Quantum field theory and the Bieberbach conjecture

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The Bieberbach conjecture

The Bieberbach conjecture tells us how fast the Taylor expansion coefficients of a holomorphic univalent function, on the unit disc($|z| < 1$) grows, namely

$$f(z) = z + \sum_{p=2}^{\infty} b_p z^p \qquad \boxed{|b_n| \leq n, \qquad \forall n \geq 2}$$

The Koebe growth theorem gives us two sided bounds on $|f(z)|$

$$\frac{|z|}{(1+|z|)^2} \leq |f(z)| \leq \frac{|z|}{(1-|z|)^2}$$

A function is univalent on a domain D if it is holomorphic, and one-to-one, i.e. for all $z_1, z_2 \in D$, $f(z_1) = f(z_2)$ if $z_1 = z_2$.

Fully crossing symmetric amplitude

Fully crossing symmetric amplitude can be written as

$$\mathcal{M}(s_1, s_2) = \sum_{p=0,q=0}^{\infty} \mathcal{W}_{p,q} x^p y^q$$

$$x = -(s_1 s_2 + s_2 s_3 + s_3 s_1), \; y = -s_1 s_2 s_3$$

We are using convention

$$s_1 = s - \frac{4m^2}{3}, s_2 = t - \frac{4m^2}{3}, s_3 = -s_1 - s_2$$

We parametrize s_1, s_2 such that

$$x = \frac{-27a^2\tilde{z}}{(\tilde{z}-1)^2}, \qquad y = \frac{-27a^3\tilde{z}}{(\tilde{z}-1)^2}, \qquad a = y/x$$

Now amplitude is a function $a, \tilde{z}$, namely $\mathcal{M}(\tilde{z}, a)$. We can Taylor expand it

$$\mathcal{M}(\tilde{z}, a) = \sum_{n=0}^{\infty} \alpha_n(a) a^{2n} \tilde{z}^n$$

$\alpha_n(a)$ are related to Wilson coefficients

Wilson coefficients

$$\alpha_p(a) a^{2p} = \sum_{n=0}^{p} \sum_{m=0}^{n} \mathcal{W}_{n-m,m} a^m (-1)^{p-n} (-27)^n \, a^{2n} \binom{-2n}{p-n}$$

Quantum field theory and the Bieberbach conjecture

Crossing symmetric dispersion relation

$$\mathcal{M}(\tilde{z},a)=\alpha_0+\frac{1}{\pi}\int_{\frac{2\mu}{3}}^{\infty}\frac{ds_1'}{s_1'}\mathcal{A}\left(s_1';s_2^{(+)}\left(s_1',a\right)\right)H\left(s_1',\tilde{z}\right)\longrightarrow H(s_1',\tilde{z})=\frac{27a^2\tilde{z}\left(2s_1'-3a\right)}{27a^3\tilde{z}-27a^2\tilde{z}s_1'-(1-\tilde{z})^2\left(s_1'\right)^3}$$

$A(s_1;s_2)$ is the s-channel discontinuity

Crossing symmetric dispersion relations and unitarity enable us to prove the followings

Remember that amplitude

$$\mathcal{M}(\tilde{z},a)=\sum_{n=0}^{\infty}\alpha_n(a)a^{2n}\,\tilde{z}^n$$

1) The Bieberbach bound

$$\boxed{\left|\frac{\alpha_n(a)a^{2n}}{\alpha_1(a)a^2}\right|\leq n,\quad\forall n\geq 2}$$

2) The Koebe two sided bound on amplitude

$$\boxed{\frac{|\tilde{z}|}{(1+|\tilde{z}|)^2}\leq\left|\frac{M(\tilde{z},a)-\alpha_0}{\alpha_1(a)a^2}\right|\leq\frac{|\tilde{z}|}{(1-|\tilde{z}|)^2}}$$

3) For small a, the $\frac{M(\tilde{z},a)-\alpha_0}{\alpha_1(a)a^2}$ is Univalent

The bounds on Wilson coefficients

From $|\alpha_2(a)a^4| \leq 2$ we get $-\frac{9}{16\mathrm{m}^2} \leq \frac{\mathcal{W}_{01}}{\mathcal{W}_{10}} \leq \frac{9}{8m^2}$

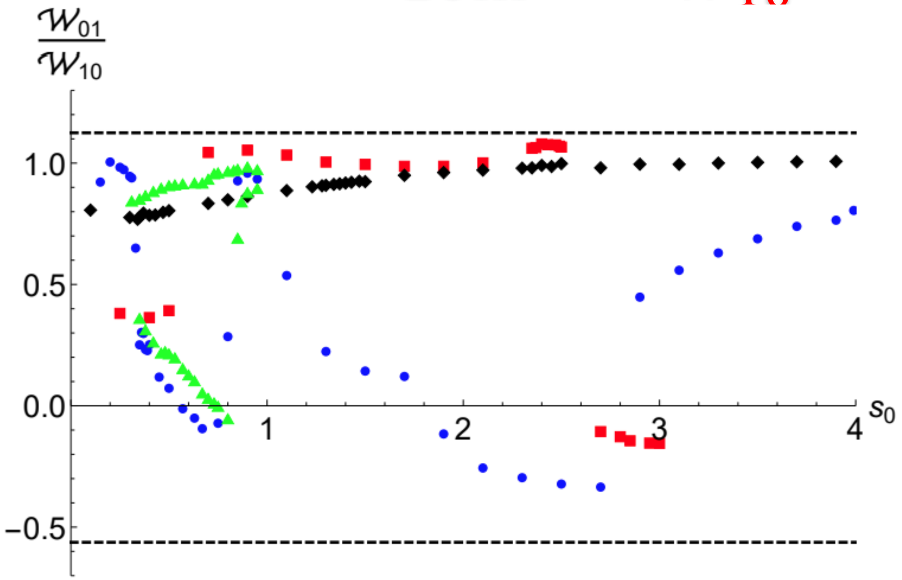

Figure 3: Ratio of $\frac{\mathcal{W}_{0,1}}{\mathcal{W}_{1,0}}$ obtained from the S-matrix bootstrap. The horizontal axis is the Adler zero s_0. The green points are for the pion lake [19]. The blue and red points are for the upper and lower river boundaries [20, 21] while the black points are for the line of minimum averaged total cross section S-matrices [21].

The bounds on Wilson coefficients

On going work by Aninda, Prashant shows more tighter bounds on Wilson coefficients using the theory of Typically Real Univalence functions

$w_{p,q} = \frac{W_{p,q}}{W_{1,0}}$	TR_U	$SDPB$	$Pion$
w_{01}	-0.5625	-0.5625	-0.335
w_{11}	-0.13186	-0.1318	-0.846
w_{02}	-0.089	-0.1268	-0.056
w_{20}	0	0	
w_{21}	-0.025955	-0.25955	-0.0157
w_{12}	-0.0223	-0.02789	-0.0121
w_{30}	0	0	0.001236
w_{03}	-0.00664	-0.00156	-0.24303
w_{31}	-0.0047	-0.0047	-0.000515
w_{40}	0	0	0.000091
w_{50}	0	0	

Lower bounds

$w_{p,q} = \frac{W_{p,q}}{W_{1,0}}$	TR_U	$SDPB$	$Pion$
w_{01}	1.125	1.939	1.07
w_{11}	0.105	0.216	0.0918
w_{02}	0.0396	0.0296	0.0182
w_{20}	0.140625	0.140625	
w_{21}	0.0183	0.023	0.005995
w_{12}	0.013735	0.0111	0.06525
w_{30}	0.0197	0.0197754	0.017065
w_{03}	0.00712	0.0071	0.004636
w_{31}	0.00884	0.00218	0.0027
w_{40}	0.00278	0.00278	0.00228
w_{50}	0.000391	0.000391	

Upper bounds

Thank You