Moduli spaces in heterotic string theory

(moduli space of certain instanton connections on manifolds with $\mathcal{G}$-structures)

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Collaborations with

E Svanes, JHEP **1410** (2014) 123; JHEP **1412** (2014) 008

E Svanes and E Hardy, JHEP **1601** (2016) 049

M Larfors and E Svanes,
Adv. Theor. Math. Phys. **19** 837-903 (2015);
in Proc N Hitchin's 70th bday conf (2018) (ArXiV 1709.06974); JHEP **1611** (2016) 016; Commun. Math. Phys. (2017);
and in progress

P Candelas and J McOrist,
Commun. Math. Phys. **356** (2017) 567-612; ArXiV 1810.00879;

MA Fiset ArXiV 1809.01138, and in progress

A Ashmore, R Minasian, C Strickland-Constable and E Svanes, JHEP 1810 (20)

M Larfors, M Magill E Svanes, hepth 1904.01027

Introduction

General context: interested in effective field theory derived from compactifications of heterotic string theories and the CFTs related to these, when we preserve the minimun possible amount of supersymmetry.

What are the mathematical structures encountered?

Introduction

Let Y be a d-dimensional Riemannian manifold and V be a vector bundle on Y.

We have the following mathematical objects on these theories:

- Riemannian metric g_{mn} on Y
- a "scalar" ϕ (the dilaton)
- Gauge fields A for the gauge group $G \subseteq E_8 \times E_8$.
  So, V is a vector bundle on Y with connection A with structure group G contained in $E_8 \times E_8$.
- 3-form H, “the flux”, defined by

$$H = \mathrm{d}B + \frac{\alpha'}{4}(\mathcal{CS}[A] - \mathcal{CS}[\Theta]) \, , \quad \mathcal{CS}[A] = \mathrm{tr}(A \wedge \mathrm{d}A + \frac{2}{3} A \wedge A \wedge A) \, ,$$

  where Θ is a connection on TY, and $\alpha' \neq 0$ is a constant.

  Note that the B-field is not gauge invariant!

Introduction

We have: $[\,Y, (V, A), (TY, \Theta), H\,]$

Mathematically: supersymmetry constrains the geometry

$$\text{heterotic } \mathcal{G} \text{ structure on } [\,Y, (V, A), (TY, \Theta), H\,]$$

and we want to study the geometry and quantum moduli of these compactifications (including α' and non-perturbative corrections).

Goal in physics: construct the quantum effective field theory and this is largely determined by the geometry and moduli of these compactifications. For example: we want to find the massless spectrum, the (effective) 4 or 3 dimensional Lagrangian, correlation functions, and to study dualities....

Introduction

I will focus today in the case $d = 7$ and mainly on the supergravity side (the $d = 6$ can be deduced from this!):

A reason:
Much less is known about 3-dim $N = 1$ supergravity+YM.
The three dimensional space time is AdS_3 or 3-dim Minkowski.

Recall: Compactifications to four dimensions with $N = 1$ supersymmetry give strong constrains on the geometry and on the moduli space. In particular, the moduli space must be complex and Kähler.

Compactifications to three dimensions with $N = 1$ supersymmetry, are not as constrained and we know much less about the geometry of the moduli space.

Outline

- Heterotic G_2 systems $[\,(Y,\varphi),(V,A),(TY,\Theta),H\,]$
  T Friederich & S Ivanov 2001& 2003; J Gauntlett, N Kim, D Martelli, & D Waldram 2001; J Gauntlett, D Martelli, & D Waldram 2004; P Ivanov & S Ivanov 2005; A Lukas & C Matti 2010; J Gray, M Larfors, D Lüst, 2012; XD, M Larfors & E Svanes 2014
- The tangent space of the moduli space of heterotic G_2 systems
  - Moduli space of (Y,φ)
    Gibbons, Page and Pope, D Joyce: S Karigiannis, S Grigorian, C Leung....; XD, E Svanes and M Larfors
  - Moduli space of vector bundles (V,A) over (Y,φ)
    Donaldson & Thomas; C Leung & Karigiannis; XD, E Svanes and M Larfors; ...
  - Moduli space of heterotic G_2 systems
    A Clarke, M García Fernández & C Tipler 1607.01219; XD, M Larfors & E Svanes 1607.03473 & 1704.08177 & in progress
    MA Fiset, C Quigley, E Svanes 1710.06865; XD & MA FIset, in progress
- Outlook and open problems

Key ideas

- The conditions for the quadruple $[(Y,\varphi),(V,A),(TY,\Theta),H]$ to admit a heterotic structure are equivalent to a differential $\mathcal{D}$ which satisfies $\check{\mathcal{D}}^2=0$ acting on forms with values on a bundle $\mathcal{Q}$ on Y which is topologically

$$\mathcal{Q}=TY\oplus \mathrm{End}(TY)\oplus \mathrm{End}(V)$$

- The infinitesimal moduli of the heterotic structure correspond to classes in

$$H^1_{\check{\mathcal{D}}}(Y,\mathcal{Q})\ .$$

- Global moduli space: Maurer Cartan equations? what is the geometric structure? what are the singularities and what happens there? quantum corrections? duality symmetries?

The geometry of Y

Supersymmetry requires that Y has an integrable G_2 structure.

A manifold with a G_2 structure is a seven dimensional manifold Y which admits a smooth positive three form φ.

In fact, any 7-dimensional manifold which is spin and orientable (that is its first and second Stiefel-Whitney classes are trivial) admits a G_2 structure.

The three form φ determines a Riemannian metric g_φ and a four form $\psi = *\varphi$.

The geometry of Y

Consider now the exterior derivative of φ and ψ can be decomposed into irreducible G_2 representations

$$\begin{aligned} d\varphi &= \tau_0\, \psi + 3\, \tau_1 \wedge \varphi + *\tau_3 \\ 35 &= \quad 1 \quad + \quad 7 \quad + \; 27 \\ d\psi &= 4\, \tau_1 \wedge \psi + *\tau_2 \\ 21 &= \quad 7 \quad + \; 14 \end{aligned}$$

where $\tau_i \in \Omega^i(Y)$ (the torsion classes) are determined by the G_2 structure φ on Y.

An integrable G_2 structure satisfies $\quad \tau_2 = 0$.

The geometry of Y

We can write the structure equations when $\tau_2 = 0$ as

$$d\varphi = i_{T(\varphi)}(\varphi) \qquad d\psi = i_{T(\varphi)}(\psi)$$

where

$$T(\varphi) = \frac{1}{6}\,\tau_0\,\varphi - \tau_1 \lrcorner \psi - \tau_3 \ .$$

Remarks:

- $T(\varphi) = 0$ means $\tau_i = 0, \forall i$.
  In this case Y has G_2 holonomy ($d\varphi = 0$ and $d\psi = 0$)
- $T(\varphi)$ is the torsion of the unique connection ∇ compatible with the integrable G_2 structure ($\nabla\varphi = 0$, $\nabla\psi = 0$) which is totally antisymmetric.

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The geometry of Y

Question: is there a differential $\check{\mathrm{d}}$ with $\check{\mathrm{d}}^2 = 0$ which encodes this geometry?

That is, is there an analogue for (Y, φ) of the Dolbeault differential $\bar{\partial}$ with $\bar{\partial}^2 = 0$ for a complex manifold (X, J)?

The geometry of Y

Canonical G_2 cohomology

Reyes-Carrion, 93;Fernandez-Ugarte, 98

Consider the differential operator $\check{\mathrm{d}}$ defined by

$$\check{\mathrm{d}}_0 = \mathrm{d}\,, \quad \check{\mathrm{d}}_1 = \pi_7 \circ \mathrm{d}\,, \quad \check{\mathrm{d}}_2 = \pi_1 \circ \mathrm{d}$$

We have

$$\check{\mathrm{d}}^2 = 0 \quad \Longleftrightarrow \quad \tau_2 = 0$$

and a canonical cohomology $H^*_{\check{\mathrm{d}}}(Y)$ on a manifold Y with an integrable G_2 structure.

Constraints on V

Supersymmetry imposes conditions on the curvature F of the Yang-Mills connection:

$$F \wedge \psi = 0 \ .$$

That is, the connection A on the bundle V is an instanton.

One can construct an operator $\check{\mathrm{d}}_A$ which acts on forms with values on $\mathrm{End}(V)$ where

$$\mathrm{d}_A \alpha = \mathrm{d}\alpha + A \wedge \alpha + (-1)^k \alpha \wedge A \ , \qquad \alpha \in \Omega^k(Y, \mathrm{End}(V))$$

We have

$$\check{\mathrm{d}}_A^2 = 0 \quad \Longleftrightarrow \quad \tau_2 = 0 \quad \text{and} \quad F \wedge \psi = 0$$

This leads to cohomology groups $H^*_{\check{\mathrm{d}}_A}(Y, \mathrm{End}(V))$.

A Further Constraint on $[(Y, \varphi), (V, A)]$
The anomaly cancelation condition

The anomaly cancelation condition: $H = T(\varphi)$

$$dB + \frac{\alpha'}{4}(\mathcal{CS}[A] - \mathcal{CS}[\Theta]) = \frac{1}{6}\,\tau_0\,\varphi - \tau_1 \lrcorner \psi - \tau_3$$

Moreover: a solution of the supersymmetry conditions, which also satisfies the anomaly cancelation automatically satisfies the equations of motion iff the connection Θ on TY satisfies

$$R(\Theta) \wedge \psi = 0\ .$$

That is, Θ must be an instanton. (Hull, Ivanov, Martelli and Sparks)

Summary

A heterotic G_2 system is a quadruple

$$[(Y,\varphi),(V,A),(TY,\Theta),H]$$

where:

- (Y,φ) is a manifold with an integrable G_2 structure
- (V,A) and (TY,Θ) have instanton connections.
- The anomaly cancelation condition is satisfied: $H = T(\varphi)$

$$\mathrm{d}B + \frac{\alpha'}{4}(\mathcal{CS}[A] - \mathcal{CS}[\Theta]) = \frac{1}{6}\,\tau_0\,\varphi - \tau_1 \lrcorner \psi - \tau_3$$

Key ideas

- The heterotic G_2 system $[(Y,\varphi),(V,A),(TY,\Theta),H]$ is equivalent to a the existence of a differential $\mathcal{D}$ such that $\check{\mathcal{D}}^2=0$ on forms with values in a bundle $\mathcal{Q}$ on Y which is topologically

$$\mathcal{Q} = TY \oplus \mathrm{End}(TY) \oplus \mathrm{End}(V).$$

- The infinitesimal moduli of the heterotic structure are given by

$$H^1_{\check{\mathcal{D}}}(Y,\mathcal{Q}) \; .$$

The heterotic G_2 system as a differential

Consider the linear operator

$$\mathcal{D} = \begin{pmatrix} \mathrm{d}_\zeta & \mathcal{R} & -\mathcal{F} \\ \mathcal{R} & \mathrm{d}_\Theta & 0 \\ \mathcal{F} & 0 & \mathrm{d}_A \end{pmatrix}$$

acting on forms with values in $\mathcal{Q} = TY \oplus \mathrm{End}(TY) \oplus \mathrm{End}(V)$ where

$\mathrm{d}_\zeta M = \mathrm{d}M + \zeta \wedge M\,, \; M \in \Omega^k(Y, TY)$, $\quad \zeta$ a connection on TY with torsion $-T(\varphi)$

$\mathrm{d}_A \alpha = \mathrm{d}\alpha + A \wedge \alpha + (-1)^k \alpha \wedge A\,, \quad \alpha \in \Omega^k(Y, \mathrm{End}(V))$, and similar for d_Θ

The heterotic G_2 system as a differential

$\mathcal{F}$ (and similarly for $\mathcal{R}$) is a linear map defined by

$$\mathcal{F} : \Omega^k(Y, TY) \oplus \Omega^k(Y, \mathrm{End}(V)) \longrightarrow \Omega^{k+1}(Y, \mathrm{End}(V)) \oplus \Omega^{k+1}(Y, TY)$$

$$\begin{pmatrix} M \\ \alpha \end{pmatrix} \mapsto \begin{pmatrix} \mathcal{F}(M) \\ \mathcal{F}(\alpha)^a \end{pmatrix} = \begin{pmatrix} (-1)^k M^a \wedge F_{ab} \mathrm{d}x^b \\ (-1)^k \frac{\alpha'}{4} g^{ab} \, \mathrm{tr}(\alpha \wedge F_{bc} \mathrm{d}x^c \end{pmatrix}$$

The heterotic G_2 system as a differential

Theorem Let Y be a manifold with a G_2 structure φ, V a bundle on Y with connection A, and TY the tangent bundle of Y with connection Θ. Let ζ be the connection one-form on TY defined earlier. Consider the exterior covariant derivative $\mathcal{D}$ defined above. Then

$$\check{\mathcal{D}}^2 = 0 \iff \big([Y,\varphi],[V,A],[TY,\Theta],H\big) \text{ is a heterotic system}$$

where

$$\check{\mathcal{D}}_0 = \mathcal{D}\ , \quad \check{\mathcal{D}}_1 = \pi_7 \circ \mathcal{D}\ , \quad \check{\mathcal{D}}_2 = \pi_1 \circ \mathcal{D}\ .$$

The heterotic G_2 system as a differential

Comments on the proof: Consider

$$\mathcal{D}^2 = \begin{pmatrix} \mathrm{d}_\zeta^2 + \mathcal{R}^2 - \mathcal{F}^2 & \mathrm{d}_\zeta\mathcal{R} + \mathcal{R}\mathrm{d}_\Theta & -(\mathrm{d}_\zeta\mathcal{F} + \mathcal{F}\mathrm{d}_A) \\ \mathcal{R}\mathrm{d}_\zeta + \mathrm{d}_\Theta\mathcal{R} & \mathcal{R}^2 + \mathrm{d}_\Theta^2 & -\mathcal{R}\mathcal{F} \\ \mathcal{F}\mathrm{d}_\zeta + \mathrm{d}_A\mathcal{F} & \mathcal{F}\mathcal{R} & -\mathcal{F}^2 + \mathrm{d}_A^2 \end{pmatrix}$$

It is not too hard to see that $\check{\mathcal{D}}^2 = 0$ is satisfied for the heterotic G_2 system as long as the Bianchi identity of the anomaly is satisfied and

$$\check{\mathrm{d}}_A(\check{\mathcal{F}}(M)) + \check{\mathcal{F}}(\check{\mathrm{d}}_\zeta(M)) = 0\ .$$

This is true due to the BI for F: $\mathrm{d}_A F = 0$.

The converse however is more involved: in particular the vanishing of the (1,1) entry implies the Bianchi identity of the anomaly.

Infinitesimal deformations of (Y, φ)

Consider a family $(Y, \varphi(t))$ with an integrable G_2 structure with $(Y, \varphi(0)) = (Y, \varphi)$.
Idea: study integrable G_2 structures in terms of $M_t \in \Omega^1(Y, TY)$,

$$\partial_t \varphi = \frac{1}{2} M_t^a \wedge \varphi_{abc} \, \mathrm{d}x^{bc} = i_{M_t}(\varphi)$$

$$\partial_t \psi = \frac{1}{3!} M_t^a \wedge \psi_{abcd} \, \mathrm{d}x^{bcd} = i_{M_t}(\psi)$$

Deformations preserving the integrability of the G_2 structure are given by

$$i_{\check{\mathrm{d}}_\varsigma M_t}(\psi) = 0 \, ,$$

Diffeomorphisms: $\mathcal{L}_V \psi = i_{\check{\mathrm{d}}_\varsigma V}(\psi)$

Infinitesimal deformations of (Y, φ)

The dimension of the space of infinitesimal deformations of integrable G_2 structures, $\mathcal{TM}_0$, is in general is not finite.

An exception (of course!): Y has G_2 holonomy
In this case

$$\mathcal{TM}_0 = H^1_{\check{\mathrm{d}}_\varsigma}(Y, TY)$$

precisely matches the deRham cohomology $H^3(Y)$.

Deformations of $[(Y, \varphi), (V, A)]$

Consider now deformations of $[(Y, \varphi), (V, A)]$

Want:

- deformations M of the integrable G_2 structure φ on Y which preserve the integrability of the G_2 structure together with
- deformations $\alpha \in \Omega^1(Y, \mathrm{End}(V))$ of the instanton connection A such that simultaneous deformations of φ and A preserve the instanton condition $F \wedge \psi = 0$ on V.

Deformations of $[\,(Y, \varphi), (V, A)\,]$

Varying the instanton equation $F \wedge \psi = 0$ we find

$$\check{d}_A(\alpha_t) = -\check{\mathcal{F}}(M_t) = -\pi_7(\mathcal{F}(M_t)) \ .$$

where

$$\begin{aligned} \mathcal{F} : \quad \Omega^k(Y, TY) &\longrightarrow \Omega^{k+1}(Y, \mathrm{End}(V)) \\ M &\mapsto \mathcal{F}(M) = (-1)^k \, i_M(F) \ . \end{aligned}$$

$\check{\mathcal{F}}$ maps M_t into a two form with values in $\mathrm{End}(V)$ which is exact in $\check{d}_A$-cohomology.

Deformations of $[(Y,\varphi),(V,A)]$

Then we have so far

$$\check{\mathrm{d}}_A(\alpha_t) = -\check{\mathcal{F}}(M_t) \quad \text{and} \quad i_{\check{\mathrm{d}}_\varsigma M_t}(\psi) = 0$$

which gives

$$\mathcal{TM}_1 = H^1_{\check{\mathrm{d}}_A}(Y, \mathrm{End}(V)) \oplus \ker \check{\mathcal{F}}\ , \qquad \ker \check{\mathcal{F}} \subseteq \mathcal{TM}_0$$

Again: there is no reason why the dimension should be finite (except in the case where Y has G_2 holonomy)

Deformations of the heterotic G_2 system

Moduli

Consider the action of $\mathcal{D}$ on one forms $\mathcal{Z}$ with values in

$$\mathcal{Q} = TY \oplus \mathrm{End}(TY) \oplus \mathrm{End}(V)$$

$$\mathcal{D}\mathcal{Z} = \begin{pmatrix} \mathrm{d}_\zeta M + \mathcal{R}(\kappa) - \mathcal{F}(\alpha) \\ \mathrm{d}_\Theta \kappa + \mathcal{R}(M) \\ \mathrm{d}_A \alpha + \mathcal{F}(M) \end{pmatrix} , \quad \mathcal{Z} = \begin{pmatrix} M \\ \kappa \\ \alpha \end{pmatrix}$$

Equations for moduli: $\check{\mathcal{D}}\mathcal{Z} = 0$

Deformations of the heterotic G_2 system

In particular,

$$\check{\mathrm{d}}_\zeta\, M_t + \check{\mathcal{R}}(\kappa_t) - \check{\mathcal{F}}(\alpha_t) = 0$$

turns out to be the same equation as

$$i_{\check{\mathrm{d}}_\zeta M_t}(\psi) = 0\ .$$

by the anomaly cancelation condition, if we identify the degrees of freedom corresponding to the antisymmetric part of M_t, with the (covariant) variations $\mathcal{B}_t$ of the B field

$$\partial_t H = \mathrm{d}\mathcal{B}_t + \frac{\alpha'}{2}\left(\mathrm{tr}(\alpha_t \wedge F) - \mathrm{tr}(\kappa_t \wedge R(\Theta))\right)\ .$$

Deformations of the heterotic G_2 system

Then

$$\mathcal{TM} = H^1_{\check{\mathcal{D}}}(Y, \mathcal{Q}) \ .$$

The $\check{\mathcal{D}}$-exact forms correspond to diffeomorphisms of Y and gauge transformations.

$\mathcal{TM}$ is finite dimensional.

Main result

heterotic structure on $(Y, [V, A], [TY, \Theta], H)$ $\longleftrightarrow$ special structure $\check{D}$ on $\mathcal{Q}$

infinitesimal moduli of the heterotic structure (massless spectrum) — infinitesimal deformations of the special structure on $\mathcal{Q}$ $\left(H^1_{\check{\mathcal{D}}}(Y, \mathcal{Q}) \right)$

Outlook and open problems

- We have derived the same results from a superpotential

$$W = \frac{1}{2}\int_Y e^{-2\phi}\left((H + h\varphi)\wedge\psi - \frac{1}{2}\,\mathrm{d}\varphi\wedge\varphi\right)$$

XD, M Larfors, E E Svanes, M Magill (1904.01027)

- Examples???

G2 holonomy: Joyce; Joyce & Karigiannis; Corti, Haskins, Nordström & Pacini; Braun; etc

Instatons on G2 holonomy manifolds: Wapulski; Sa Earp; Menet, Nordström & Sa Earp; ...

Fernández, Ivanov, Ugarte & Villacampa 2011; Fernández, Ivanov, Ugarte & Vassilev 2015

- And who can compute the cohomologies?

- Better understanding of the structure of moduli space of heterotic G_2 systems (Y, V, TY, H): What is the mathematical structure?

Outlook and open problems

- Global questions: metric? geometrical structure? singularities of the moduli space? higher order deformations and obstructions?
- $SU(3)$ case:
  - Analogue of the Maurer-Cartan equation and Kodaira-Spencer theory leads to L_∞-algebras and an analogue of the holomorphic Chern-Simons theory

    A. Ashmore, XD, R. Minasian, C. Strickland-Constable, E. Svanes (ArXiV 1806.08367)

    See also M. García Fernández, C. Rubio, C. Shahbazi, C. Tipler, 1803.01873, 1807.10329.
  - Universal Bundle P. Candelas, XD, J McOrist, R. Sisca (ArXiV 1810.00879)

Outlook and open problems

- CFT and σ-model perspective

  MA Fiset, C Quigley, E Svanes 1710.06865; Melnikov, Minasian & Sethi; XD, MA Fiset 1809.01138& in progress

- Quantum corrections? We have world sheet instanton corrections and NS5branes. For example

$$\mathrm{d}H = \frac{\alpha'}{4}(\mathrm{tr}F^2 - \mathrm{tr}R^2) + \Sigma$$

What are the non perturbative corrections the moduli space? What are the generalisations of the Donaldson-Thomas invariants?

Outlook and open problems

- Concept of mirror symmetry? Dualities?

$$\Gamma[\,(Y,\varphi),(V,A),(TY,\Theta),H\,]=\Gamma[\,(Y',\varphi'),(V',A'),(TY',\Theta'),H'\,]$$

- Relation with Type-II, *M*-theory and *F*-theory?