Space-Time Action for G_2 Compactifications in Superspace

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Overview

I. Given a supersymmetric string theory or M-theory compactification in the supergravity approximation, can it be corrected order by order in α' (or the inverse radius) to give a solution of the corrected equations of motion and supersymmetry transformations? ⟹ Type II string theory on G_2 manifolds

II. What is the manifestly supersymmetric complete space-time action for an arbitrary string theory or M-theory compactification? ⟹ M-theory on G_2 manifolds

I. Type II String Theory on G_2 Manifolds

General Remarks

A vacuum of type II supergravity is

$$R^{1,2} \times M$$

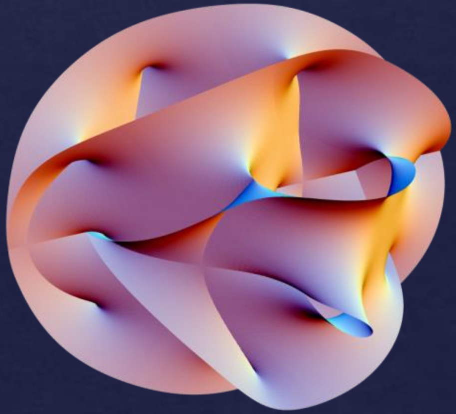
Minkowski space

Compact 7d manifold
$R_{ab} = 0$

parallel spinor

Supersymmetry in space-time $\nabla_a \eta = 0$

The metric on M has G_2 holonomy...



Calabi-Yau, *Spin*(7) manifold

Tools

Given a 7d spin manifold M there is a unit spinor η

$$\varphi_{abc} = \eta^T \Gamma_{abc} \eta$$

unit spinor $\eta^T \eta = 1$

and a 4-form...

$$\psi_{abcd} = \eta^T \Gamma_{abcd} \eta$$

…related by

$$\psi = *\varphi$$

$$g_{ab} = g_{ab}[\varphi] = (\det s)^{-1/9} s_{ab}$$

Metric

$$s_{ab} = -\frac{1}{144} \varphi_{amn} \varphi_{bpq} \varphi_{rst} \varepsilon^{mnpqrst}$$

If the metric has G_2 holonomy

$$d\varphi = 0$$

$$d\psi = 0$$

In general, if the manifold is spin (but the spinor might not be covariantly constant) then the space has a G_2 structure and forms can be decomposed into irreducible representations of G_2

$$d\varphi = \tau_0 \psi + 3\tau_1 \wedge \varphi + *\tau_3$$

$$d\psi = 4\tau_1 \wedge \psi + \tau_2 \wedge \varphi$$

$\tau_0, \tau_1, \tau_2, \tau_3$ are torsion classes

Leading Order Correction

Gravitino supersymmetry transformation

$$\delta\psi_a = \nabla_a\eta + \boxed{A_a\eta + iB_a{}^b\Gamma_b\eta}$$

α' corrections

- In 7d dimensions spinors have 8 real components.
  A basis is $\{\eta, \Gamma_a\eta\}$, a=1,...,7.

- The coefficients are tensors $A_a = A_a[\varphi] \quad B_a{}^b = B_a{}^b[\varphi]$

At $O(\alpha'^3)$

$$A_a = 0$$

$$B_a{}^b = \alpha'^3\varphi_{acd}\nabla^c\left(\frac{1}{32g}\varepsilon^{dc_1\ldots c_6}\varepsilon^{bd_1\ldots d_6}R_{c_1c_2d_1d_2}R_{c_3c_4d_3d_4}R_{c_5c_6d_5d_6}\right)$$

Supersymmetric Vacuum

To order α'^3

$$\delta\psi_a = \boxed{\nabla'_a \eta'} + \boxed{A_a[\varphi]\eta + iB_a{}^b[\varphi]\Gamma_b \eta} = 0$$

Primed quantities include corrections: $\eta' = \eta + O(\alpha'^3)$

φ, η of G_2 holonomy manifold

Set up PDE

$$\boxed{\begin{aligned} d\varphi' &= \alpha[\varphi] \\ d\psi' &= \beta[\varphi] \\ \psi' &= *'\varphi' \end{aligned}}$$

$$\alpha_{abcd} = 8A_{[a}\varphi_{bcd]} - 8B_{[a}{}^e\psi_{bcd]e}$$

$$\beta_{abcde} = 10A_{[a}\psi_{bcde]} - 40B_{[ab}\varphi_{cde]}$$

A necessary and sufficient condition for this PDE to be solvable is that α and β should be exact.

To order α'^3 we can check this explicitly.

The PDE for φ' is solvable!

K. B., D. Robbins,
E. Witten, 1404.2460

All Orders in α’

Using induction over the order in α' it is possible to show that a solution of the supersymmetry conditions exists to any order in α' provided the corresponding

$$d\varphi'|_n = \alpha[\varphi']|_n = d\chi_n$$

$$d\psi'|_n = \beta[\varphi']|_n = d\xi_n$$

$$\psi' = *'\varphi'$$

contains the unknown which is the order n of φ’

involves φ’ to order less than n

Exactness of α and β is not only necessary but also sufficient....

There exists a solution of δψ=0 to all orders in α’!

K. B., D. Robbins, E. Witten, 1404.2460

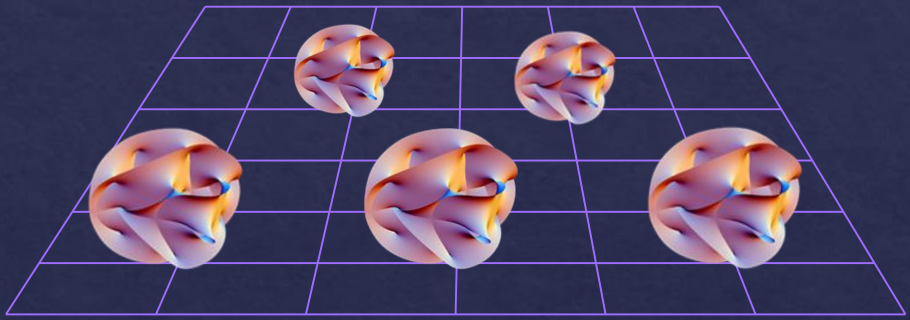
II. The Space-Time Action of M-theory Compactified to 4d in N=1 Superspace

M-Theory on a G_2 Manifold

We wish to describe the fluctuations around the background...

$$R^{1,3} \times M$$

compact G_2 manifold



... these include massless states as well as massive KK modes.

Guiding Principles

4d supersymmetry + Locality

Assemble fields into 4d superfields

Keep locality along space-time and *M*.

$$\phi = \phi(x, y)$$

$$C = \frac{1}{3!} C_{abc}(x, y) dy^a \wedge dy^b \wedge dy^c$$

11d fields decompose into many 4d fields

$$C_{MNP}, G_{MN} \rightarrow \begin{cases} C_{abc}, C_{ab\mu}, C_{a\mu\nu}, C_{\mu\nu\rho} \\ g_{ab}, g_{a\mu}, g_{\mu\nu} \end{cases}$$

Manifest Global 4d Supersymmetry

The coordinates of flat 4d superspace are $\left(x^m, \theta^\mu, \bar{\theta}_{\dot{\mu}}\right)$

Superfields are functions of these coordinates...

Chiral superfields

$\bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} - i\theta^\alpha \sigma^m_{\alpha\dot{\alpha}} \partial_m$

$$\bar{D}_{\dot{\alpha}} \Phi = 0$$

$$\Phi\left(x,\theta\right) = \mathcal{C}\left(x_+\right) + \sqrt{2}\theta\psi\left(x_+\right) + \theta\theta F\left(x_+\right)$$

$$x^m_\pm = x^m \pm i\theta\sigma^m\bar{\theta}$$

$$\mathcal{C}_{abc}(x,y) = \hat{\varphi}_{abc}(x,y) + iC_{abc}(x,y)$$

To leading order in α': $\hat{\varphi} = \varphi$

Action For Chiral Superfields

$$I = \frac{1}{2}\int d^4x\left[K\left(\Phi, \Phi^+\right)\right]\Big|_D + \int d^4x\left[f(\Phi)\right]\Big|_F + c.c.$$

Lagrangian density for bosonic fields

$$L = -\int_{M\times M} d^7y\,d^7y'\,\frac{\delta^2 K}{\delta\mathcal{C}(y)\,\delta\mathcal{C}(y')}\left(\partial_\mu\mathcal{C}(y)\partial^\mu\mathcal{C}(y') - F(y)F(y')\right)$$

$$+2\,\mathrm{Re}\int_M d^7y\,\frac{\delta f(\mathcal{C})}{\delta\mathcal{C}(y)}F(y)$$

Superpotential

A good candidate is $$f\left(\Phi\right) = \beta \int_M \Phi \wedge d\Phi$$

constant

In a supersymmetric ground state

$$\frac{\delta f}{\delta \Phi} = 0 \Rightarrow d\Phi = 0 \Rightarrow d\hat{\varphi} = 0, G_4 = 0$$

Comparing with the previous results $d\varphi' = \alpha = d\chi$

$$\hat{\varphi} = \varphi' - \chi$$

There is a closed 3-form!

Kähler Form

$\mathcal{C}_{abc} = \varphi_{abc} + iC_{abc}$ are coordinates of an infinite dimensional Kaehler manifold. Eleven-dimensional gauge transformations

$$\delta C = d\Lambda$$

$\Lambda \in V$ the space of 2-forms mod closed 2-form

…gives rise to isometries of the metric.

The Kähler form is invariant and as a result there is a moment map (a concept we borrow from symplectic geometry). As we show in more detail in our paper the vanishing of the moment map implies

$$\mu = 0 \Rightarrow \nabla_a \left(\frac{\delta K}{\delta \mathcal{C}_{abc}(y)} \right) = 0$$

Closed 4-form!

Needles to say it would be interesting to derive these conditions from a Kaluza-Klein reduction of M-theory. We envision this as a two step process:

1) we rewrite the action of 11d supergravity in a form that displays manifest N=1 supersymmetry in 4d.

2) non-renormalization theorems should then give us information about which results hold to all orders in perturbation theory.

Kaluza-Klein Reduction of M-Theory

Bosonic Fields

Fields are decomposed into a 4+7 split:

$$C_{MNP} \rightarrow \quad C_{abc}, C_{ab\mu}, C_{a\mu\nu}, C_{\mu\nu\rho}$$

$$G_{MN} = \begin{pmatrix} h_{\mu\nu} + g_{cd} A_\mu^c A_\nu^d & g_{bc} A_\mu^c \\ g_{ac} A_\nu^c & g_{ab} \end{pmatrix}$$

$M, N = 0, \dots, 10$ $\quad$ μ,ν=0,..,3 $\quad$ $a, b = 4, \dots, 10$

Symmetries: $\begin{cases} C \rightarrow C + d\Lambda \\ x^M \rightarrow x^M - \xi^M \end{cases}$

4d system is very complicated but known in detail...

4 Summary

As a summary we present a concrete example. The space-time effective action for eleven-dimensional supergravity compactified to four dimensions is

$$
\begin{aligned}
S = &- \frac{1}{8\kappa^2}\int dv\, h^{\alpha\beta}\left(\frac{1}{2}g^{ab}g^{cd} + g^{ac}g^{bd}\right)\mathcal{D}_\alpha g_{ab}\mathcal{D}_\beta g_{cd} \\
&+ \frac{1}{2\kappa^2}\int dv \left(h^{\beta\mu}h^{\gamma[\rho}h^{\alpha]\nu} - \frac{1}{2}h^{\alpha\mu}h^{\beta[\nu}h^{\gamma]\rho}\right)\mathcal{D}_\alpha h_{\beta\gamma}\mathcal{D}_\mu h_{\nu\rho} \\
&+ \frac{1}{4\kappa^2}\int dv f\left[g^{ab}h^{\alpha[\beta}h^{\mu]\nu}\hat{\nabla}_a h_{\alpha\beta}\hat{\nabla}_b h_{\mu\nu} - h^{\alpha\beta}\left(\frac{1}{2}g^{ab}g^{cd} + g^{ac}g^{bd}\right)\hat{\nabla}_a h_{\alpha\beta}\hat{\nabla}_b g_{cd}\right. \\
&\left. + \left(g^{pt}g^{qu}g^{rs} - \frac{1}{2}g^{ps}g^{qt}g^{ru} + g^{pr}g^{qu}g^{st}\right)\hat{\nabla}_r g_{pq}\hat{\nabla}_u g_{st}\right] - \frac{1}{8\kappa^2}\int dv f^{-1}\left(\mathcal{F}^a_{\mu\nu}\right)^2 \\
&- \frac{1}{24\kappa^2}\int dv \left[\left(\mathcal{D}_\mu \mathtt{C}_{abc} - 3\partial_{[a}\mathtt{C}_{bc]\mu}\right)^2 + 4f\left(\hat{\nabla}_{[a}\mathtt{C}_{bcd]}\right)^2\right] \\
&- \frac{1}{16\kappa^2}\int dv f^{-1}\left(\mathcal{F}_{\mu\nu ab} + \mathcal{F}^c_{\mu\nu}\mathtt{C}_{abc}\right)^2 - \frac{1}{24\kappa^2}\int dv\left[f^{-2}\left(\mathtt{F}_{\mu\nu\rho a}\right)^2 + \frac{f^{-3}}{4}\left(\mathtt{F}_{\mu\nu\rho\sigma}\right)^2\right] \\
&- \frac{55}{2^8 3^3 \kappa^2}\int dx^{\mu\nu\rho\sigma}dy^{abcdefg}\mathtt{F}_{[\mu\nu\rho\sigma}\mathtt{F}_{abcd}\mathtt{C}_{efg]}.
\end{aligned}
\tag{4.1}
$$

$\alpha, \beta, \ldots$ are space-time indices

a,b,c,… are 7d indices

K. B, M. Becker, D. Robbins, 1412.8198

Goal: Write this Action in Superspace

Kinetic Terms

Use the Kaehler potential

$$K = -\frac{3}{\kappa^2}\int_M d^7 y \sqrt{g(F)}$$

$$g_{ab} = g_{ab}[\varphi] = (\det s)^{-1/9} s_{ab}$$

Metric

$$s_{ab} = -\frac{1}{144}\varphi_{amn}\varphi_{bpq}\varphi_{rst}\varepsilon^{mnpqrst}$$

F is a real superfield whose bottom components is φ

$$F_{abc} = \frac{1}{2i}\left(\Phi_{abc} - \bar{\Phi}_{abc}\right) - 3\partial_{[a}V_{bc]}$$

Real superfield for $C_{ab\mu}$

The kinetic terms obtained from M-theory compactification are

projections on different G_2 irreps

$$S_{kin} = \frac{1}{24\kappa^2}\int\sqrt{g}\left[\frac{4}{3}\left(\pi_1\partial_\mu\varphi\right)^2 - \left(\pi_{27}\partial_\mu\varphi\right)^2\right] +$$

$$\frac{1}{24\kappa^2}\int\sqrt{g}\left\{-\left[\pi_1(\partial_\mu C - 3\partial C_\mu)\right]^2 - \left[\pi_7(\partial_\mu C - 3\partial C_\mu)\right]^2 - \left[\pi_{27}(\partial_\mu C - 3\partial C_\mu)\right]^2\right\}$$

Expanding in components the kinetic terms obtained from superspace are

$$S_{kin} = \frac{1}{24\kappa^2}\int\sqrt{g}\left[\frac{4}{3}\left(\pi_1\partial_\mu\varphi\right)^2 + \left(\pi_7\partial_\mu\varphi\right)^2 - \left(\pi_{27}\partial_\mu\varphi\right)^2\right] +$$

$$\frac{1}{24\kappa^2}\int\sqrt{g}\left\{-\left[\pi_1(\partial_\mu C - 3\partial C_\mu)\right]^2 + \left[\pi_7(\partial_\mu C - 3\partial C_\mu)\right]^2 - \left[\pi_{27}(\partial_\mu C - 3\partial C_\mu)\right]^2\right\}$$

This coefficient only agrees after integrating out auxiliary fields in the gravity multiplet.

Potential

The potential for the scalar from the metric can be nicely expressed in terms of torsion classes

Scalar curvature of a G_2 structure manifold (Bryant)

$$S_{pot} = \frac{1}{2\kappa^2}\int d^7y\sqrt{g}\left(\frac{21}{8}|\tau_0|^2 + 30|\tau_1|^2 - \frac{1}{2}|\tau_3|^2 - \frac{1}{2}|\tau_2|^2\right)$$

Get contributions from the superpotential

$$W = \pm\frac{1}{8\kappa^2}\int \varphi d\varphi$$

Get contributions from the superpotential and from integrating out D_{ab} which is the auxiliary field in the real superfield for $C_{ab\mu}$

This result agrees precisely with the superspace result!

Tensor Hierarchy and Chern-Simons Actions in Superspace

References:
K. B., M. Becker, W. D. Linch and D. Robbins, 1601.03066, 1603.07362

1) In the first paper we embedded the tensor hierarchy consisting of all fields descending from the M-theory three-form

$$C_{MNP} \to C_{abc}, C_{ab\mu}, C_{a\mu\nu}, C_{\mu\nu\rho}$$

…and the corresponding abelian gauge transformations

$$\delta C = d\Lambda$$

into superspace.

We explicitly constructed the supersymmetrized Chern-Simons action…

$$S = -\frac{1}{12\kappa^2}\mathrm{Re}[i\int d^4x d^2\theta(\boxed{2\Phi EG + \Phi W^\alpha W_\alpha} + 2\Sigma^\alpha E W_\alpha) -$$

$$\frac{1}{12\kappa^2}\int d^4x d^4\theta[-2\hat{\Phi}UH + V\hat{E}H + (VD^\alpha U - D^\alpha VU)W_\alpha +$$

$$(V\bar{D}_{\dot\alpha}U - \bar{D}_{\dot\alpha}VU)\bar{W}^{\dot\alpha} - \Sigma^\alpha U D_\alpha U - \bar{\Sigma}_{\dot\alpha}U\bar{D}^{\dot\alpha}U - X\hat{E}U]$$

2) In the second paper we coupled this system to the non-abelian gauge field arising from the metric.

Stay Tuned! More To Come...