0$ and spin $\ell$
  - $\frac{1}{N^2} a_{\delta,\ell}$
  - these operators appear at order $\frac{1}{N^2}$ hence $a_{\delta,\ell} \geq 0$ for unitarity
- At finite $\lambda$, there is a correction to the dimension of double trace operators $\gamma^{ex}_{n,l}$

# Main goal

The rest of the talk is devoted to:

- Find solutions $\mathcal{A}(u, v)$ of the crossing equation in the regime in which single trace operators appear in the OPE
- Understand in which way the presence of such operators affects the anomalous dimension $\gamma^{ex}_{n,l}$ of double trace, twist 4 operators

# Mellin amplitude

- Assume that the correlator at order $\frac{1}{N^2}$ admits a Mellin representation of the form

$$\mathcal{A}(u,v) = \frac{1}{(2\pi i)^2} \int \underbrace{\Gamma^2(x+2)\Gamma^2(y+2)\Gamma^2(z+2)}_{\text{poles at x=-2,...: twist of double trace ops}} M(x,y,z) u^{-x} v^{-y} dx dy$$

- Only two independent variables: $z = -2 - x - y$
- Crossing relations imply that M(x,y,z) is completely symmetric in $x$, $y$ and $z$
- Single poles of $M(x,y,z)$ at $x = -\tau/2$ correspond to the presence in the OPE of new operators of twist $\tau$

## Simple example

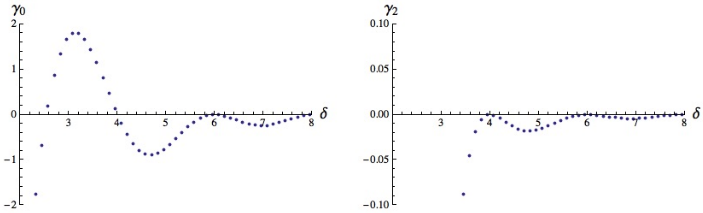
- Consider the presence of a single scalar operator of dimension $\delta$ with OPE coefficient $\frac{1}{N^2} a_{\delta,0}$.
- The Mellin amplitude consistent with crossing symmetry and containing a pole at $x = -\frac{\delta}{2}$ is

$$M_\delta(x,y,z) = h^{(0)}_{\delta,0} \left( \frac{1}{x+\delta/2} + \frac{1}{y+\delta/2} + \frac{1}{z+\delta/2} \right)$$

  where $h^{(0)}_{\delta,0} = a_{\delta,0} f(\delta) \geq 0$

- It is possible to compute the contribution to the anomalous dimension $\gamma^{ex}_{0,\ell}$ as a function of $\delta$

# Anomalous dimension




- $\gamma^{ex}_{0,0}$ is negative for $\delta > 4$
- $\gamma^{ex}_{0,\ell}$ for $\ell > 0$ is always negative

## More general case

- Consider the more general exchange of an operator of twist $\tau$, spin $\ell$ with all its descendants

$$M_{\tau}^{(\ell)}(x,y) = a_{\tau,\ell} \sum_k \alpha_k^{(\ell)} \left( \frac{P_{\ell+\tau}^{(\ell)}(y,z)}{x+\tau/2+k} + \frac{P_{\ell+\tau}^{(\ell)}(x,z)}{y+\tau/2+k} + \frac{P_{\ell+\tau}^{(\ell)}(x,y)}{z+\tau/2+k} \right)$$

  - $k$ is the descendant level
  - $P_{\ell+\tau}^{(\ell)}(y,z)$ is related to the Mack polynomial (Mellin representation of the superconformal block)
- the behaviour of $\gamma_{0,\ell}^{ex}$ is the same as for the single scalar.
- Warning: There is an ambiguity of adding a symmetric polynomial of degree $\ell - 1$ which is set to 0 for the moment. Notice that for $\ell = 0$ there is no ambiguity.

## General exchange

- For a new single trace primary operator of twist $\tau$ and spin $\ell$, the correction to the anomalous dimension of double trace operators has the property:
    - $\gamma^{ex}_{0,\ell} > 0$ if $\tau < 4$
    - $\gamma^{ex}_{0,\ell} \leq 0$ if $\tau \geq 4$

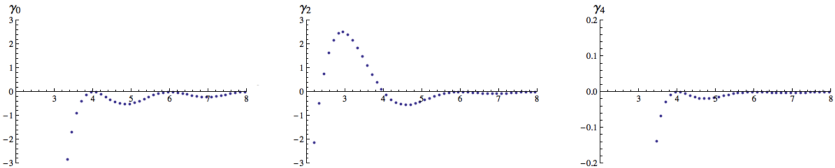



Fig.Example of $\ell = 2$

# Positivity constraints

- To focus on the anomalous dimension of twist 4 operators we need to consider terms proportional to $u^2$ (or $x = -2$) obtaining

$$\gamma_{0,j}^{ex} \sim -\int dy \Gamma^2(y+2)\Gamma^2(2-y)\mathcal{F}_j(y)P_{\ell+\tau}^{(\ell)}(y,-y)$$

  where $\mathcal{F}_j(y)$ are the continuos Hahn polynomials.
- it is possible to check that
  - $\mathcal{F}_j(y) = P_{j+4}^{(j)}(y,-y)$
  - $P_{\ell+\tau}^{(\ell)}(y,-y) = \sum_{j=0}^{\ell} c_j(\tau)\mathcal{F}_j(y)$ where $c_j(\tau) \geq 0$ for $\tau \geq 4$

## Positivity constraints

If all the single trace operators have $\tau \geq 4$, the contribution to the whole Mellin amplitude at $x = -2$ (and hence $z = -y$) satisfies

$$M(-2, y, -y) = \sum_{n=0,2,\cdots} \underbrace{c_n}_{c_n \geq 0} \mathcal{F}_n(y) \; \rightarrow \gamma^{ex}_{0,\ell} \leq 0$$

- Notice that due to the symmetries of $M(x, y, z)$, it follows that

  $$M(x, -2, -x) = \sum_{n=0,2,\cdots} c_n \mathcal{F}_n(x)$$

  with $c_n \geq 0$.

## Strong positivity constraints

- Assume that at fixed spin the minimal twists are ordered in such a way that $\tau_0 < \tau_2 < \cdots < 4 < \tau_{\ell^*} < \tau_{\ell^*+2} \ldots$
- This implies that $M(x, -2, -x) = \sum_{n=0,2,\cdots} c_n \mathcal{F}_n(x)$ with $c_{\ell^*}, c_{\ell^*+2}, \cdots$ positive
- Moreover: $\gamma^{ex}_{0,\ell} \leq 0$ for $\ell \geq \ell^*$.

◦ The same constraints can be derived by assuming Regge behaviour for the Mellin amplitude.

# Comments on the spectrum

How does this reflect on the spectrum?

- The dimension of double trace twist 4 operators is given by

$$\Delta_{0,\ell} = 4 + \ell - \underbrace{\frac{1}{N^2}\frac{96}{(\ell+1)(\ell+6)}}_{\text{SUGRA correction}} + \frac{1}{N^2}\gamma^{ex}_{0,\ell}$$

- If the single trace operator with minimal twist $\tau_{min}$ has
  - $\tau_{min} < 4$ then $\tau_{min}$ is the minimum twist for fixed spin
  - $\tau_{min} \geq 4$ then $4 - \gamma^{sugra}_{0,\ell^*}$ is an upper bound for the twist, since $\gamma^{ex}_{0,\ell} \leq 0$ in this regime
- This upper bound has been observed with the numerical bootstrap.

Beem, Rastelli, van Rees

# Causality constraints

- Causality constraints on effective field theories have been studied by Adams, Arkani-Hamed, Dubovsky, Nicolis and Rattazzi.
- The S-matrix of a low energy effective field theory should satisfy certain positivity constraints if the theory has a consistent UV completion.
- In the forward limit $t \to 0$ the regular part of the S-matrix has an expansion

$$\mathcal{T}(s, 0, -s) = \alpha + \beta s^2 + \gamma s^4 + \cdots$$

where all the coefficients $\alpha, \beta, \cdots$ are non-negative

## Causality constraints

- In our formalism, the connection between the S-matrix and the $\frac{1}{N^2}$ contribution of the Mellin amplitude is through the flat space limit:

$$\mathcal{T}(s,t,u) = -2 \lim_{\lambda\to\infty} \lambda^{3/2} \oint \frac{d\alpha}{2\pi i} \frac{e^{-\alpha}}{\alpha^6} M\left(\frac{\sqrt{\lambda}s}{\alpha}, \frac{\sqrt{\lambda}t}{\alpha}, \frac{\sqrt{\lambda}u}{\alpha}\right)$$

- In the forward limit the Mellin amplitude have an expansion:

$$M\left(\frac{\sqrt{\lambda}s}{\alpha}, 0, -\frac{\sqrt{\lambda}s}{\alpha}\right) = c_0 + c_1\frac{\lambda s^2}{\alpha^2} + c_2\frac{\lambda^2 s^4}{\alpha^4} + \cdots$$

  where the coefficients $c_i = c_i(\lambda)$. Only the leading term survives in the flat space limit and is poitive.
- Positivity constraints in the flat space limit are equivalent to the ones of causality.

# Conclusions

In this talk I discussed

- solutions at order $\frac{1}{N^2}$ to crossing symmetry of a four point function of $\frac{1}{2}$-BPS operators in $\mathcal{N} = 4$ SYM with gauge group $SU(N)$
- in particular the regime in which the OPE contains single trace operators as well as double trace operators
- the anomalous dimensions of double trace operators of twist 4 get corrections with a definite sign which can be tracked back to positivity properties of Mack polynomials
- the relation between the positivity constraints and causality of the S-matrix of effective field theories.