Geometric Hints of Non-perturbative Topological Strings

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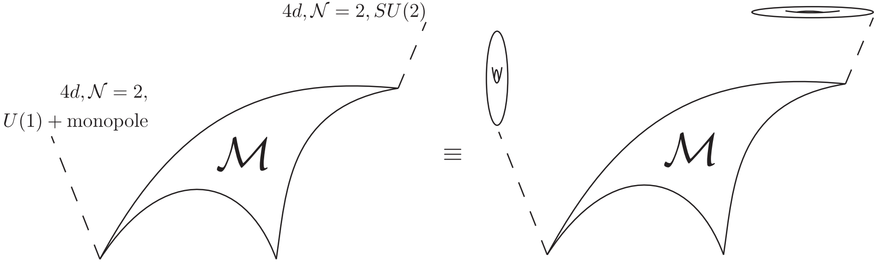
CENTER OF MATHEMATICAL SCIENCES AND APPLICATIONS

This talk is based on work with

- Shing-Tung Yau (Harvard) and Jie Zhou (Perimeter)
- Earlier work with: D. Länge, P. Mayr, H. Movasati, E. Scheidegger

Physical moduli spaces can be geometrized



Seiberg and Witten '94

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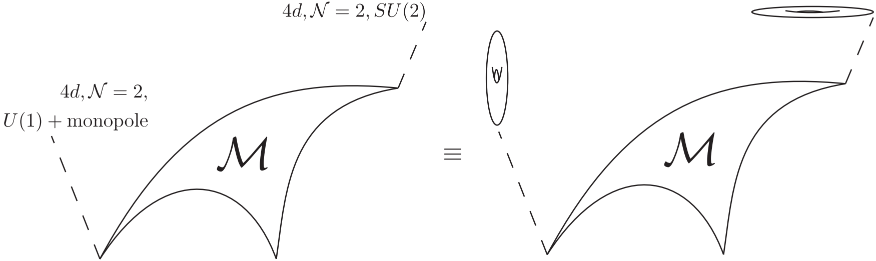

Seiberg and Witten '94

- Can be obtained from type IIB string theory on CY Y
  Klemm, Lerche, Mayr, Vafa, Warner '96

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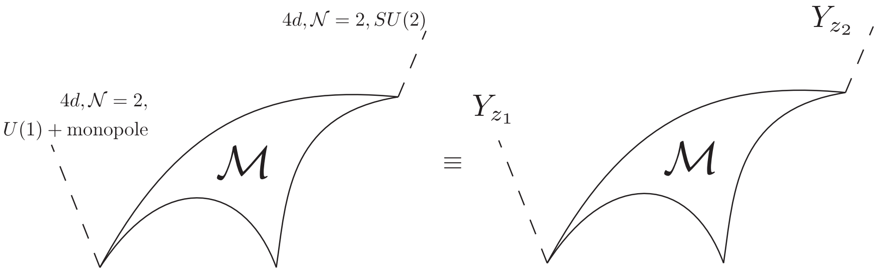

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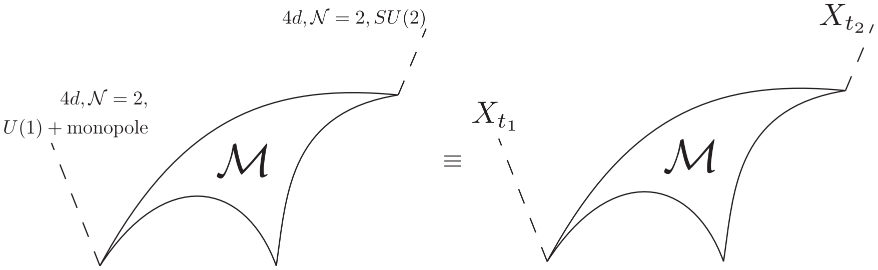


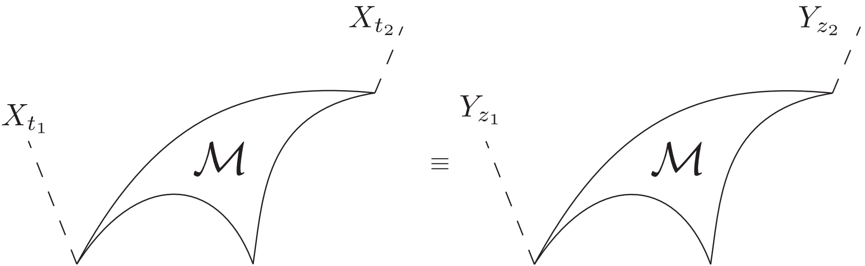
Seiberg and Witten '94

- Can be obtained from type IIB string theory on CY Y

  Klemm, Lerche, Mayr, Vafa, Warner '96
- Can be obtained from type IIA string theory on CY X

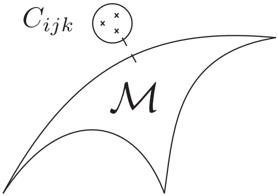
  Katz, Klemm & Vafa '96, Katz, Mayr & Vafa '97

Mirror symmetry puts forward surprising insights into math and physics



- Mirror symmetry maps variations of the symplectic structure of X to variations of the complex structure of Y

Gauge theory data is encoded in 3-point function



- C_{ijk} is the 3-pt correlator on a sphere of an $\mathcal{N} = (2, 2)$ SCFT.
- $C_{ijk} = \eta_{kl} C^{l}_{ij}$ is related chiral ring structure constants $\phi_i \phi_j = C^{l}_{ij} \phi_l$
- The SCFT can be realized as a non-linear sigma model into $X(Y)$ and a topological theory can be defined Witten '88
- C_{ijk} is part of the data of a flat connection on $\mathcal{M}$.

Topological string theory probes higher genus mirror symmetry

- Topological string amplitudes are global objects on the moduli spaces with interesting limits Bershadsky, Cecotti, Ooguri & Vafa (1993)

$$Z = \exp\left(\sum_{g,n} \lambda^{2g-2} \frac{1}{n!} \mathcal{F}^{g}_{i_1\dots i_n}(z; t_*, \bar{t}_*) x^{i_1} \dots x^{i_n}\right) ,$$

- A limit of $\mathcal{F}^g$ encodes generating functions of higher genus Gromov-Witten invariants of X
- The perturbative expansion is asymptotic as it is for *physical* string theory Gross & Periwal (1988), Shenker (1990), lecture notes of Marino (2012)

A non-perturbative formulation of topological strings is needed

- $Z_{DT} = Z_{GW}$ Maulik, Nekrasov, Okounkov & Pandharipande '03
  $Z_{BH} = |Z_{top}|^2$ Ooguri, Strominger & Vafa '04
- Non-perturbative completion from ABJM/spectral theory Marino et al. '10-'16, Grassi, Hatsuda, Marino '14
- Techniques of transseries and resurgence Couso-Santamaria, Edelstein, Schiappa, Vonk '13,'14

A universal, intrinsic differential equation in λ can be obtained

- The special geometry of $\mathcal{M}$ leads to a polynomiality of $\mathcal{F}^g$
- Some universal monomials can be obtained to all genera
- These monomials are governed by a differential equation in λ

Outline

The geometry of $\mathcal{M}$ is special

Special geometry data

- $\mathcal{M}$ is the moduli space of complex (Kähler) structures of CY $Y(X)$
- $z^i, i = 1, \ldots, n = \dim(\mathcal{M})$, local coordinates
- $\mathcal{L} \to \mathcal{M}$ a hermitian line bundle, with metric $e^{-K} = ||\Omega||^2 , \Omega \in \Gamma(\mathcal{L})$
- $G_{i\bar{\jmath}} = \partial_i \partial_{\bar{\jmath}} K$. a Kähler metric
- $C_{ijk} \in \Gamma(\mathcal{L}^2 \otimes \mathrm{Sym}^3 T^* \mathcal{M})$, Yukawa coupling or three-point function

Flatness of tt^* connections gives special geometry

D_i denotes the covariant derivative with the connections

$$\Gamma^k_{ij} = G^{k\bar{k}} \partial_i G_{j\bar{k}} \quad K_i := \partial_i K$$

The action of the chiral and anti-chiral ring give the flat tt^* connection:

$$[\nabla_i, \nabla_{\bar{\jmath}}] = 0\,, \quad \nabla_i = D_i - C_i$$

Curvature has a special form

$$-R_{i\bar{\imath}}{}^l{}_j = [\bar{\partial}_{\bar{\imath}}, D_i]^l{}_j = \bar{\partial}_{\bar{\imath}} \Gamma^l_{ij} = \delta^l_i G_{j\bar{\imath}} + \delta^l_j G_{i\bar{\imath}} - C_{ijk} \overline{C}^{kl}_{\bar{\imath}}\,,$$

Anomaly equation provides recursive information on higher genus amplitudes

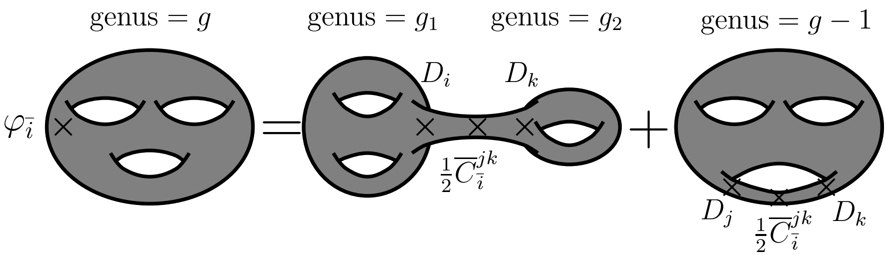
$$Z = \exp\left(\sum_{g,n} \lambda^{2g-2} \frac{1}{n!} \mathcal{F}^{g}_{i_1 \dots i_n}(z; t_*, \bar{t}_*) x^{i_1} \dots x^{i_n}\right)$$

- $\mathcal{F}^{g}_{i_1 \dots i_n} \neq 0$ for $2g - 2 + n > 0 \quad \mathcal{F}^0 := C_{ijk}$
- $\mathcal{F}^{g}_{j i_1 \dots i_n} := D_j \mathcal{F}^{g}_{i_1 \dots i_n}$
- $$\bar{\partial}_{\bar{\imath}} \mathcal{F}^{(1)}_j = \frac{1}{2} C_{jkl} \overline{C}^{kl}_{\bar{\imath}} + (1 - \frac{\chi}{24}) G_{j\bar{\imath}}$$

Anomaly equation provides recursive information on higher genus amplitudes

Anomaly comes from contributions at the boundary of Riemann surface moduli spaces

$$\bar{\partial}_{\bar{i}}\mathcal{F}^{g} = \frac{1}{2}\bar{C}_{\bar{i}}^{jk}\left(\sum_{g_1+g_2=g} D_j\mathcal{F}^{(g_1)}D_k\mathcal{F}^{(g_2)} + D_jD_k\mathcal{F}^{(g-1)}\right),$$



Bershadsky, Cecotti, Ooguri & Vafa (1993)

Recursion leads to Feynman diagrams

$$\overline{C}_{\bar\imath\bar\jmath\bar k} = e^{-2K} D_{\bar\imath} D_{\bar\jmath} \bar\partial_{\bar k} S,$$

$$\partial_{\bar\imath} S^{ij} = \bar{C}_{\bar\imath}^{ij}, \qquad \partial_{\bar\imath} S^{j} = G_{i\bar\imath} S^{ij}, \qquad \partial_{\bar\imath} S = G_{i\bar\imath} S^{i}.$$

One can recursively integrate the anomaly equation:

$$\bar\partial_{\bar\imath} \mathcal{F}^{g} = \frac{1}{2} \bar{C}_{\bar\imath}^{jk} \left(\sum_{g_1+g_2=g} D_j \mathcal{F}^{(g_1)} D_k \mathcal{F}^{(g_2)} + D_j D_k \mathcal{F}^{(g-1)} \right),$$

Iterative use of $[\bar\partial_{\bar\imath}, D_i]^l{}_j$ leads to Feynman diagrams with S^{ij}, S^i, S propagators and $\mathcal{F}^{(g)}_{i_1 \ldots i_n} = D_{i_1} \ldots D_{i_n} \mathcal{F}^{(g)}$ vertices.

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One can recursively integrate the anomaly equation:

$$\bar{\partial}_{\bar{\imath}}\Big(\mathcal{F}^{(g)} - \frac{1}{2} S^{jk} \big(\sum_{g_1+g_2=g} D_j \mathcal{F}^{(g_1)} D_k \mathcal{F}^{(g_2)} + D_j D_k \mathcal{F}^{(g-1)} \big)\Big)$$
$$= -\frac{1}{2} S^{jk} \bar{\partial}_{\bar{\imath}} \big(\sum_{g_1+g_2=g} D_j \mathcal{F}^{(g_1)} D_k \mathcal{F}^{(g_2)} + D_j D_k \mathcal{F}^{(g-1)} \big),$$

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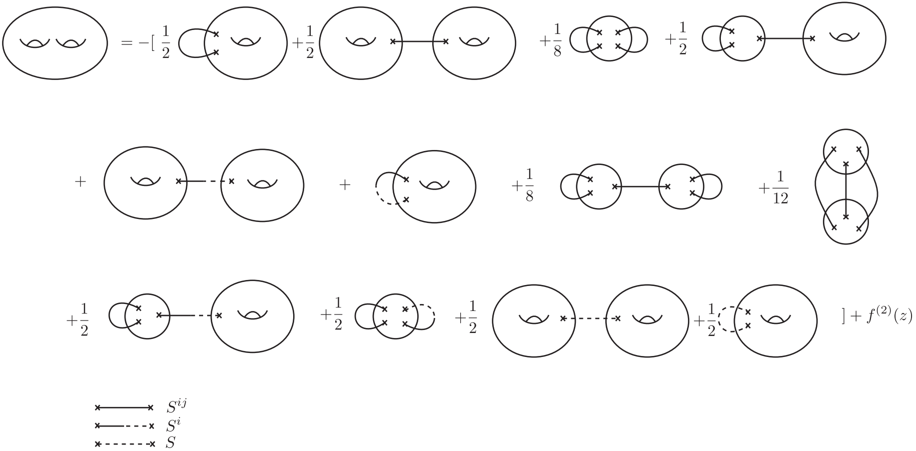
Recursion leads to Feynman diagrams



Bershadsky, Cecotti, Ooguri & Vafa (1993)

Yamaguchi & Yau discover polynomial structure of amplitudes

Yamaguchi & Yau (2004)

$$\bar{\partial}_{\bar{z}}\mathcal{F}^{g}=\frac{1}{2}\bar{C}_{\bar{z}}^{zz}\left(\sum_{g_1+g_2=g}D_z\mathcal{F}^{(g_1)}D_z\mathcal{F}^{(g_2)}+D_zD_z\mathcal{F}^{(g-1)}\right),$$

All nonholomorphic dependence of the amplitudes for the quintic is captured by the connections and multiderivatives thereof

$$A_p=G^{z\bar{z}}(z\partial_z)^pG_{z\bar{z}}\quad\text{and}\quad B_p=e^{K}(z\partial_z)^pe^{-K},\qquad p=1,2,3,\ldots$$

finitely many of these generate a differential ring, $\mathcal{F}_n^g$ are polynomials in these generators.

Related work: Hosono, Saito & Takahashi (99), Hosono (02), Huang & Klemm (06), Aganagic, Bouchard & Klemm (06)

Polynomial structure can be generalized

The polynomial structure was generalized to all Calabi-Yau threefolds MA & Länge (2007)

Polynomiality

$\mathcal{F}^{(g)}_{i_1\ldots i_n}$ are degree $3g-3+n$ inhomogeneous polynomials in the non-holomorphic generators (S^{ij}, S^i, S, K_i) where degrees $(1,2,3,1)$ are assigned to the generators respectively. *With coefficients and ambiguities which are rational functions in the algebraic moduli*

Show by induction

- Show that D_i[generator] is again expressible in terms of generators and increases the grading by 1.
- Show that the initial correlation functions have that property and proceed using the anomaly equations.

The differential ring of generators closes

Differential ring closes on non-holomorphic generators

$$
\begin{aligned}
D_i S^{jk} &= \delta_i^j S^k + \delta_i^k S^j - C_{imn} S^{mj} S^{nk} + h_i^{jk}, \\
D_i S^j &= 2\delta_i^j S - C_{imn} S^m S^{nj} + h_i^{jk} K_k + h_i^j, \\
D_i S &= -\frac{1}{2} C_{imn} S^m S^n + \frac{1}{2} h_i^{mn} K_m K_n + h_i^j K_j + h_i, \\
D_i K_j &= -K_i K_j - C_{ijk} S^k + C_{ijk} S^{kl} K_l + h_{ij},
\end{aligned}
$$

MA & Länge (2007)

Initial correlation function

- $\mathcal{F}_i^{(1)} = \frac{1}{2} C_{ijk} S^{jk} + (1 - \frac{\chi}{24}) K_i + f_i^{(1)}$

The anomaly becomes a polynomial recursion

The equation splits into two sets of equations

$$\begin{aligned} \frac{\partial \mathcal{F}^{(g)}}{\partial S^{ij}} &= \frac{1}{2} \sum_{g_1+g_2=g} D_i \mathcal{F}^{(g_1)} D_j \mathcal{F}^{(g_2)} + \frac{1}{2} D_i D_j \mathcal{F}^{(g-1)}, \\ 0 &= \frac{\partial \mathcal{F}^{(g)}}{\partial K_i} + S^i \frac{\partial \mathcal{F}^{(g)}}{\partial S} + S^{ij} \frac{\partial \mathcal{F}^{(g)}}{\partial S^j} \end{aligned}$$

MA & Länge (2007)

Polynomial structure allows computations

Boundary conditions are the name of the game

- BCOV recursion is the only method to solve higher genus topological strings on compact Calabi-Yau manifolds. For example the quintic (BCOV (93), g=2, Katz, Klemm & Vafa (99) g=4)
- Application of leading physically expected singular behavior Huang & Klemm

$$F^g(t_{con}) \sim \frac{1}{t_c^{2g-2}} + \mathcal{O}(1)$$

- YY Polynomial structure + boundary conditions + A-model provide enough information to solve the quintic to high genus. Huang, Klemm & Quackenbush (06), g=51

One can also tackle more difficult compact geometries

$K3$ fibrations Haghighat & Klemm, Elliptic fibrations MA, Scheidegger; Huang, Katz & Klemm

Polynomial ring recovers and generalizes quasi modular forms

- MA, Scheidegger, Yau, Zhou (2013) For non-compact geometries with known duality groups, $\mathcal{M}$ is identified with a modular curve, polynomial ring becomes the ring of quasi modular forms of Kaneko & Zagier, analogous construction of compact CY should give the generalization
- MA, Movasati, Scheidegger & Yau (2014) The polynomial generators can be thought of as coordinates on a larger moduli space, giving topological string theory an algebraic description.

Outline

Some characteristic monomials appear at every genus

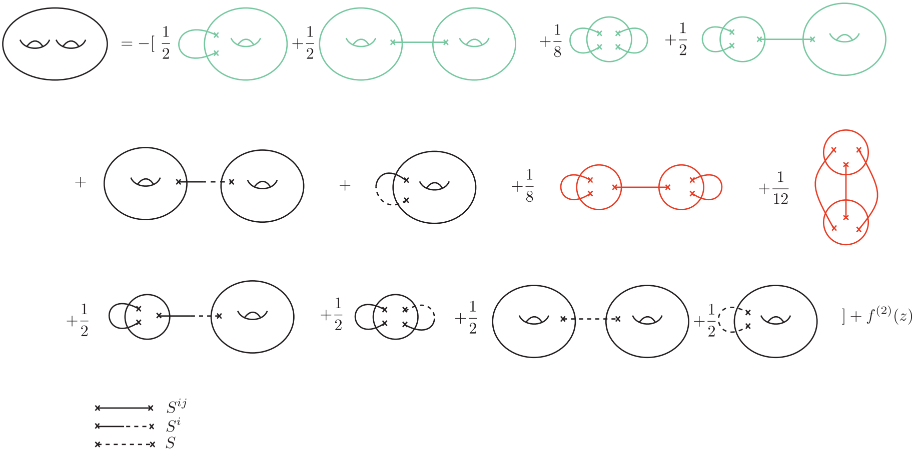
- Take a one-dimensional slice of $\mathcal{M}$, z local coordinate
- Consider the highest degree term of $\mathcal{F}^g$ in S^{zz}:

$$f(z)(S^{zz})^{3g-3}$$

,

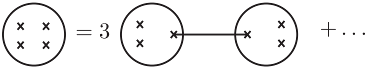
- $f(z) = a_g \, C_{zzz}^{2g-2}, a_g \in \mathbb{Q}$.
- What can we say about these?

Vertices of Feynman diagrams are further decomposed

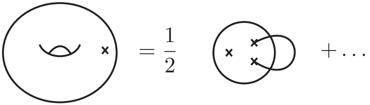


Vertices of Feynman diagrams are further decomposed

$$D_z C_{zzz} = 3 C_{zzz}^2 S^{zz} + \partial_z C_{zzz} - 3 s_{zz}^z C_{zzz} - 4 K_z C_{zzz} \, .$$



$$\mathcal{F}_z^1 = \frac{1}{2} C_{zzz} S^{zz} + \left(1 - \frac{\chi}{24}\right) K_z + f^1(z)$$



Anomaly equation for these monomials can be studied

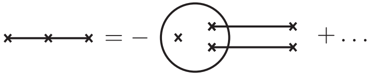
- Define

$$\mathcal{F}_s = \sum_{g=2}^{\infty} \lambda^{2g-2} a_g C_{zzz}^{2g-2} (S^{zz})^{3g-3}$$

- Use:

$$\sum_{g=2}^{\infty} \lambda^{2g-2} S^{zz} \frac{\partial \mathcal{F}^{(g)}}{\partial S^{zz}} = \sum_{g=2}^{\infty} \lambda^{2g-2} \frac{S^{zz}}{2} \left(\sum_{h=1}^{g-1} D_z \mathcal{F}^{(h)} D_z \mathcal{F}^{(g-h)} + D_z D_z \mathcal{F}^{(g-1)} \right) .$$

$$D_z S^{zz} = 2S^z - C_{zzz} (S^{zz})^2 + h_z^{zz}$$



Equation in the coupling is an Airy equation

- Define $\lambda_s^2 := \lambda^2 C_{zzz}^2 (S^{zz})^3$
- $$\mathcal{Z}_{top,s} = \exp \sum_{g=2}^{\infty} \lambda_s^{2g-2} a_g$$
- Make the change of variables: $z = (2\lambda_s^2)^{-\frac{2}{3}}, v = 2^{-\frac{1}{3}} e^{\frac{1}{3\lambda_s^2}} \lambda_s^{\frac{1}{3}} \mathcal{Z}_{top,s}$.
- From the summation of the anomaly equation the following can be obtained
$$\left(\partial_z^2 - z\right) v(z) = 0\,.$$
This is an Airy differential equation!

Related work for ABJM Matrix model: Fuji, Hirano, Moriyama '11

Strong coupling expansion can be obtained

- The full solution is:

$$\mathcal{Z}_{top,s} = \frac{e^{-\frac{1}{3\lambda_s^2}}}{\lambda_s} \left(I_{\frac{1}{3}} \left(\frac{1}{3\lambda_s^2} \right) + \zeta K_{\frac{1}{3}} \left(\frac{1}{3\lambda_s^2} \right) \right) , \quad \zeta \in \mathbb{C} \, .$$

- Near $\lambda = \infty$:

$$\mathcal{F}_s = -\frac{1}{3\lambda_s^2} - \frac{1}{3} \ln \lambda_s + 6^{-\frac{2}{3}} \frac{1 - \frac{\pi}{\sqrt{3}}\zeta}{\frac{2\pi}{\sqrt{3}}\zeta} \lambda_s^{-\frac{4}{3}} + \mathcal{O}(\lambda_s^{-4}) \, .$$

Conclusion

Summary

- The chiral ring leads to a differential ring of functions on $\mathcal{M}$
- Topological string theory is polynomial in the generators of this ring
- The coefficients of some monomials can be determined to all genus
- A universal Airy differential equation can be obtained for the topological string partition function in a limit
- A second non-perturbative solution appears

Outlook

- Geometric meaning of λ?
- Relation to non-perturbative definition of top. strings by Hatsuda, Grassi and Marino (2014)
- Physical and enumerative meaning of the strong coupling expansion

Thanks to ...

- collaborators D. Länge, P. Mayr, H. Movasati, E. Scheidegger, S. T. Yau and J. Zhou
- You for your attention