Some mathematical applications of little string theory

Mina Aganagic

UC Berkeley

Based on joint works with

Edward Frenkel and Andrei Okounkov, to appear

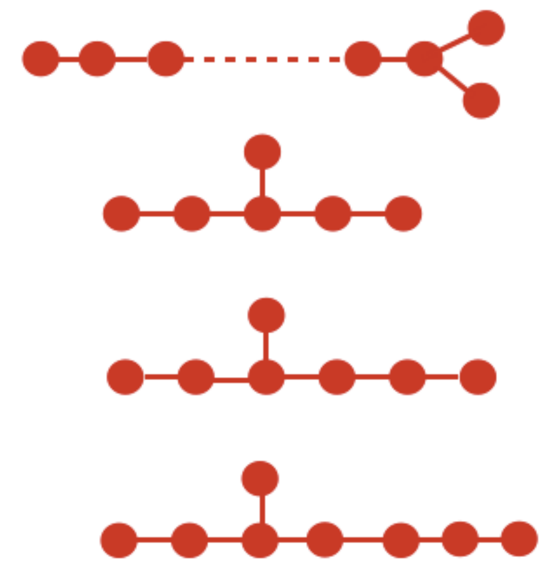
Andrei Okounkov, arXiv:1604.00423 [math.AG],

Nathan Haouzi, arXiv:1506.04183 [hep-th]

String theory predicts existence of a remarkable quantum field theory in six dimensions,

the (2,0) superconformal field theory.

The theory is labeled by a simply-laced Lie algebra, $\mathbf{g}$.



$\mathbf{g} = A_n$

$\mathbf{g} = D_n$

$\mathbf{g} = E_6$

$\mathbf{g} = E_7$

$\mathbf{g} = E_8$

It is remarkable, in part, because it is expected to play an important role in pure mathematics:

Geometric Langlands
Program

Knot Categorification
Program

The fact that the theory has no classical limit,

makes it hard to extract its predictions.

AGT correspondence,

after Alday, Gaiotto and Tachikawa,

serves well to illustrate

both the mathematical appeal of the theory

and the difficulty of working with it.

The AGT correspondence states

that the partition function of the $\mathbf{g}$-type (2,0) SCFT,

on a six manifold of the form

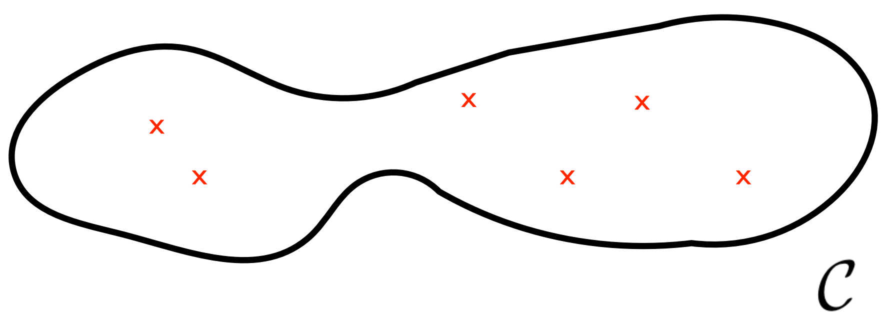
$$M_6 = \mathcal{C} \times \mathbb{R}^4$$

is a conformal block on the Riemann surface $\mathcal{C}$

of a vertex operator algebra which is also labeled by $\mathbf{g}$:

$\mathcal{W}(\mathbf{g})$ -algebra

The correspondence further relates defects of the 6d theory to vertex operators of the $\mathcal{W}(\mathbf{g})$-algebra, inserted at points on $\mathcal{C}$.



If one is to take the conjecture at its face value,
it is hard to make progress on it.

Since we do not know how to describe the (2,0) SCFT,
we cannot formulate or evaluate its partition function in any generality.

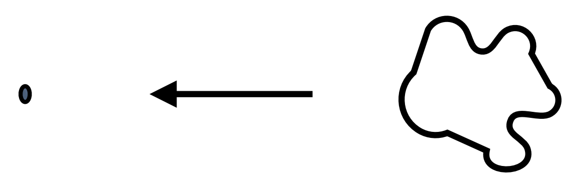
(Exceptions are $\mathbf{g} = A_1$, or special choices of defects.)

I will argue in this talk that one can make progress

by replacing

the 6-dimensional conformal field theory,

which is a point particle theory,



by the 6-dimensional string theory which contains it,

the (2,0) little string theory.

It turns out that there is

a little string version of the AGT correspondence

which one can make

precise in a very general setting.

The correspondence can be proven explicitly.

On the W-algebra side, one replaces the ordinary W-algebra by its “q-deformation”.

The deformed W-algebra is the one defined by Frenkel and Reshetikhin in the 90’s.

The correspondence is between

the partition function of the

$\mathbf{g}$-type 6d little string theory on

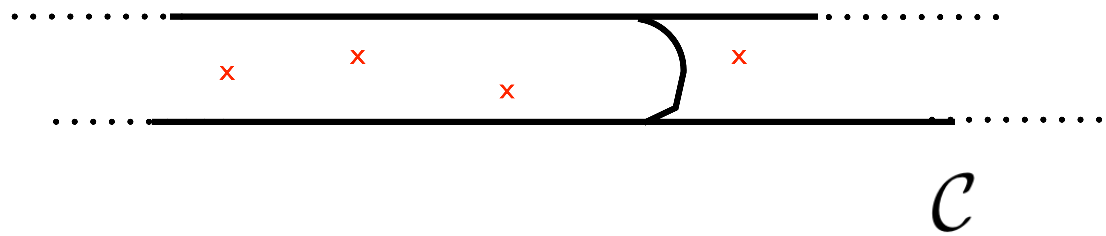
$$M_6 = \mathcal{C} \times \mathbb{R}^4$$

and

q-conformal block of the deformed $\mathcal{W}(\mathbf{g})$ algebra on $\mathcal{C}$

The Riemann surface $\mathcal{C}$

can be taken to be either a cylinder, or a torus.



We will take $\mathcal{C}$ to be a cylinder, since the torus case follows by additional identifications.

On each side of the correspondence, one replaces a theory with conformal symmetry,

with its mass deformation.

The conformal symmetry is broken in either case,

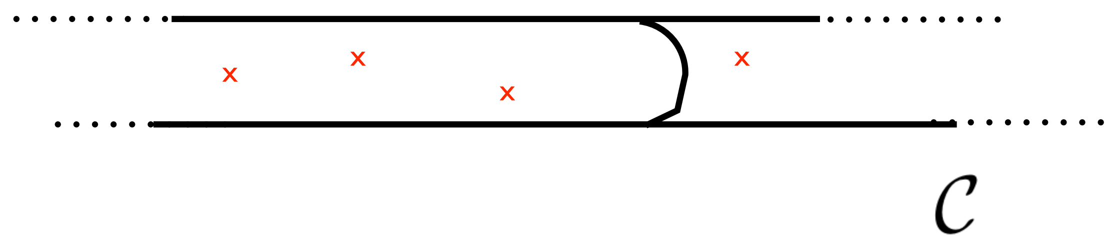
but in a canonical way.

The partition function of the

$\mathfrak{g}$-type little string on $\mathcal{C} \times \mathbb{R}^4$

with arbitrary collections of defects

at points on $\mathcal{C}$ and filling the $\mathbb{R}^4$



turns out to be the same as the partition function of

$\mathfrak{g}$-type quiver gauge theory with 8 supercharges,

on $S^1 \times \mathbb{R}^4$.

The mechanism for this is akin to localization.

The 6d little string partition function we compute is trivial in the absence of defects.

The quiver gauge is the theory on the co-dimension two defects of the little string theory on $\mathcal{C}$.

In the point particle limit,

the localization is not as useful:

The theory on the defects of the (2,0) CFT

has no known direct description.

In particular, it is not a gauge theory.

This will lead to a correspondence between:

q-deformed conformal block

of the $\mathcal{W}(\mathbf{g})$ -algebra with collection of vertex operators

at points on $\mathcal{C}$,

and

partition function

of a corresponding $\mathbf{g}$ -type quiver gauge theory on $S^1 \times \mathbb{R}^4$

which we will be able to prove.

In the rest of the talk, I will describe the correspondence and its proof in more detail.

Then, I will describe another application of little string theory, to the geometric Langlands program.

To define the $\mathbf{g}$ -type little string theory on

$$\mathcal{C} \times \mathbb{R}^4$$

one starts with the 10-dimensional IIB string theory on

$$Y \times \mathcal{C} \times \mathbb{R}^4$$

where Y is the ADE surface singularity of type $\mathbf{g}$

The resolution of the singularity at the origin of

$$Y$$

gives a collection of vanishing 2-cycles intersecting according to the Dynkin diagram of $\mathbf{g}$

$\mathbf{g} = D_n$

The 6d little string theory,

is a six dimensional string theory

obtained by taking the limit of IIB string theory on

$$Y \times \mathcal{C} \times \mathbb{R}^4$$

where one keeps only the degrees of freedom

supported at the singularity.

The little string limit involves sending the string coupling constant to zero, but keeping the string scale m_s finite.

The defects of little string theory originate as

D-branes of the ten dimensional IIB string,

which survive the limit.

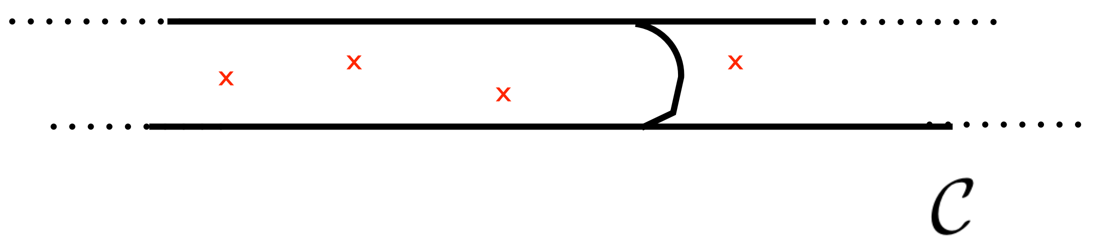
In string theory on

$$Y \times \mathcal{C} \times \mathbb{R}^4$$

the defects we need

are D5 branes wrapping (non-compact) 2-cycles of Y,

at points on $\mathcal{C}$,



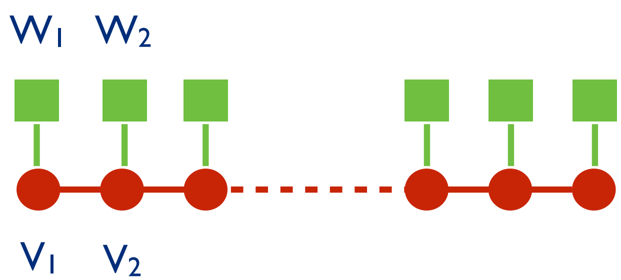
and filing the $\mathbb{R}^4$.

The theory on the defect D5 branes

is a

quiver gauge theory,

with quiver diagram based on the Dynkin diagram of g:



$$\mathbf{g} = A_n$$

The fact that one gets

a quiver gauge theory

on $S^1 \times \mathbb{R}^4$,

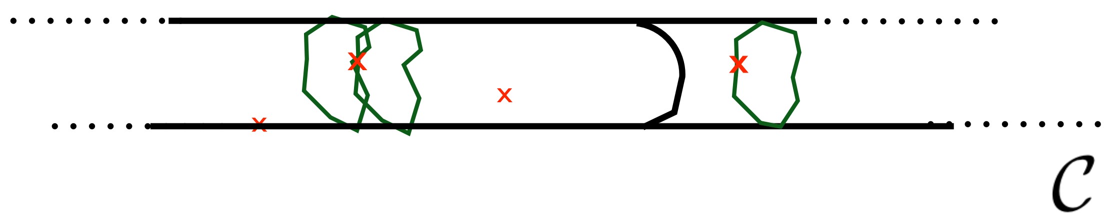
rather than the more obvious one,

on $\mathbb{R}^4$,

is due to a stringy effect.

In a string theory,

one has to include the winding modes of strings around $\mathcal{C}$



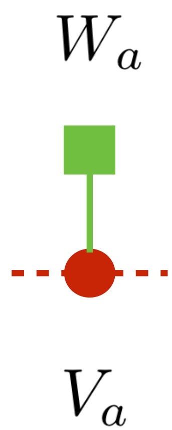
These turn the theory on the defects supported on $\mathbb{R}^4$,

to a five dimensional quiver gauge theory on

$$S^1 \times \mathbb{R}^4$$

where the S^1 is the T-dual of the circle in $\mathcal{C}$.

The dimension vectors



$$\dim(W_a) = m_a$$

$$\dim(V_a) = d_a$$

are determined by the classes of 2-cycles in Y which support the D5 branes, The gauge group of the theory

$$G = \prod_a U(d_a)$$

originates from the D5 branes

supported on compact two-cycles in Y;

The matter fields come from strings at the intersections of the branes.

The flavor symmetry group

$$G_F = \prod_{a=1}^{n} U(m_a)$$

comes from the gauge symmetry group of the non-compact D5 branes.

We need not an arbitrary quiver gauge theory,

but rather that which describes defects in the 6d little string on

$$\mathcal{C} \times \mathbb{R}^4$$

which preserve 4d conformal invariance

in the low energy limit.

To get a single puncture on $\mathcal{C}$ in the conformal limit,

one has to start with a collection of $n+1$ non-compact D5 branes

$$n \leq \mathrm{rk}(\mathbf{g})$$

whose relative separations on $\mathcal{C}$ vanish in the conformal limit

and add compact D5 branes so that the net D5 brane charge is zero.

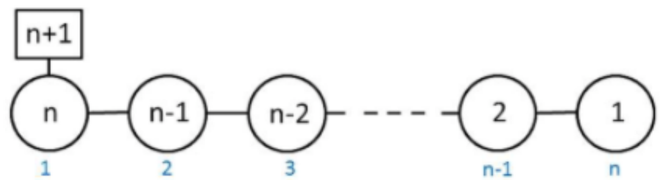
The possible choices can be classified,
making use of the relation of geometry of
the surface Y and representation theory of $\mathbf{g}$.

$H_2(Y, \partial Y, \mathbb{Z})$, containing classes of non-compact D5 branes,
is the same as weight lattice of $\mathbf{g}$.

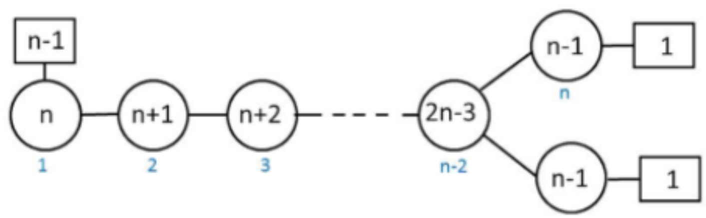
To get a single puncture defect, it turns out one should choose
$n+1$ elements of the weight lattice of $\mathbf{g}$, $n \leq \mathrm{rk}(\mathbf{g})$,
which sum up to zero,
and each of which lies in the Weyl orbit of a fundamental weight.

For example, quiver gauge theories on D5 branes corresponding to “full punctures” are:

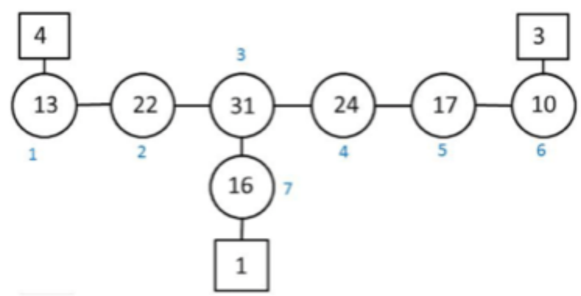
A_n



D_n



E_7



If we consider several defects on $\mathcal{C}$ instead of one, the ranks of the gauge and flavor symmetry group simply add.

The quiver gauge theory partition function on $S^1 \times \mathbb{R}^4$

can be computed using localization,

as Nekrasov and also Pestun explained.

Mathematically,

this leads to K-theoretic version of instanton counting.

Localization

lets one express the partition function as a

sum over the fixed points in instanton moduli space,

labeled by tuples of 2d Young diagrams,

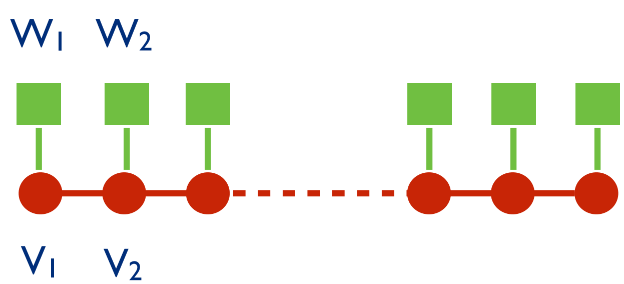
$$\{R\} = \{R_{a,i}\}$$

$$a = 1, \ldots \mathrm{rk}(\mathbf{g}) \qquad i = 1, \ldots d_a$$

The contribution of each fixed point

$$I_{\{R\}}$$

can be read off from the quiver,



$$\mathbf{g} = A_n$$

as a product of contributions of the nodes and the arrows.

In the end we sum over all the fixed points.

$$Z = \sum_{\{R\}} e^{\tau \cdot R} \, I_{\{R\}}$$

The parameters which enter the gauge theory partition function

$$\sum_{\{R\}} e^{\tau \cdot R}\, I_{\{R\}}(q,t;g,x)$$

have a geometric interpretation in string theory.

The partition function

$$Z = \sum_{\{R\}} e^{\tau \cdot R} \; I_{\{R\}}(q,t;g,x)$$

depends on:

q, t for rotations of two complex planes in $M = \mathbb{C}^2$

x for positions of non-compact D5 branes

g for positions of compact D5 branes

The gauge coupling parameters

$$\tau$$

are associated to the moduli of little string theory,

coming from sizes of vanishing 2-cycles in Y.

As soon as we resolve the singularities of Y,
by giving the two-cycles non-zero area,
the bulk of the 6d theory is abelianized.

All the relevant dynamics of the little string theory
is localized on the D5-branes.

The partition function of little string theory on M_6
with the corresponding collection of defects on $\mathcal{C}$
is the quiver gauge theory partition function.

Now, let me describe the deformed $\mathcal{W}(\mathbf{g})$ -algebra.

corresponding to a simple Lie algebra $\mathbf{g}$.

It is defined by Frenkel and Reshetikhin

in the "free-field formalism".

One starts with a free field algebra, with generators

$$e_a[k], \quad k \in \mathbb{Z}$$

labeled by the nodes of the Dynkin diagram, $\quad a = 1, \ldots, \mathrm{rk}(\mathbf{g})$

with commutation relations in terms of

a deformed Cartan matrix $\; C_{ab}(q,t)$

$$[e_a[k], e_b[m]] = \frac{1}{k}(q^{\frac{k}{2}} - q^{-\frac{k}{2}})(t^{\frac{k}{2}} - t^{-\frac{k}{2}})C_{ab}(q^{\frac{k}{2}}, t^{\frac{k}{2}})\delta_{k,-m}$$

The $\mathcal{W}_{q,t}(\mathbf{g})$ algebra itself is defined

as the set of vertex operators of the free field algebra which

commute with the screening charges

$$Q_a = \int dx \; S_a(x)$$

where

$$S_a(x) =: \exp\Big(\sum_{k\neq 0} \frac{e_a[k]}{q^{\frac{k}{2}} - q^{-\frac{k}{2}}} e^{kx}\Big) :$$

are the screening vertex operators.

Taking the limit

$$t = q^\beta, \qquad q \to 1$$

the deformed $\mathcal{W}(\mathbf{g})$ algebra becomes the ordinary one

$$\mathcal{W}_{q,t}(\mathbf{g}) \qquad \to \qquad \mathcal{W}_\beta(\mathbf{g})$$

containing the Virasoro algebra as a subalgebra,
with central charge depending on β

General q-conformal blocks of the W-algebra

are correlators of vertex operators,

$$\langle\mu|V_{\alpha_1}(z_1)\ldots V_{\alpha_k}(z_k)\ \prod_{a=1}^{n} Q_a^{N_a}|\mu'\rangle$$

where $V_\alpha(z)$ are built out of the free fields

and depend on continuous momenta

$$\alpha \in \mathbb{C}^{\mathbf{rk}(\mathbf{g})}$$

.... and the state $|\mu\rangle$, labeled by the weight $\mu \in \mathbb{C}^{\mathrm{rk}(\mathbf{g})}$,

generates the Verma module representation of the algebra.

It is defined by

$$e_a[k]|\mu\rangle = 0 \quad \text{for} \quad k > 0, \qquad e_a[0]|\mu\rangle = \mu_a|\mu\rangle$$

There is as of now, neither a math nor physics definition
of what it means to be a general q-deformed chiral vertex operator algebra.

Correspondingly, Frenkel and Reshetikhin did not
define q-deformations of general vertex operators
of the W-algebra.

My student Nathan Haouzi and I showed the following….

For each defect of little string,

corresponding to the collection of $n+1$ non-compact D5 branes,

there exist deformed vertex operators

$$V_{\alpha}(z) \equiv: \prod_{i=0}^{n} V_{\omega_i}(x_i) :$$

defined in terms of collections of $n+1$ weights ω_i we had before,

which become the primary vertex operators of $\mathcal{W}_{\beta}(\mathbf{g})$,

in the conformal limit,

$$t = q^{\beta}, \qquad q \to 1$$

$$x_{a+1}/x_a = q^{\alpha_a}, \qquad x_0 = z$$

with α and z fixed

The corresponding q-correlators

$$\langle \mu | V_{\alpha_1}(z_1) \ldots V_{\alpha_k}(z_k) \prod_{a=1}^{n} Q_a^{N_a} | \mu' \rangle$$

are in fact contour integrals, since

$$Q_a = \int dy\, S_a(y)$$

To specify the q-conformal block, we need to specify the contour.

We show that there exist choices of contours

such that the q-conformal block

$$\langle \mu | V_{\alpha_1}(z_1) \ldots V_{\alpha_k}(z_k) \;\; \prod_{a=1}^{n} Q_a^{N_a} | \mu' \rangle$$

equals the little string theory partition function,

$$\sum_{\{R\}} e^{\tau \cdot R} \; I_{\{R\}}(q, t; g, x)$$

The weight of the Verma module $|\mu\rangle$ is

the modulus μ of the 6d theory,

related to coupling constant of the D5 brane theory,

by $q^{\mu} = e^{\tau}$.

The conformal limit is the strong coupling limit of the D5 brane gauge theory,

$q \to 1$ with μ fixed.

The little string version of the correspondence is simple to prove:

The sum over the poles

$$\langle\mu|V_{\alpha_1}(z_1)\dots V_{\alpha_k}(z_k)\ \prod_{a=1}^{n} Q_a^{N_a}|\mu'\rangle\ \equiv \int \prod_{a=1}^{n}\ dy^{N_a}\ y^{\mu}\ \mathcal{I}(q,t;y,x)$$

in the contour prescription to evaluate the conformal block

$$\sum_{\{R\}} e^{\tau\cdot R}\ I_{\{R\}}(q,t;g,x)$$

is the sum over instantons, term by term.

$$\mathrm{res}_{\{R\}}\mathcal{I}(q,t;y,x) = I_{\{R\}}(q,t;g,x)$$

Another application of little string theory is to the

geometric Langlands correspondence.

Geometric Langlands correspondence
was formulated in the early '90s by
Beilinson and Drinfeld.

In the same work, they explained that
one can phrase the correspondence
in the language of 2d conformal field theory.

Geometric Langlands

can be interpreted as the correspondence

between conformal blocks on a Riemann surface $\mathcal{C}$,

associated to a Langlands dual pair of Lie algebras

$^L\mathbf{g}$ and $\mathbf{g}$

The electric side are the conformal blocks of the affine current algebra

$$ {}^{L}\hat{\mathbf{g}} $$

at the critical level $k = -h$ (infinite coupling) .

On the magnetic side, are the conformal blocks of the

$$ \mathcal{W}_{\beta}(\mathbf{g}) $$

algebra in the classical, $\beta \to \infty$, limit.

The proof of the

geometric Langlands correspondence

was given in this context,

by Belinson and Drinfeld,

and

by Frenkel with Gaitsgory and Vilonen.

There are two ways in which one may try to generalize this.

First, it is natural to deform away from
the critical level k or equivalently, to finite β.

Second, it is natural to replace the conformal chiral algebras by their
q-deformed counterparts.

The first deformation

is the "quantum Langlands correspondence."

In the abelian case, it was proven by Polishchuk and Rothstein.

For $\mathbf{g} = A_1$, some partial results were obtained by

Feigin, Frenkel and Stoyanovsky, and also Teschner, and others.

The rest is open.

It turns out that one can
implement both generalizations,
and it is easiest to do it
at the same time.

In a joint work with Edward Frenkel and Andrei Okounkov,

we formulate the

quantum q-Langlands correspondence.

It relates the deformed conformal blocks of

the quantum affine current algebra

electric

$$U_\hbar({}^L\hat{\mathbf{g}})$$

corresponding to ${}^L\hat{\mathbf{g}}$ at level k ,

and the q-conformal blocks of the deformed W-algebra

magnetic

$$\mathcal{W}_{q,t}(\mathbf{g})$$

where:

$$t = \hbar, \qquad q = \hbar^{(k+h)}$$

We prove the

quantum q-Langlands correspondence

for any simply laced Lie algebra,

i.e. when

$$^{L}\mathbf{g} = \mathbf{g}$$

The q-conformal blocks of

electric

$$U_{\hbar}({}^{L}\hat{\mathbf{g}})$$

magnetic

$$\mathcal{W}_{q,t}(\mathbf{g})$$

chiral algebras

arize as partition functions of little string theory,

with co-dimension four defects.

Take the little string theory associated

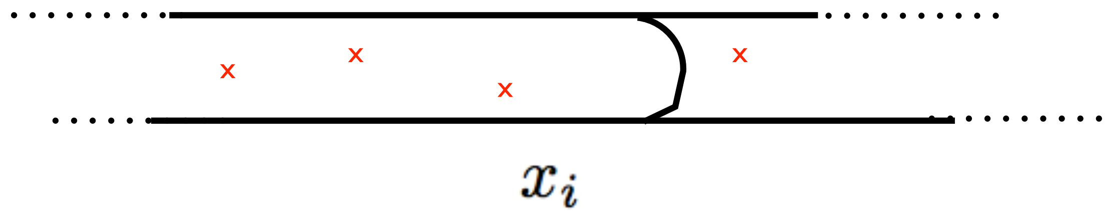
to the simply laced Lie algebra $\mathbf{g}$

on

$$M_6 = \mathcal{C} \times \mathbb{C} \times \mathbb{C}$$

just as before.

The defects we need are self-dual strings supported at points on $\mathcal{C}$



and on one of the two complex planes in

$$M_6 = \mathcal{C} \times \mathbb{C} \times \mathbb{C}$$

In the present notation, this is the plane rotated by q.

From perspective of IIB string on

$$Y \times M_6$$

the defect strings come from D3 branes,

supported on 2-cycles in Y, and the chosen 2-plane in M_6.

The partition function of the 6d little string theory,
as before, localizes to the partition function
of the gauge theory on the defects.

The gauge theory is
a tree dimensional $\mathfrak{g}$ -type quiver gauge theory,
with N=4 supersymmetry
on $\mathbb{C} \times S^1$.

Depending on the type of boundary conditions at infinity, the partition function of the quiver gauge theory on

$$\mathbb{C} \times S^1$$

generates the

${}^L\mathbf{V}_i$ or $\mathbf{V}_i$

electric magnetic

q-conformal blocks

More precisely, each electric q-conformal block of $U_\hbar({}^L\hat{\mathbf{g}})$

is vector valued. From

$${}^L\mathbf{V}_i$$

we can generates all the components of the vector,

by differentiating it with respect to x_i 's,

or placing insertions at $0 \in \mathbb{C}$.

The proof of the quantum q-Langlands correspondence is in terms of an explicit linear map between the electric and the magnetic conformal blocks.

$$ {}^{L}\mathbf{V}_i = \sum_j \mathfrak{P}^j{}_i \, \mathbf{V}_j $$

The linear map we need to establish the correspondence was constructed in a joint paper with Andrei Okounkov for any simply laced Lie algebra.

The matrix

$$\mathfrak{P}^j{}_i$$

computes the partition function

of the quiver gauge theory on $T^2 \times I$,

with the two sets of boundary conditions at the ends of the interval.

It also has a geometric meaning, as the

elliptic stable envelope

of X, the Higgs branch of the 3d gauge theory.

It generalizes the stable envelopes in cohomology and K-theory of X

due to Maulik and Okounkov.

To study non-simply laced Lie algebras.

one adds a twist that ends up permuting the nodes of the Dynkin diagram

as we go once around the origin of the complex $\mathbb{C}$ plane in

$$M_6 = \mathcal{C} \times \mathbb{C} \times \mathbb{C}$$

which supports the defects.

An important generalization
of the geometric Langlands program
is to include ramifications.

This simply corresponds to including
D5 brane defects from the first half of the talk.

As Kapustin and Witten explained,
the geometric Langlands correspondence
is related to
S-duality of
N=4 super-Yang-Mills theory.

While many aspects of S-duality

can be understood within N=4 SYM theory,

or using the (2,0) CFT

compactified on a two-torus.....

….to derive S-duality of N=4 SYM theory

one needs

little string theory,

as was shown by Vafa showed in '97.

The role of little string theory in understanding S-duality

explains

why one is able to make progress

on the geometric Langlands problem,

in the context of the

quantum q-Langlands correspondence.