Space-Time, Quantum Mechanics

and

Scattering Amplitudes

Nima Arkani-Hamed
Strings 2013

The last year has seen a flurry of activity in

Computing $\longleftrightarrow$ understanding

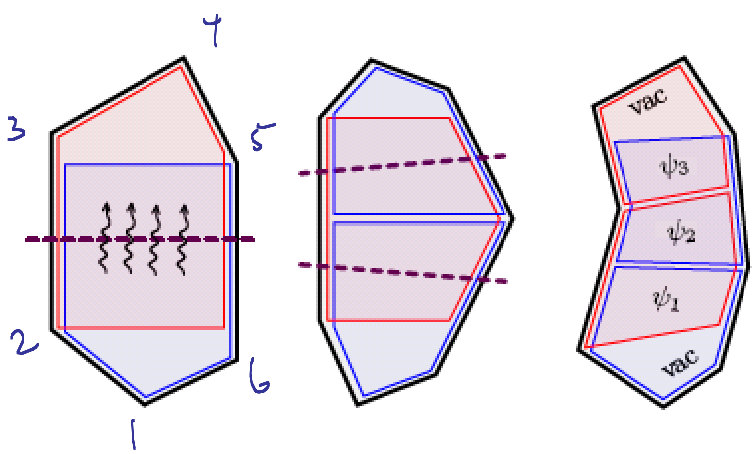
scattering amplitudes.

Some Highlights ...

Tour-de-Force Computations

* Multi-Regge limit of $\mathcal{N}=4$ SYM [Dixon, Duhr; Pennington]
  to 10 Loops, then all loops!
* 6D SYM diverges @ 6-loops
  [Bern, Carrasco, Dixon, Douglas, von Hippel, Johansson]
  (Michael's talk)
* Surprisingly [?] good behavior of
  $\mathcal{N}<8$ SUGRA [Bern, Davies, Dennen, Huang]

*Breakthrough in $\mathcal{N}=4$ SYM:
using integrability to compute amps
@ finite coupling



[Basso, Sever, Vieira]

(Pedro's talk)

"Pentagon Transitions" as building blocks

* Breakthrough in formulating $\mathcal{N}=8$ SUGRA as a novel string theory in Twistor Space

$$\mathcal{M}_n = \sum_{d=0}^{\infty} \int \frac{\prod_{\mathrm{a}=0}^{d} \mathrm{d}^{4|8}\mathcal{Z}_{\mathrm{a}}}{\mathrm{vol}\,\mathrm{GL}(2;\mathbb{C})} \det{}'(\Phi)\det{}'(\widetilde{\Phi}) \prod_{i=1}^{n} \mathrm{d}^2\sigma_i\, \delta^2(\lambda_i - \lambda(\sigma_i)) \exp[\![\mu(\sigma_i)\tilde{\lambda}_i]\!]$$

$$\Phi_{ij} = \frac{\langle ij\rangle}{(ij)} \qquad \text{for } i \neq j, \qquad\qquad \widetilde{\Phi}_{ij} = \frac{[ij]}{(ij)} \qquad \text{for } i \neq j,$$

$$\Phi_{ii} = -\sum_{j\neq i} \left\{ \Phi_{ij} \frac{\prod_{k\neq i}(ik)}{\prod_{l\neq j}(jl)} \prod_{\mathrm{a}=0}^{n-d-2} \frac{(jp_{\mathrm{a}})}{(ip_{\mathrm{a}})} \right\}, \qquad\qquad \widetilde{\Phi}_{ii} = -\sum_{j\neq i} \widetilde{\Phi}_{ij} \prod_{\mathrm{a}=0}^{d} \frac{(j\,p_{\mathrm{a}})}{(i\,p_{\mathrm{a}})},$$

[Hodges, Cachazo-Geyer, Cachazo-Skinner; + Mason; Cachazo; Skinner]

(David's talk)

Beginning to expose rich properties of string pert. theory:

* Trees for all n [Mafra, Schlotterer, Stieberger]

* String YM amps as Mellin transform of SUGRA amps [?!] [Stieberger, Taylor]

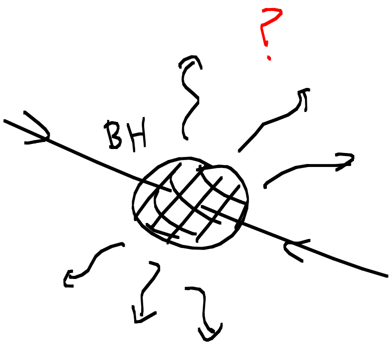
* "Motivic" structure of string trees [Schlotterer, Stieberger; Drummond, Ragoucy]

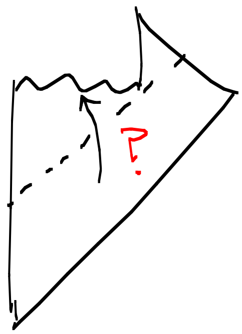
Instead of "reviewing everything" [impossible], I will talk about a broad theme unifying a number of threads of progress:

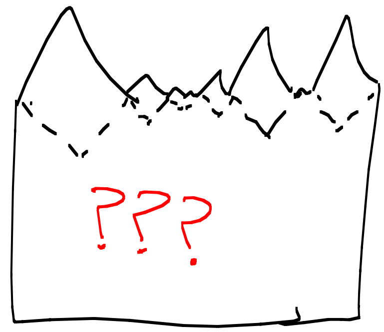
Deeper appreciation for power of
Deeper understanding of new origins for

Locality + Unitarity

Motivation







Emergent Space-Time

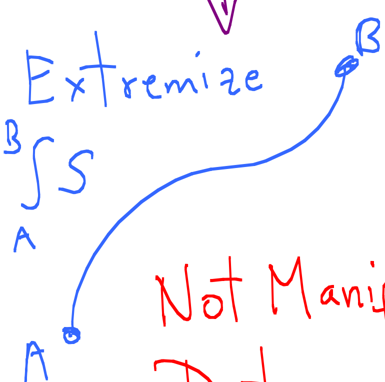
Emergent QM(?)

Classical $\longrightarrow$ Quantum

$F = ma$: Manifestly Deterministic

Reformulate

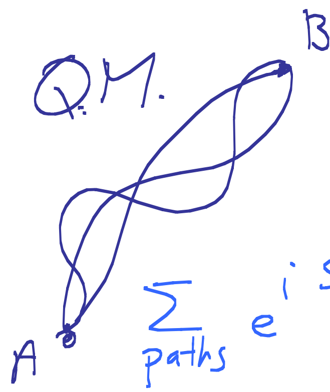
Extremize $\int_A^B S$



Not Manifestly Deterministic

Deform $\longrightarrow$

Q.M.



$\sum_{\text{paths}} e^{iS/\hbar}$

Strategy

1. Reformulate QFT, Eviscerating Locality + Unitarity → see them arise as emergent phenomena

2. Find Natural Deformation

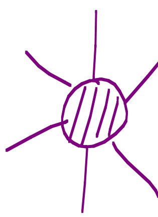
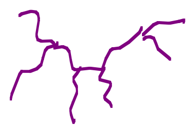
Step 1

0. Planar $\mathcal{N}=4$ SYM
1. Non-planar
2. Non-supersymmetric
3. Gravity
4. Perturbative Strings
5. AdS/CFT

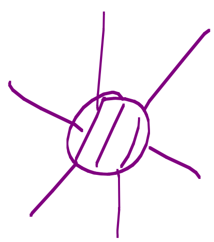
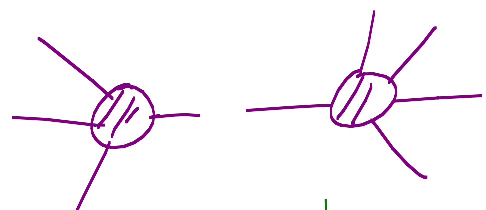
⋮

Locality + Unitarity

* Obviously, hard-wired into usual QFT formalism

e.g. trees  $= \sum_i$  makes it obvious:

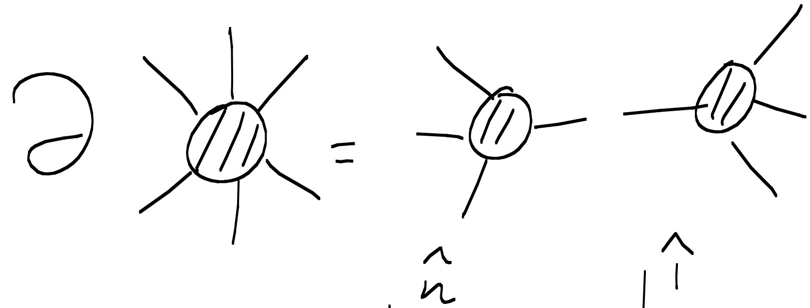
Locality: Poles where $\left(\sum_{L} p_i\right)^2 \to 0$

Unitarity: ∂  $=$ 

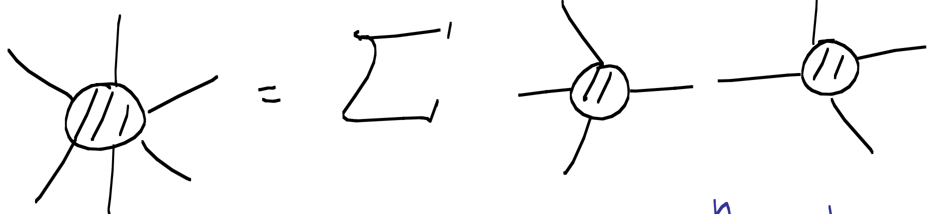
Factorization

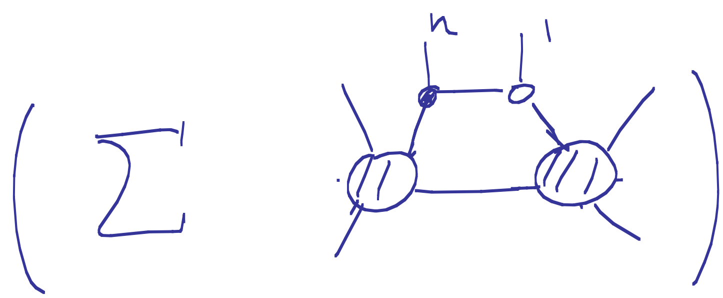
Power of Exploiting singularities

BCFW



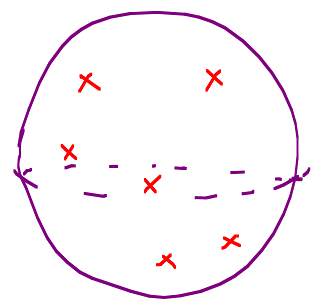
$\longrightarrow$

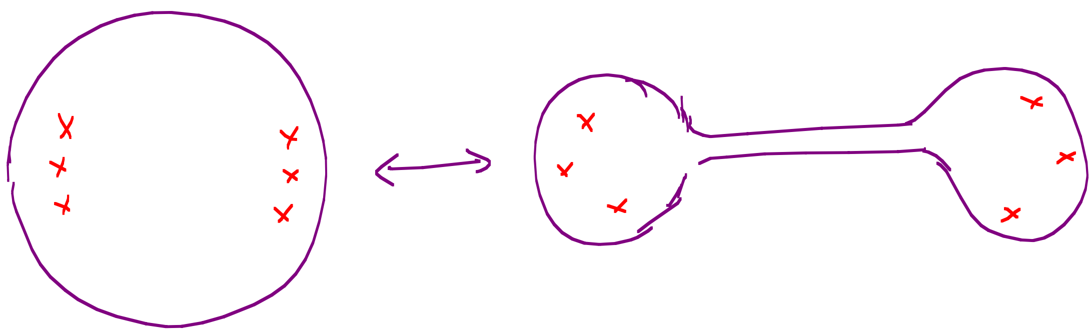




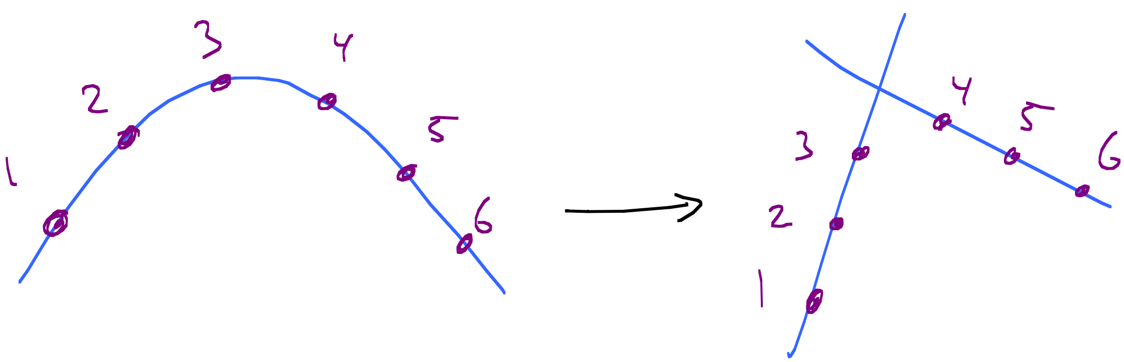
Other Representations

* Classic Stringy





* Twistor Stringy [Witten]



[To get exactly right singularities is non-trivial, what are rules in twistor space? Is there twistor-string for eg ϕ^3 theory?]

At loop level: textbook loc. + unit. about positions of branch-cuts + discontinuity structure.

TO EXPLOIT

* 60's Weinberg comment "hopelessly advanced complex analysis"

* Today: some exists! "Symbology" [Goncharov, Spradlin, Vergu, Volovich.]

* OPE's
* Collinear Limits
* Integrability, Grassmannian Structure, ...

$\downarrow$

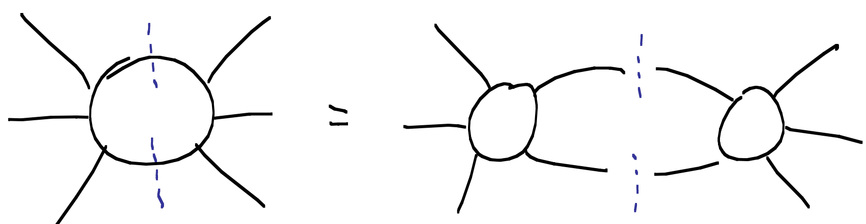
"Bootstrap" for planar $\mathcal{N}=4$ SYM

[Alday, Maldacena, Gaiotto, Sever, Vieira]

[Caron-Huot, He]

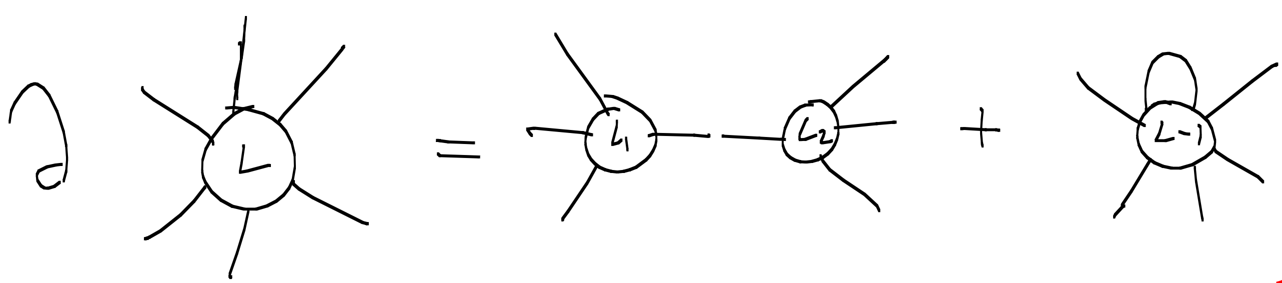
[Basso, Sever, Vieira]

L + U have an even simpler avatar in "cut-structure" of "integrand" for amplitudes

e.g. 

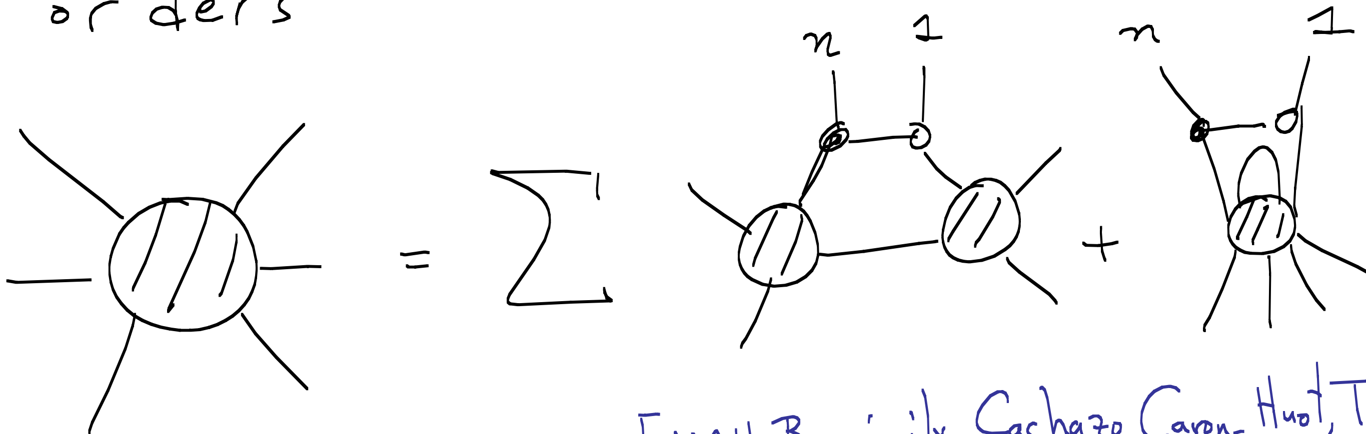
+ powerful "generalized" Unitarity [Bern Dixon Dunbar Kosower]

In planar $\mathcal{N} \geq 2$ theories,
most transparent + sharp statement
for THE integrand:



{Famous Structure in Many Settings!}

Can be EXPLOITED to recursively determine integrand to all-loop orders



[NAH, Bourjaily, Cachazo, Caron-Huot, Trnka]

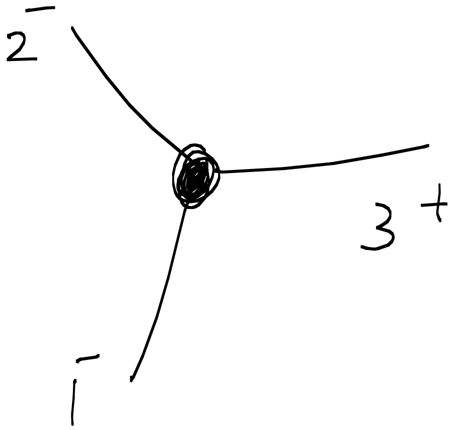
Planar $\mathcal{N}=4$ SYM

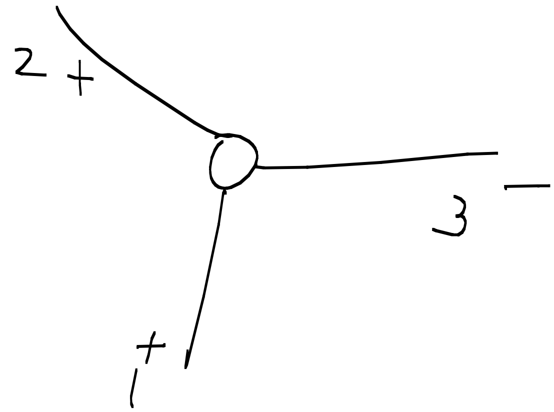
Kinematics I

- Massless $p_a^{\alpha\dot{\alpha}} = \lambda_a^{\alpha} \tilde{\lambda}_a^{\dot{\alpha}}$
- Color-stripped $\mathcal{M}_n[\lambda_a, \tilde{\lambda}_a; h_a]$
- Maximal SUSY: on-shell superspace

$$|\tilde{\eta}\rangle = |+1\rangle + \tilde{\eta}^I |{+\tfrac{1}{2}}\rangle_I + \frac{\tilde{\eta}^I \tilde{\eta}^J}{2!} |0\rangle_{IJ} + \cdots + \tilde{\eta}^4 |{-1}\rangle$$

- $\mathcal{M}_{n,\hat{k}}[\lambda_a, \tilde{\lambda}_a, \tilde{\eta}_a]$: cyclically invariant





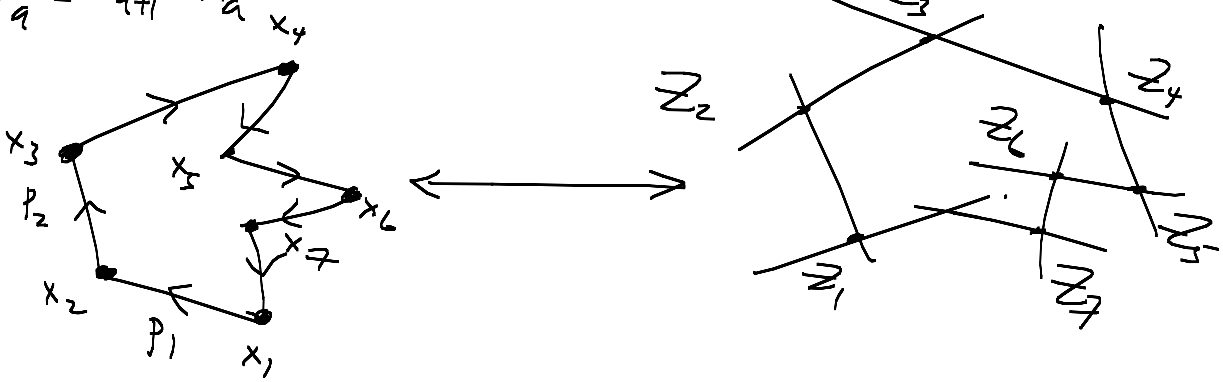
$$\tilde{\lambda}_1 \propto \tilde{\lambda}_2 \propto \tilde{\lambda}_3 \qquad\qquad \lambda_1 \propto \lambda_2 \propto \lambda_3$$

$$\frac{\langle 12\rangle^3}{\langle 13\rangle\langle 23\rangle} \qquad\qquad \frac{[12]^3}{[13]\,[23]}$$

Nailed By Poincaré

Kinematics II

$$p_a^\mu = x_{a+1}^\mu - x_a^\mu$$



[Hodges]

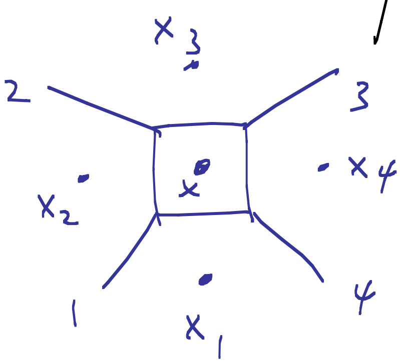
$$Z_a = \begin{pmatrix} \lambda_a \\ \mu_a \\ \hdashline \eta_a \end{pmatrix} \qquad Z_a \sim t_a Z_a$$

"momentum twistors"

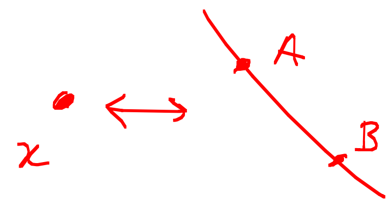
$$\tilde{\lambda}_a = \frac{\langle a-1\, a\rangle \mu_{a+1} + \langle a+1\, a-1\rangle \mu_a + \langle a\, a+1\rangle \mu_{a-1}}{\langle a-1\, a\rangle \langle a\, a+1\rangle}$$

Kinematics III

Planar Loop Integrand:

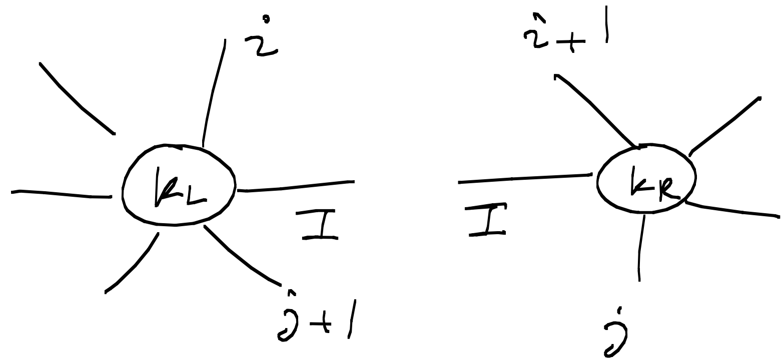


$$\frac{d^4x}{(x-x_1)^2 \cdots (x-x_4)^2}$$



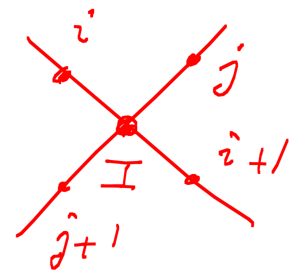
$$\mathcal{M}_{n,\hat{k},\ell} = \frac{\delta^4\left(\sum p_a\right)\delta^8\left(\sum_a \lambda_a \tilde{\eta}_a\right)}{\langle 12\rangle\langle 23\rangle \cdots \langle n1\rangle}$$

$$\times \mathcal{M}_{n,k,\ell}\left[Z_a ; AB_1, AB_2, \ldots AB_\ell\right]$$



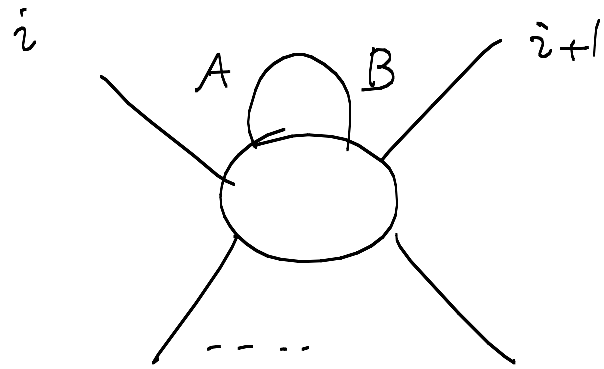
$k_L + k_R = k - 1$

Pole where $\langle i\ i{+}1\ j\ j{+}1 \rangle \longrightarrow 0$

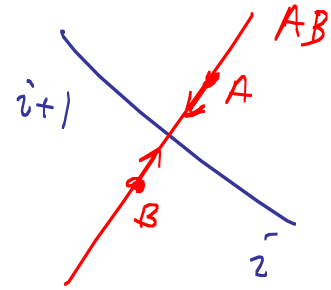


Residue:

$$\{i\ i{+}1\ j\ j{+}1\} \times M_L[j{+}1, \dots, i, I] \times M_R[I, i{+}1, \dots, j]$$

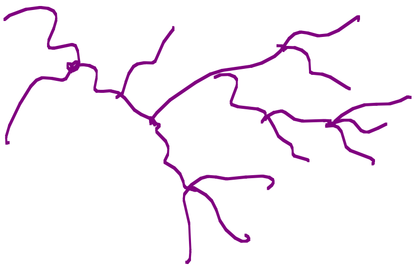


Pole when $\langle AB\, i\, i{+}1 \rangle \longrightarrow 0$

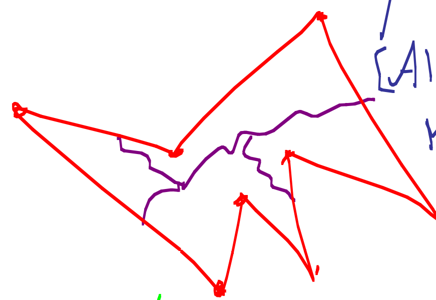


Residue : $\displaystyle\int_{GL(2)} d^{4|4}Z_A\, d^{4|4}Z_B\, \mathcal{M}_{\ell-1}[A, B, \ldots]$

"Amplitude"



"Wilson-Loop"

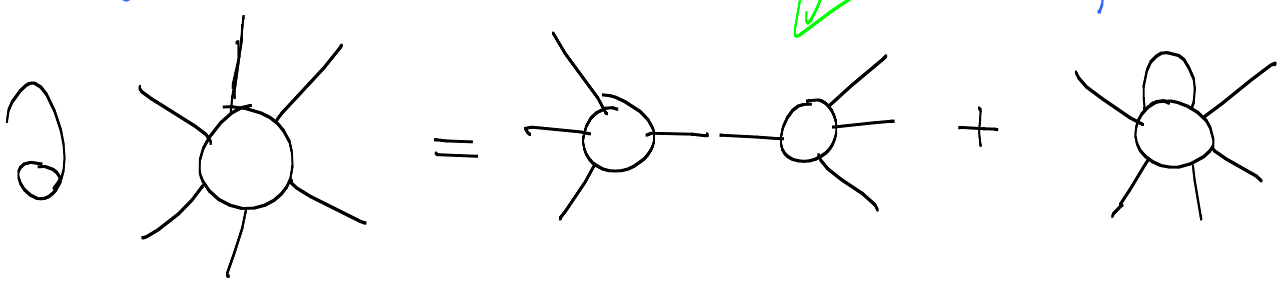


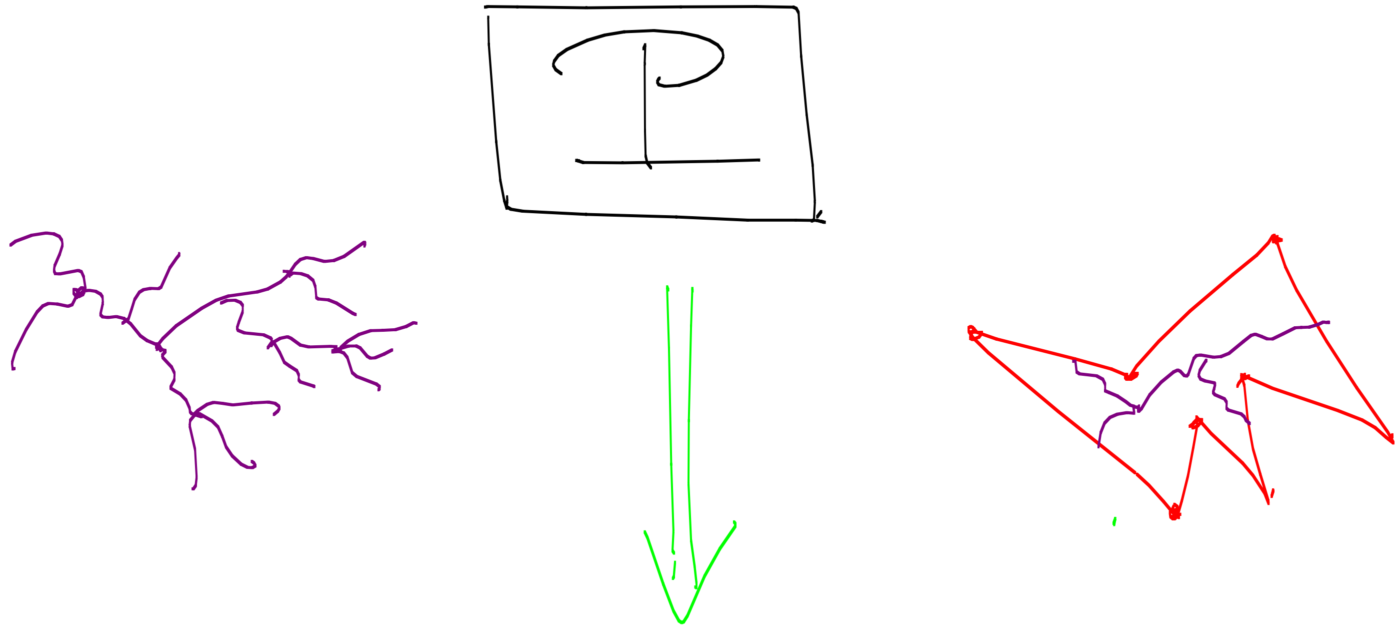
[Alday-Maldacena, Mason-Skinner, Caron-Huot]

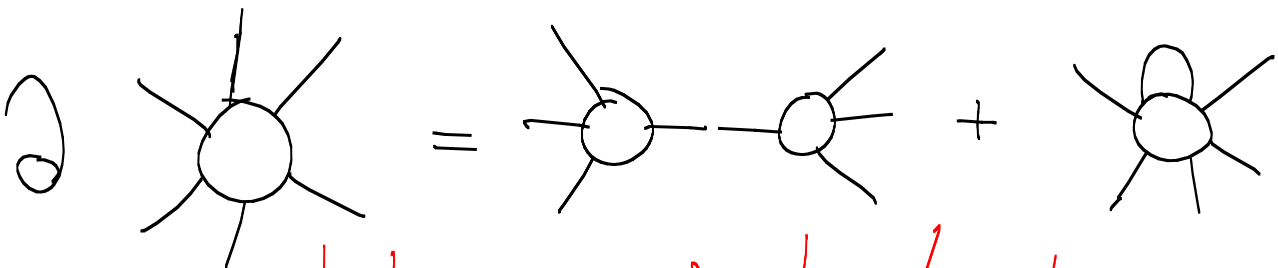
Conformal Manifest

Dual Conformal Manifest

[Drummond Henn Korchemsky Sokatchev]



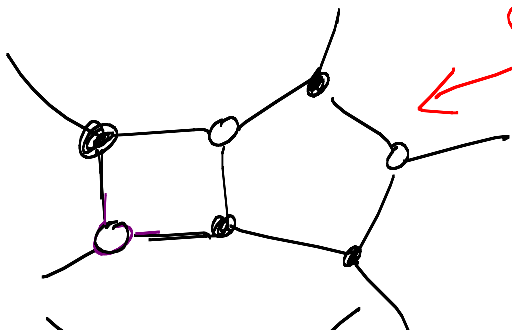


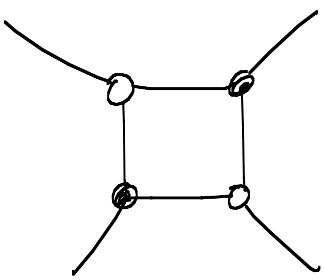


- Locality + Unitarity Emergent
- Infinite Yangian Symmetry Manifest

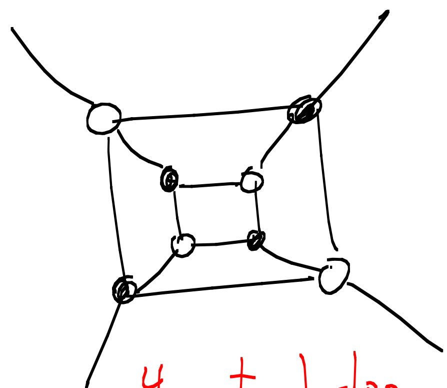
On-Shell Diagrams

All internal lines <u>on-shell</u>





4 pt tree



4-pt 1-loop

[NAH
Bourjaily
Cachazo
Goncharov
Postnikov
Trnka
strings '12]

NO "VIRTUAL PARTICLES"

Positive Grassmannian

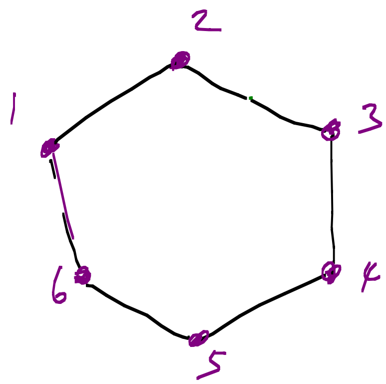
$G(k,n)$: k-planes in n-dimensions

$$C_{\alpha a} = \begin{array}{c} \\ k \end{array} \overset{\longleftarrow\; n \;\longrightarrow}{\begin{pmatrix} c_1 & c_2 & \cdots & c_n \end{pmatrix}} \Big/ GL(k)$$

$G^+(k,n)$: all minors $(a_1 \cdots a_k) > 0$

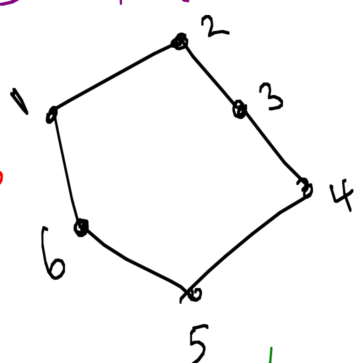
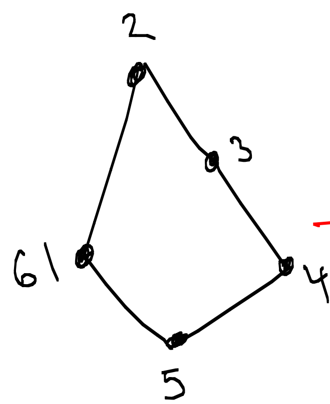
for $a_1 < a_2 \cdots < a_k$

$G^+(3,n)$



$\longrightarrow$ Convex Polygon

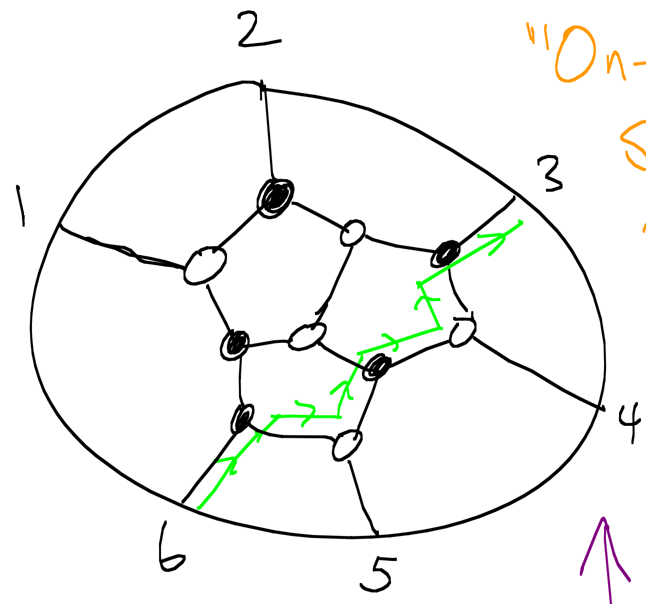
Boundaries:

$\longrightarrow$  $\longrightarrow$  $\longrightarrow$ ...

$\longrightarrow$ Consecutive linear dependencies $\longrightarrow$ Permutations

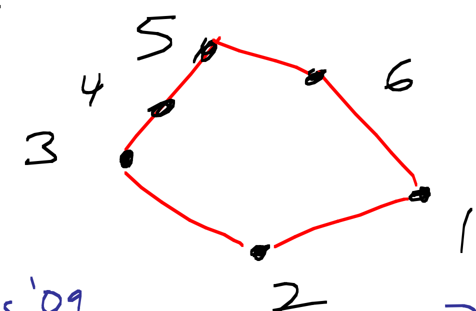
$1 \to 4$
$2 \to 6$
$3 \to 5$
$4 \to 7$
$5 \to 8$
$6 \to 9$

Affine Permutation



"On-shell" Spacetime Picture

Cell of Positive Grassmannian



[NAH, Cachazo, Cheung, Kaplan strings '09
NAH, Bourjaily, Cachazo, Goncharov, Postnikov, Trnka strings '12]

Yangian Invariance

$\uparrow$

Positive Diffeomorphisms

* So: a beautiful story for building blocks of amplitude

– but why do we combine them as dictated by loop BCFW recursion?

To make result Local + Unitary!

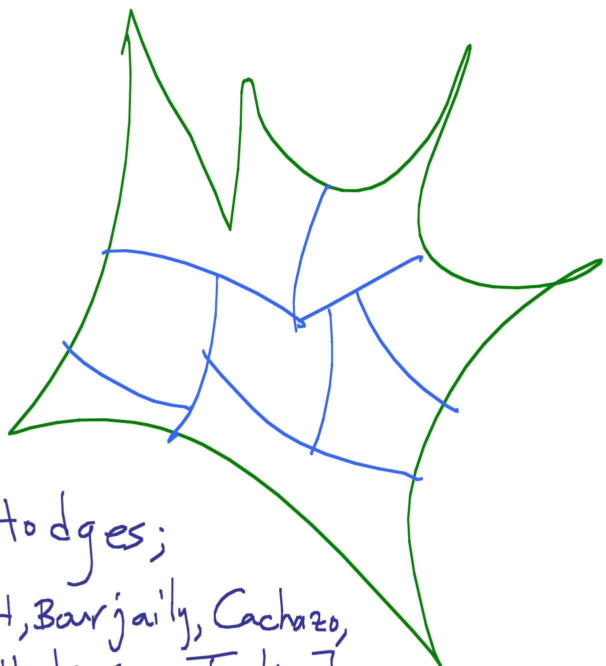
- Simplest example: NMHV tree

$$[1\,2\,3\,4\,5] = \int \frac{dc_1}{c_1} \cdots \frac{dc_5}{c_5} \Big/ GL(1) \; \delta^{4|4}\left(c_1 Z_1 + \cdots + c_5 Z_5\right) = \frac{\delta^4\left(\langle 1234\rangle \eta_5 + \cdots\right)}{\langle 1234\rangle \cdots \langle 5123\rangle}$$

- $$\mathcal{M}^{\text{tree}}_{\text{BCFW}} = \sum_{i<j} [1\; i\; i{+}1\; j\; j{+}1]$$

[WHY THIS COMBINATION?]

Vague Fantasy



[Hodges;
NAH, Bourjaily, Cachazo,
Hodges + Trnka]

Amplitude is "the volume" of "some region" in "some space".
Many "triangulations".

Locality + Unitarity Emerge Somehow

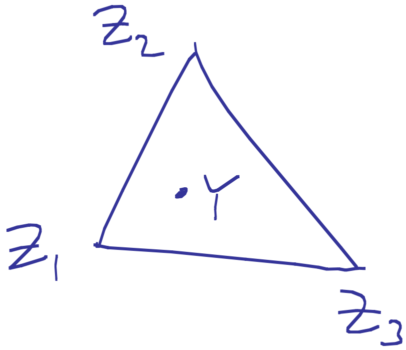
Locality + Unitarity

from

Positivity

w/ Jaroslav Trnka
to appear soon

Triangles $\longrightarrow$ Positive Grassmannian



$$Y^I = c_1 Z_1^I + c_2 Z_2^I + c_3 Z_3^I$$

$(c_1 c_2 c_3)/GL(1)$, $c_a > 0$

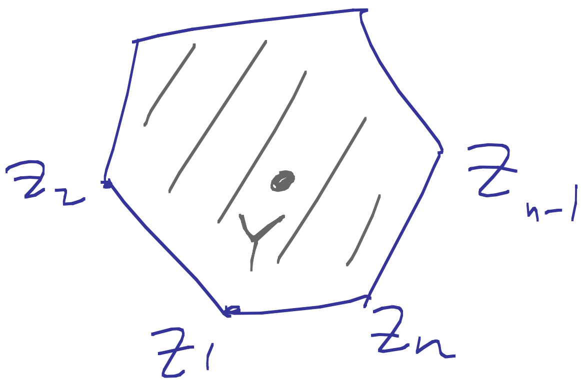
$\longrightarrow$ $(c_1 \dots c_n)/GL(1)$ $\quad c_a > 0$ $\quad$ Simplex

$\longrightarrow$ $(\vec{C}_1 \dots \vec{C}_n)/GL(k)$, $(c_{a_1} \dots c_{a_k}) > 0$

$a_1 < \dots < a_k$

Positive Grassmannian

Polygons



$$\langle Z_{a_1} Z_{a_2} Z_{a_3} \rangle > 0$$

$$a_1 < a_2 < a_3$$

key point

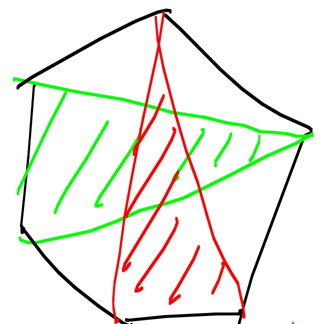
$$Y^I = c_1 Z_1^I + \dots + c_n Z_n^I$$

$$\left.\begin{array}{l}(c_1 \dots c_n) \in G^+(1,n) \\ (Z_1, \dots, Z_n) \in G^+(3,n)\end{array}\right. \longrightarrow \text{Region in } G(1,3)$$

$$Y^I = c_a Z_a^I$$

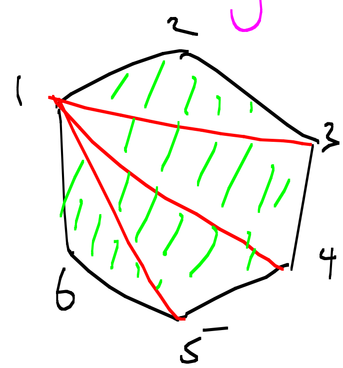
2D $\quad$ $(n-1)$-d

* 2D cells of $G^+(1,n) \longrightarrow$

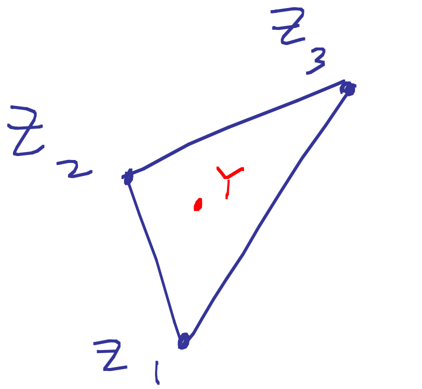


"Triangles"

* Triangulation, e.g $\sum_i (1\ i\ i+1)$

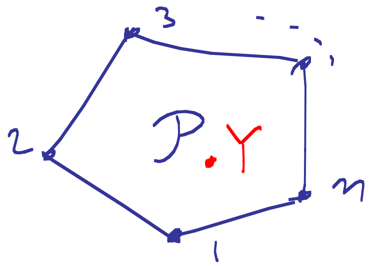


"Volume" as a Form



$$\Omega = \frac{\langle Y\,dY\,dY\rangle\,\langle 123\rangle^2}{\langle Y12\rangle\langle Y23\rangle\langle Y31\rangle}$$

$$= \frac{dx}{x}\frac{dy}{y}\,, \quad Y = Z_1 + xZ_2 + yZ_3$$



Ω_P : Form with log. sing. on ∂P. Immediate from triangulation.

[Form Ω_P makes sense for general complex Z's]

First Generalization

$$\boxed{Y^I = c_a Z_a^I}$$

$I = 1, \ldots, l+m$

$\uparrow$
even

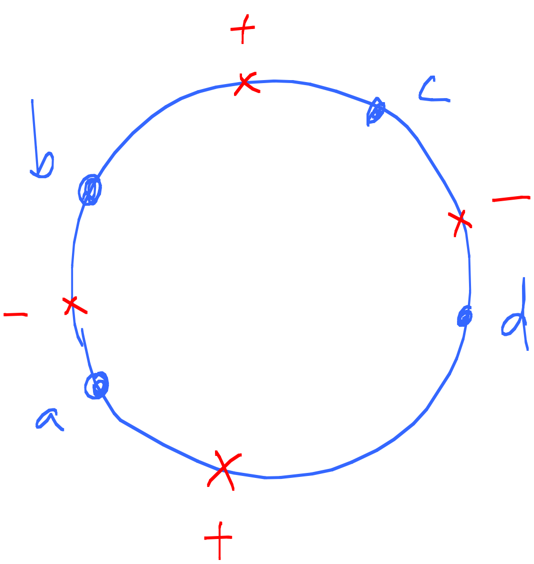
- $Z \in G^+(l+m, n)$, $c \in G^+(l, n)$

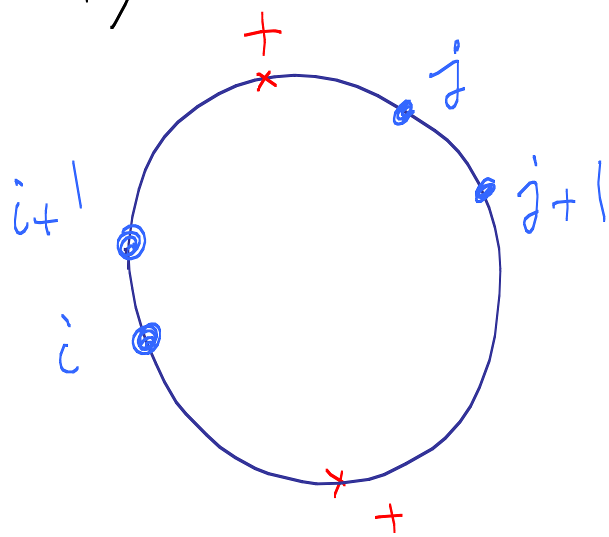
- Polygon: $m = 2$

  First new case: $m = 4$

Boundaries

$$\langle Y Z_a Z_b Z_c Z_d \rangle = c_1 \langle 1abcd \rangle + \cdots + c_n \langle nabcd \rangle$$





Boundaries are $\left| Z_i Z_{i+1} Z_j Z_{j+1} \right|$!

- Triangulation:

$$\sum_{i<j} \langle 1\, i\, i{+}1\, j\, j{+}1 \rangle$$

- In fact: $\mathcal{M}_{k=1}$ is "the volume" of this region

- Locality From Positivity

Generalization

$$\boxed{Y_\alpha^I = C_{\alpha a} Z_a^I}$$

"Polygon" in $G(K,K+4)$

Positive Grassmannian $G^+(k,n)$

External Data in $G^+(4+K,n)$

New treatment of SUSY

$$\underset{\mathbb{P}^{3|4}}{\begin{pmatrix} Z_a \\ \hline \eta_a \end{pmatrix}} \Bigg| \quad \underset{Y}{\begin{array}{c|c} & k \\ 4 & \begin{pmatrix} 0 \\ \hline 1 \\ & 1 \\ & & 1 \\ & & & 1 \end{pmatrix} \\ k & \end{array}} , \quad \underset{\mathbb{P}^{3+k}}{\begin{pmatrix} Z_a \\ \hline \phi_1 \cdot \eta_a \\ \vdots \\ \phi_k \cdot \eta_a \end{pmatrix}} = \mathcal{Z}_a$$

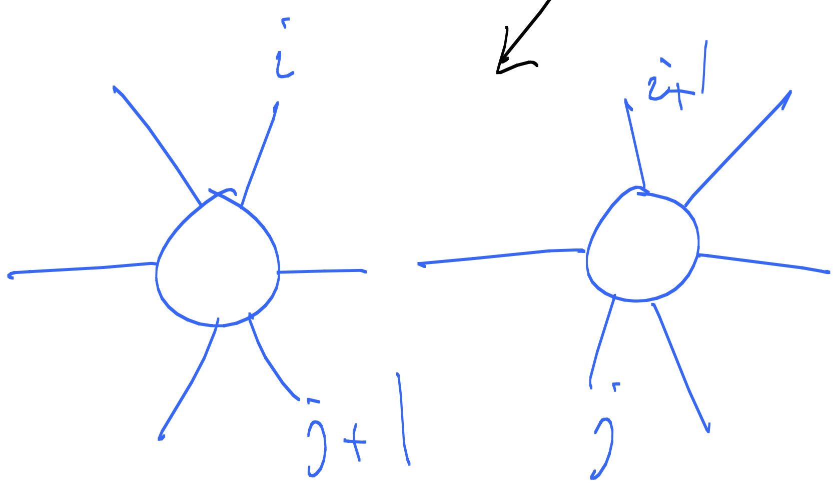
Super-Geometry $\longrightarrow$ Bosonic Geometry

$$\mathcal{M}_{n,k}[Z_a, \eta_a] = \int d^4\phi_1 \ldots d^4\phi_k \, M[\mathcal{Z}_a]$$

- Tree amplitude is "volume" of this region!
- BCFW is (one) "triangulation" [cell decomposition]

Boundaries of $P_{n,k}$

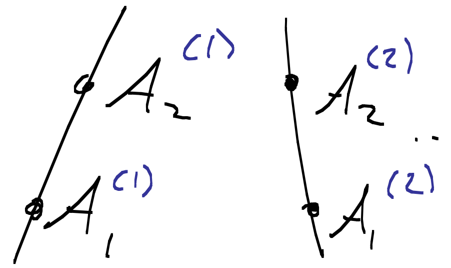
$$C = \begin{array}{c} \begin{array}{cccccccccc} i & i{+}1 & & & & & j & j{+}1 & & \end{array} \\ \left[\begin{array}{cccccccccc} * & * & 0 & 0 & 0 & 0 & * & * & 0 & 0 \\ \multicolumn{5}{c}{C_L} & \multicolumn{5}{c}{0} \\ \multicolumn{5}{c}{0} & \multicolumn{5}{c}{C_R} \end{array}\right] \end{array}$$



Unitarity From Positivity

Loops

$k=0$: Lines in $\mathbb{P}^3$



$$A_\gamma^{I\,(i)} = Z^{(i)}_{\gamma a} Z_a^{I+}$$

$$\begin{bmatrix} Z^{(1)} \\ \hline Z^{(2)} \\ \hline \vdots \\ Z^{(L)} \end{bmatrix}$$

All minors Positive!

Motivated by "Hiding Particles"

$$\begin{array}{cccc|c}
A & B & C & D & 1 \quad 2 \quad \cdots \quad n
\end{array}$$

$$\left[\begin{array}{cccc|c}
1 & 0 & 0 & 0 & \\
0 & 1 & 0 & 0 & Z^{(1)} \\
\hline
0 & 0 & 1 & 0 & \\
0 & 0 & 0 & 1 & Z^{(2)}
\end{array}\right]$$

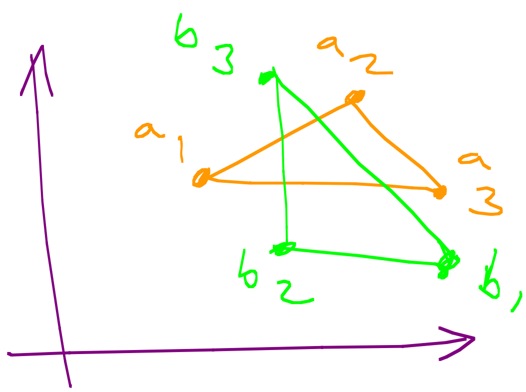
Simplest Case $n=4$

$$Z_{(i)} = \begin{bmatrix} 1 & x_i & 0 & -z_i \\ 0 & y_i & 1 & w_i \end{bmatrix}$$

$$\boxed{\vec{a}_i = \begin{pmatrix} x_i^+ \\ y_i^+ \end{pmatrix}, \quad \vec{b}_i = \begin{pmatrix} w_i^+ \\ z_i^+ \end{pmatrix}, \quad (\vec{a}_i - \vec{a}_j)\cdot(\vec{b}_i - \vec{b}_j) < 0}$$

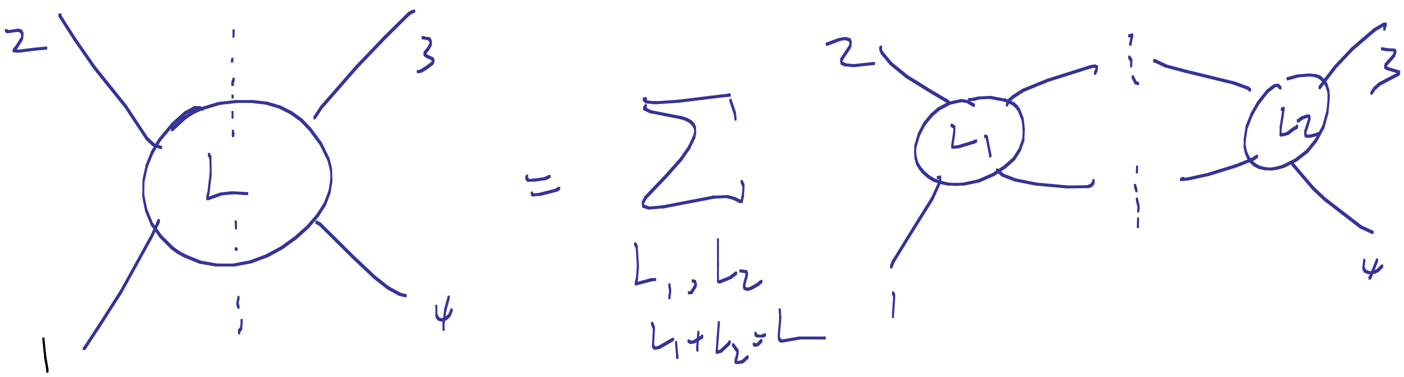
"Triangulate"

= 4-pt to all loop order!



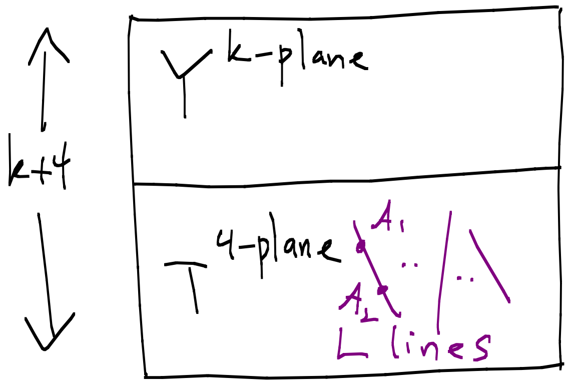
A junior highschool geometry problem

Simple geometric identity from positivity



Textbook Unitarity from Positivity

P is for Positive

- 

$$Y^I_\alpha = C_{\alpha a} Z^{+I}_a$$

$$\mathcal{A}^{(i)}_\gamma = \mathcal{D}^{(i)}_{\gamma a} Z^{+I}_a$$

$$\overset{\leftarrow\; n \;\rightarrow}{\begin{pmatrix} C \\ \hline \mathcal{D}^{(1)} \\ \hline \vdots \\ \mathcal{D}^{(L)} \end{pmatrix}}$$

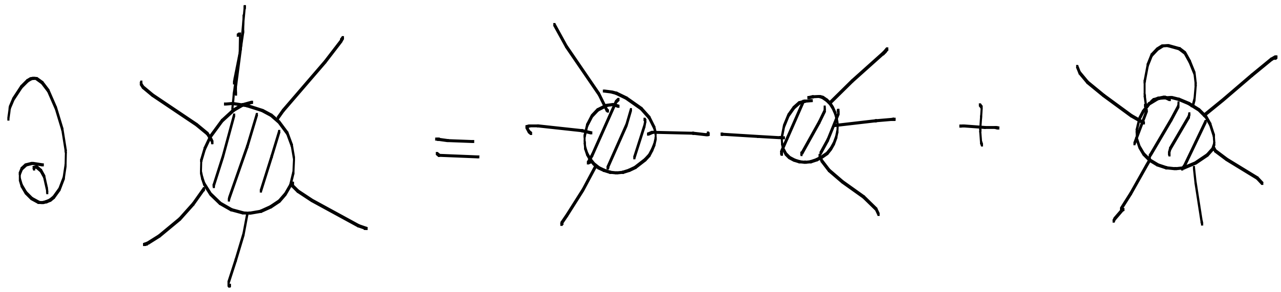
All minors w/ $C > 0$

$\mathcal{P}_{n,k,L}$

- $\Omega_{n,k,L}$: top-form w/ log sing. on $\partial \mathcal{P}_{n,k,L}$

- $$\mathcal{M}_{n,k,L} = \int d^4\phi_1 \cdots d^4\phi_k \, \Omega_{n,k,L} \, \delta^{4k}(Y; Y_0).$$

$$Y_0 = \left(\begin{array}{ccc|c} 1 & & & 0\,0\,0\,0 \\ & \ddots & & \vdots \\ & & 1 & 0\,0\,0\,0 \end{array} \right), \quad Z_a = \begin{pmatrix} Z_a \\ \hdashline \phi_1 \cdot \eta_a \\ \vdots \\ \phi_k \cdot \eta_a \end{pmatrix}$$



Follows From Positive Geometry

- All symmetries manifest
- Determining integrand reduced to "triangulating" $\mathcal{P}_{n,k,L}$

[BCFW one triangulation - not "best" one at tree level]

* This simple mathematical structure gives a complete, autonomous definition of all scattering amplitudes in Planar $\mathcal{N}=4$ SYM, totally free of usual QFT language: no Feynman Diagrams, not even on-shell diagrams, recursion relations etc.

(Momentum) Twistor Space : Kinematics

Grassmannian Positivity : Dynamics

Indication for important role of positivity in final results: fingerprint appearance of "cluster variables"

[Golden, Goncharov, Spradlin, Vergu, Volovich]

We Now Have

A first baby example of Emergent Locality + Unitarity. Unlike conventional formulation of QFT, they don't fight.

Instead emerge together, from Positivity

* This structure is as of yet un-known to the mathematicians (indeed surprising to many that it can even exist).

Expect rich geometrical — combinatorial properties in its cell structure

Still something missing

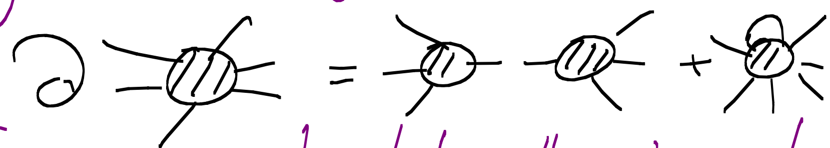
* "Integral representation" for $\Omega_{n,k,L}$

(i.e. direct defn of "volume" without "triangulation")

* With this, suspect we can give non-pert defn – FORMALLY do loop integrals – make contact with AdS/CFT, integrability story etc.

General Comments/ Speculations

Beyond Dualities

* We are finding new reprentations guaranteeing  without manifest "local evolution through spacetime".

* Strings (extra d.o.f)
* Twistor Strings (loops?)
* Grassmannians (general theories?)

* Must be more! Deeper Origin?

Gravity, Strings?

* Unexplained magic even in planar SYM
  (identities between different color orderings,
  Bern-Carrasco-Johansson relations, ..)
* Similar combinatorial properties in BCFW
  (+ twistor-string connection)

Not just "Emergent Spacetime"

Space-Time + Quantum Mechanics should emerge together, inextricably linked [as in our baby $\mathcal{N}=4$ example]