M-theory and extended geometry

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Outline

Introduction and Motivation

Generalized versus extended geometry

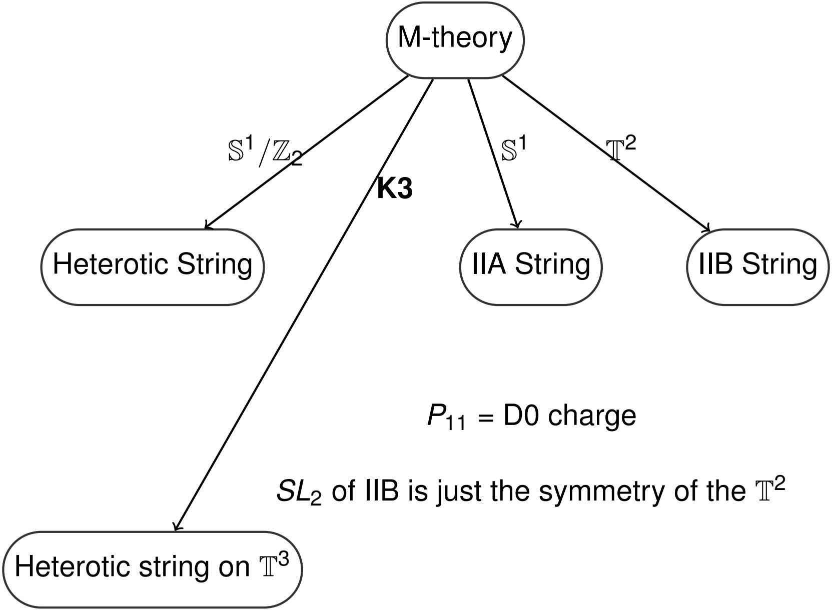
Reformulating M-theory in extended geometry

Applications

This talk is an overview of work with Malcolm Perry and others in numerous permutations beginning with arXiv1008.1763, arXiv1103.5733, arXiv1110.3097, arXiv1110.3930, arXiv1111.0459, arXiv1208.0020, arXiv:1208.5884, arXiv:1303.6727, arXiv:1305.2747 and also some forthcoming work.
For a review of the subject see arXiv:1306.2643 also an excellent review by Aldazabal, Marques and Nunez, arXiv:1305.1907.

There is a whole host of works on which extended geometry for M-theory is based. For the sake of time, let me simply note the early work of Duff, Siegel, Tseytlin, Nicolai, West and more recent works of Hohm, Hull, Zwiebach et al; Park et al; Waldram et al, Hillmann, and of course implicitly Hitchin, Gualtieri amongst many many others.

Sorry to those I havn't mentioned but should have!



P_{11} = D0 charge

SL_2 of IIB is just the symmetry of the $\mathbb{T}^2$

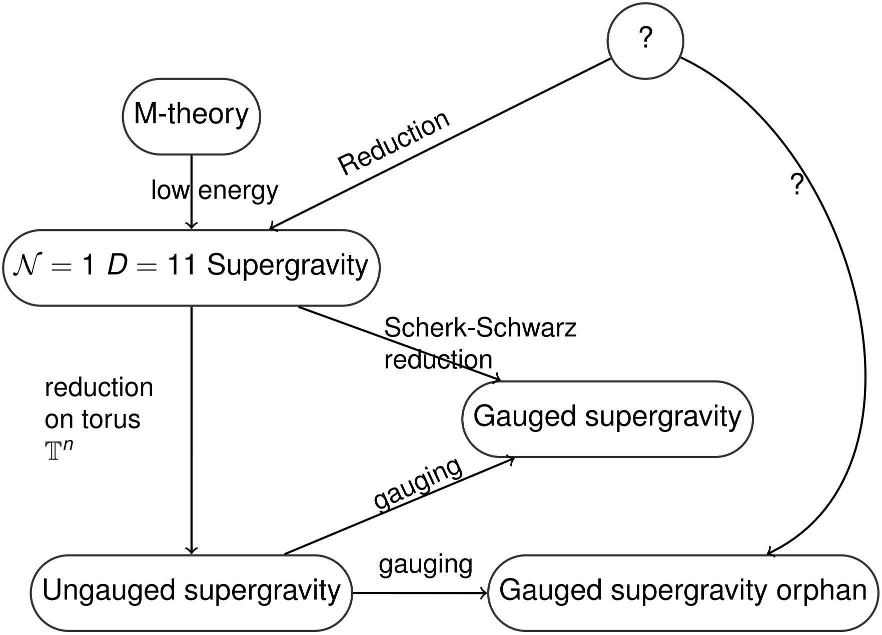
F-theory

12 dimensional

Realise the axion-dilaton of the IIB string geometrically through a torus fibred over a ten dimensional space.

Seven branes are when there are Sl_2 monodromies in the torus around some closed one cycle.

Sl_2 of IIB is again realised from the modular symmetry of the torus.



?
M-theory
Reduction
low energy
?
$\mathcal{N} = 1$ $D = 11$ Supergravity
Scherk-Schwarz
reduction
reduction
on torus
$\mathbb{T}^n$
Gauged supergravity
gauging
gauging
Ungauged supergravity
Gauged supergravity orphan

The duality symmetries in string and M-theory are much richer than the Sl_2 of IIB or the O(d,d) group of T-duality. We have the full U-duality groups of M-theory reduced on T^d.
Recall,

$$\begin{aligned} d &= 4 \qquad G = SL(5) \\ d &= 5 \qquad G = SO(5,5) \\ d &= 6 \qquad G = E_6 \\ d &= 7 \qquad G = E_7 \end{aligned} \tag{1}$$

These duality groups are manifest after dimensional reduction but it has long been conjectured that they are present in some way in the nonreduced theory.

Wish List

- Extend (and repackage) M-theory to make these duality groups manifest without any dimensional reduction.
- The repackaging should be *geometric* (in some sense) just as the Sl_2 is in M-theory or F-theory.
- Have all the usual branes be momentum modes in the extended space- equivalently have no central charges in the associated supersymmetry algebra.
- Find a parent for the orphan gauged supergravities.
- Allow monodromies in the theory so that one can describe the so called “exotic branes" of Shigemori and deBoer; just as F-theory produces seven branes.

Wishes can come true.

To do this we need to combine $g_{\mu\nu}$ and $C_{\mu\nu\rho}$ into a single geometric quantity and have diffeomorphisms and p-form gauge transformations to be combined into a generalised diffeomorphism of the extended space.

The package will be a sort of generalized geometry but in fact we will want an extended generalized geometry.

Generalized geometry for T-duality.

The tangent space is extended from $T\Lambda^1(M)$ to $T\Lambda^1(M) \oplus T\Lambda^{*1}(M)$
The metric on this generalized space is given by

$$M_{IJ} = \begin{pmatrix} g_{\mu\nu} - B^2_{\mu\nu} & B_\mu{}^\sigma \\ B^\rho{}_\nu & g^{\sigma\rho} \end{pmatrix} \tag{2}$$

This is extending just the tangent space but one could do that by extending the space itself. I will call this extended geometry though for the string it is normally called “double field theory" because the dimension of the space is doubled.

From the world sheet point of view we combine string fields and their T-duals into the same target space. Thus naturally the above metric acts on the forms:

$$dX^I = (dX^\mu, dy_\mu) \tag{3}$$

where y_μ is the string worldsheet field T-dual to X^μ.

The same may be done for membranes

Extend space to include *dual* membrane winding modes, $y_{\mu\nu}$ along with usual x^{μ} coordinates. No longer a simple doubling. Now the generalised tangent space is:

$$\Lambda^{1}(M) \oplus \Lambda^{*2}(M)\,. \tag{4}$$

The metric is given by:

$$M_{IJ} = \begin{pmatrix} g_{ab} + \frac{1}{2} C_{a}{}^{ef} C_{bef} & \frac{1}{\sqrt{2}} C_{a}{}^{kl} \\ \frac{1}{\sqrt{2}} C^{mn}{}_{b} & g^{mn,kl} \end{pmatrix}, \tag{5}$$

where $g^{mn,kl} = \frac{1}{2}(g^{mk}g^{nl} - g^{ml}g^{nk})$ and has the effect of raising an antisymmetric pair of indices.

Now, we extend the space as before, to one with coordinates $Z^I = (x^a, y_{ab})$ and demand that M_{IJ} is the metric on the space.

Now, the next step is to write down a Lagrangian such that when we remove the dependence on the novel directions:

$$\partial_{y_{ab}} = 0 \tag{6}$$

we reproduce ordinary supergravity.

This condition will then be made duality covariant, called the *section condition*.

Different solutions of the section condition then yield different duality frames.

In order to be concrete we will work with the d=4 case of SL(5). For d>5 we also need to include fivebrane modes etc. too.

So decompose the eleven dimensional space into a trivial 4+7 split. We will concentrate on the 4 dimensional space and allow arbitrary dependence on the coordinates in those directions ie. **no dimensional reduction**.

So $a = 1..4$ which implies there are 6 y_{ab} coordinates and the total extended space on which the SL(5) acts is **ten dimensional**! (in addition to the 7 directions we are ignoring).

We can construct the Lagrangian with all the right properties:

$$
\begin{aligned}
L &= \Bigg(\frac{1}{12} M^{MN} (\partial_M M^{KL})(\partial_N M_{KL}) - \frac{1}{2} M^{MN} (\partial_N M^{KL})(\partial_L M_{MK}) \\
&+ \frac{1}{12} M^{MN} (M^{KL} \partial_M M_{KL})(M^{RS} \partial_N M_{RS}) \\
&+ \frac{1}{4} M^{MN} M^{PQ} (M^{RS} \partial_P M_{RS})(\partial_M M_{NQ}) \Bigg)
\end{aligned}
$$

When M_{IJ} is independent of y_{ab} then (up to surface terms)

$$L = R - \frac{1}{48} H^2 \tag{7}$$

where $H = dC$, reproducing the ordinary Lagrangian.

This is somewhat miraculous!

- What is the SL(5) covariant section condition?
- What are its local symmetries?
- Expect that the above two questions are related

For the string case there has been a similar construction by Hull and Zwiebach using closed string field theory. They also construct a Lagrangian for the O(d,d) metric that reproduces the usual one when there is no y_a dependence but they also have an $O(d,d)$ invariant constraint, which is that fields must obey:

$$\partial_I \partial^I = \partial_{x^a} \partial_{y_a} = 0 \tag{8}$$

We begin with the generalized Lie derivative. This (when restricted) generates the usual diffeomorphisms and gauge transformations of C.
We use the following natural $SL(5)$ coordinates as follows: Z^{ab} with ab (a,b=1..5) being antisymmetric. These are related to the x^i, y_{ij} coordinates as follows:

$$z^{i5} = x^i\,, \quad z^{ij} = \epsilon^{ijlk} y_{lk} \qquad i,j,l,k = 1..4 \tag{9}$$

The generalized Lie derivative is then given by:

$$\hat{L}_X v_{ab} = X^{ef} \partial_{ef} V_{ab} + \partial_{ae} X^{ef} V_{fb} + \partial_{ef} X^{ef} V_{ab} \tag{10}$$

The algebra of $\hat{L}$ is as follows:

$$[\hat{L}_X, \hat{L}_Y] = \hat{L}_Z + \text{junk} \tag{11}$$

where

$$Z = [X,Y]_C = [a+\zeta, b+\eta] = [a,b] + L_a\eta - L_Y\zeta - \frac{1}{2}(di_a\eta - di_Y\zeta) \tag{12}$$

$[X,Y]_C$ denotes the Courant bracket.

This Courant algebra also appears when one looks at the Canonical analysis of the theory- this is because the diffeomorphism and gauge symetry algebras compose nontrivially.

What is *junk*?
This is what we would like to vanish so that we can mirror what happens in strgin theory and get a Courrant algebra for the generalized Lie derviatives.
In fact it can be shown to vanish (its a long and unpleasant expression) if the following condition holds:..

$$\epsilon^{abcde} \frac{\partial}{\partial z^{ab}} \frac{\partial}{\partial z^{cd}} = 0 \quad (13)$$

This is in terms of the $SL(5)$ coordinates on the 10 dimensional space, z^{ab}. One can go back to the x^i, y_{ij} coordinates and write the section condition in terms of those. In these coordinates it gives..

$$\frac{\partial}{\partial x^a}\frac{\partial}{\partial y_{ab}} = 0 \tag{14}$$

and

$$\epsilon_{abcd}\frac{\partial}{\partial y_{ab}}\frac{\partial}{\partial y_{cd}} = 0\,. \tag{15}$$

Obviously solved by our choice to remove dependence on y_{ab}. These equation also allow different choices of coordinates to obtain different duality frames.

These conditions have been seen before (Obers and Pioline) in the constraints on the central charges of 1/2 BPS states. Think of:

$$\frac{\partial}{\partial y_{ab}} = Z^{ab} \qquad \frac{\partial}{\partial x^a} = p_a \tag{16}$$

Other duality groups. eg $SO(5,5)$

$$\Lambda^1(M) \rightarrow \Lambda^{*2}(M) \oplus \Lambda^{*5}(M) \tag{17}$$

So we have coordinates

$$Z^I = (x^a, y_{ab}, y_{abcde}) \tag{18}$$

with $a = 1..5$, $ab = 6..15$, $abcde = 16$. Thus the space is 16 dimensional corresponding to the **16** of SO(5,5). The y_{abcde} correspond to fivebrane winding mode.

The SO(5,5) generalized metric is (upper case latin indices run from 1 to 16):

$$M_{IJ} = \begin{pmatrix} g_{ab} + \frac{1}{2} C_a{}^{ef} C_{bef} + \frac{1}{16} X_a X_b & \frac{1}{\sqrt{2}} C_a{}^{mn} + \frac{1}{4\sqrt{2}} X_a V^{mn} & \frac{1}{4} X_a \\ \frac{1}{\sqrt{2}} C^{kl}{}_b + \frac{1}{4\sqrt{2}} V^{kl} X_b & g^{kl,mn} + \frac{1}{2} V^{kl} V^{mn} & \frac{1}{\sqrt{2}} V^{kl} \\ \frac{1}{4} X_b & \frac{1}{\sqrt{2}} V^{mn} & 1 \end{pmatrix} \tag{19}$$

where we have defined:

$$V^{ab} = \frac{1}{6} \eta^{abcde} C_{cde} \,, \tag{20}$$

with η^{abcde} being the totally antisymmetric permutation symbol (it is only a tensor density and thus distinguished from the usual ϵ^{abcde} symbol) and

$$X_a = V^{de} C_{dea} \,. \tag{21}$$

We can attempt to reconstruct the dynamical theory out of this generalized metric. Consider the following Lagrangian,

$$
\begin{aligned}
L = M^{1/2}\Bigg(& \frac{1}{16} M^{MN}(\partial_M M^{KL})(\partial_N M_{KL}) - \frac{1}{2} M^{MN}(\partial_N M^{KL})(\partial_L M_{MK}) \\
& + \frac{3}{128} M^{MN}(M^{KL}\partial_M M_{KL})(M^{RS}\partial_N M_{RS}) \\
& - \frac{1}{8} M^{MN} M^{PQ}(M^{RS}\partial_P M_{RS})(\partial_M M_{NQ}) \Bigg)
\end{aligned} \tag{22}
$$

where $\partial_M = \left(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial y_{ab}}, \frac{\partial}{\partial z}\right)$.

One now imposes the condition that all fields are independent of the additional coordinates $\{y_{ab}\}, z$, ie. on all fields

$$\frac{\partial}{\partial y_{ab}} = 0\,, \qquad \frac{\partial}{\partial z} = 0\,. \tag{23}$$

We then evaluate L in terms of g_{ab} and C_{abc} with the section condition. After a long and careful calculation, the result, up to a total derivative, is

$$L = \gamma^{1/2}(R(\gamma) - \frac{1}{48}F^2)\,. \tag{24}$$

These generalized metrics and actions can then be dimensionally reduced (Thompson) to get the double field theory for the strings (Jeon et al and Hohm et al).

We have done so far all the duality groups and their actions up to E_7.

As we move beyond that, with E_8 things get more difficult (Godazgar^2, Perry appear to have successfully made this work). Higher exceptional groups of West then appear after that which means an infinite number of fields but also an infinite number of constraints. The hope is that some sense can still be made of that using these methods.

The closure of the generalised Lie derivatives can be used in each case to give the physical section condition.

$$\mathcal{L}_\xi V^M = L_\xi V^M + Y^{MN}{}_{PQ}\partial_N \xi^P V^Q \tag{25}$$

where

$$\begin{aligned}
O(n,n)_{strings}: &\qquad Y^{MN}{}_{PQ} = \eta^{MN}\eta_{PQ}\ ,\\
SL(5): &\qquad Y^{MN}{}_{PQ} = \epsilon^{iMN}\epsilon_{iPQ}\ ,\\
SO(5,5): &\qquad Y^{MN}{}_{PQ} = \frac{1}{2}(\gamma^i)^{MN}(\gamma_i)_{PQ}\ ,\\
E_{6(6)}: &\qquad Y^{MN}{}_{PQ} = 10 d^{MNR}\bar{d}_{PQR}\ ,\\
E_{7(7)}: &\qquad Y^{MN}{}_{PQ} = 12 c^{MN}{}_{PQ} + \delta_P^{(M}\delta_Q^{N)} + \frac{1}{2}\epsilon^{MN}\epsilon_{PQ}\ .
\end{aligned} \tag{26}$$

$$Y^{MN}{}_{PQ}\partial_M\partial_N = 0 \tag{27}$$

- BPS states are interesting: from this point of view they are simply null states in the extended space with the central charges being momentum in the novel extra dimensions. That is easy to see, the BPS condition

$$p^0 = Q \tag{28}$$

becomes

$$p^0 = p_{12} \tag{29}$$

- Just like how a charged object appears in ordinary Kaluza Klein theory.

Lets look for solutions of extended theories that correspond to “extended massless" states with momentum pointing in a particular direction. Something analogous to the wave solution of 11d supergravity.
Take our simple 4-d, $SL(5)$ case:
A solution to the equations of motion is:

$$\begin{aligned} d\hat{s}^2 &= (H-2)(dx^1)^2 + (2-H)\left[(dx^2)^2 + (dx^3)^2\right] + (dx^4)^2 \\ &\quad + 2(H-1)\left[dx^1 dy_{23} + dx^2 dy_{13} - dx^3 dy_{12}\right] \\ &\quad - H\left[(dy_{13})^2 + (dy_{12})^2 - (dy_{23})^2\right] + (dy_{34})^2 + (dy_{24})^2 - (dy_{14})^2 \end{aligned} \qquad (30)$$

H a harmonic function.
Found, from taking the usual wave solution in the normal space and “rotating it" using $SL(5)$ so it points along the dual directions.

What is this solution from the reduced point of view, ie. in terms of the metric and C field in 4 dimensions:

$$ds^2 = -H^{-1}(dt^2 - dz_1^2 - dz_2^2) + dw^2 \quad (31)$$

$$C_{tz_1z_2} = -\sqrt{2}(H^{-1} - 1) \quad (32)$$

That is the membrane in 4-dimensions (after changing frame with a rescaling).
Thus branes and strings are all wave solutions in the extended space. When the wave momentum points along the “dual" directions we obtain strings and branes from the the normal view point. This extends the idea of the D0 brane being momentum of the 11th dimension, all the branes now become momenta in the extended space.

Generalized Scherk-Schwarz reductions
Forget about the physical section conditon (that is like a KK reduction).
Instead allow a dependence on the extra coordinates (but they will be become restricted in a different way)
Introduce a *twist matrix* $W^I{}_{\bar{I}}$ such that:

$$V^I(x, Y) = W^I{}_{\bar{I}}(Y) V^{\bar{I}}(x) \tag{33}$$

Insert this Scherk Schwarz ansatz into the action then one obtains a doubled gauged supergravity (Aldazabal et al. Grana et al., Geissbuhler for double field theory).
The *structure constants* of the Scherk Schwarz reduction can be identified with the embedding tensor of the gauged supergravity.

- The Scherk-Schwarz reductions of the extended geometry (which by definition include a dependence on the novel dimensions) can produce all the known maximal gauged supergravities.
- The contraints on the embedding tensor, come from the constraints to make the generalized Lie derivative close and obey Jacobi.
- These include gauged supergravities that could not be obtained by ordinary Sherk-Schwarz reductions!

Conclusions

- We can construct a U-duality manifest symmetric action based on the generalized metric.
- We have a section condition whose solutions spontaneously break duality symmetry.
- Strings and branes are massless waves in this geometry with momentum pointing along the new novel directions. Just like the D0 in ordinary M-theory.
- Previous gauged supergravity orphans now have a *geometric* parent.

Insert into the generalised Lie derivative (Q independent of x):

$$\mathcal{L}_\xi Q^M = W^M_{\bar{C}} X_{\bar{A}\bar{B}}{}^{\bar{C}} \xi^{\bar{A}} Q^{\bar{B}} \tag{34}$$

where:

$$X_{\bar{A}\bar{B}}{}^{\bar{C}} = W^{\bar{C}}{}_I W^J{}_{\bar{A}} \partial_J W^I{}_{\bar{B}} + ... \tag{35}$$

Closure of the generalised Lie derviatives now just gives a quadratic constraint on $X_{AB}{}^C$:

$$\frac{1}{2}\left(X_{\bar{A}\bar{B}}{}^{\bar{C}} - X_{\bar{A}\bar{B}}{}^{\bar{C}}\right) X_{\bar{C}\bar{E}}{}^{\bar{G}} - X_{\bar{B}\bar{E}}{}^{\bar{C}} X_{\bar{A}\bar{C}}{}^{\bar{G}} + X_{\bar{A}\bar{E}}{}^{\bar{C}} X_{\bar{B}\bar{C}}{}^{\bar{G}} = 0\ . \tag{36}$$

If we define $(X_{\bar{A}})^{\bar{C}}_{\bar{B}} = X_{\bar{A}\bar{B}}{}^{\bar{C}}$ this may be written in the suggestive form

$$[X_{\bar{A}}, X_{\bar{B}}] = -X_{[\bar{A}\bar{B}]}{}^{\bar{C}} X_{\bar{C}}\ . \tag{37}$$

Thus we begin to see the structure of an algebra of gauge transformation

$$\delta_\xi V^{\bar{C}} = \xi_1^{\bar{A}} (X_{\bar{A}})_{\bar{B}}{}^{\bar{C}} V^{\bar{B}}\ . \tag{38}$$