Orbital Dynamics for LIGO/Virgo from the Double Copy and EFT

July 11, 2019
Strings 2019

Zvi Bern

ZB, C. Cheung, R. Roiban, C. H. Shen, M. Solon, M. Zeng, arXiv:1901.04424 and to appear.

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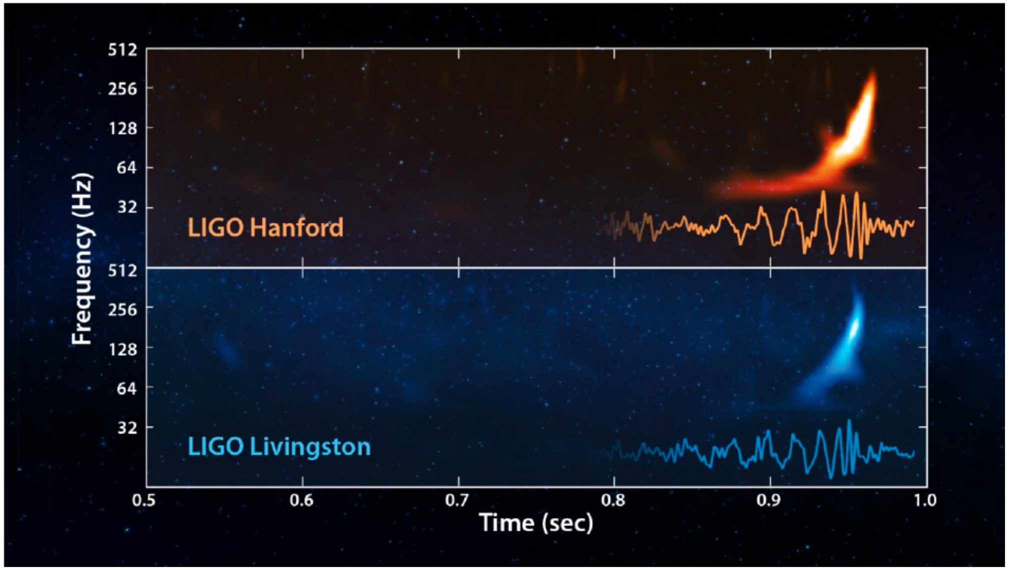
Outline

“String theory gives us new ways to think about gravity.”

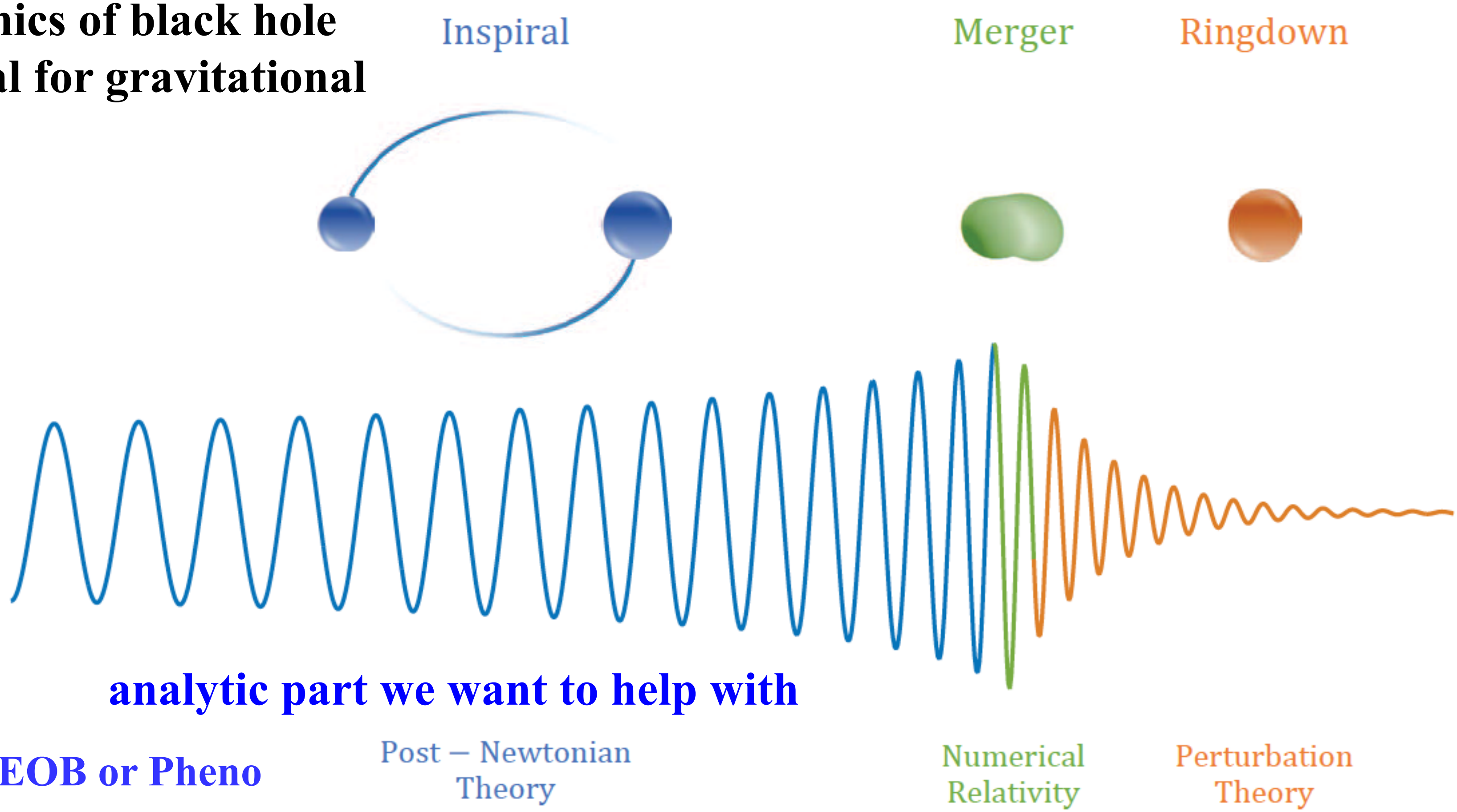
Era of gravitational wave astronomy has begun.



Ideas that flow from string theory give us a means to efficiently compute quantities of direct interest to LIGO theorists.

Goal: Improve on post-Newtonian Theory

Dynamics of black hole inspiral for gravitational waves.



analytic part we want to help with

PN + EOB or Pheno

Small errors accumulate. Need for high precision.

From Antelis and Moreno, arXiv:1610.03567

Gravity vs Gauge Theory

Consider the Einstein gravity Lagrangian

curvature

Flat-space metric

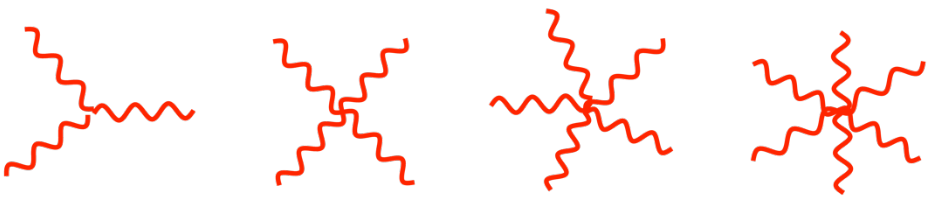
$$L_{\text{gravity}} = \frac{2}{\kappa^2}\sqrt{-g}\,R \qquad g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$$

graviton field

metric

$$\kappa^2 = 32\pi G_{\text{Newton}}$$

Infinite number of complicated interactions

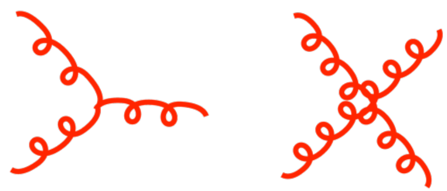


$+ \cdots$

terrible mess

Compare to gauge-theory Lagrangian on which QCD is based

$$L_{\text{YM}} = \frac{1}{g^2}F^2$$



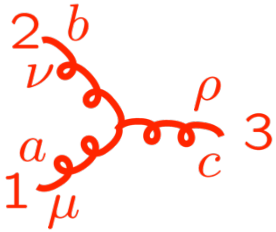
Only three and four point interactions

Gravity seems so much more complicated than gauge theory.

Gauge and gravity theories not simply related.

Three Vertices

Standard perturbative approach:



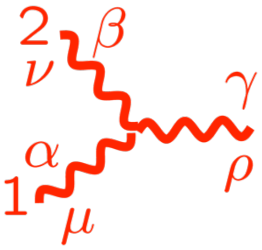
Three-gluon vertex:

$$V_{3\,\mu\nu\sigma}^{abc} = -g f^{abc}(\eta_{\mu\nu}(k_1-k_2)_\rho+\eta_{\nu\rho}(k_1-k_2)_\mu+\eta_{\rho\mu}(k_1-k_2)_\nu)$$

Three-graviton vertex:

$$k_i^2 = E_i^2 - \vec{k}_i^2 \neq 0$$

$$\begin{aligned} G_{3\mu\alpha,\nu\beta,\sigma\gamma}(k_1,k_2,k_3) = \\ \mathrm{sym}[-\tfrac{1}{2}P_3(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\sigma\gamma}) - \tfrac{1}{2}P_6(k_{1\nu}k_{1\beta}\eta_{\mu\alpha}\eta_{\sigma\gamma}) + \tfrac{1}{2}P_3(k_1\cdot k_2\eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\sigma\gamma}) \\ + P_6(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\sigma}\eta_{\beta\gamma}) + 2P_3(k_{1\nu}k_{1\gamma}\eta_{\mu\alpha}\eta_{\beta\sigma}) - P_3(k_{1\beta}k_{2\mu}\eta_{\alpha\nu}\eta_{\sigma\gamma}) \\ + P_3(k_{1\sigma}k_{2\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + P_6(k_{1\sigma}k_{1\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + 2P_6(k_{1\nu}k_{2\gamma}\eta_{\beta\mu}\eta_{\alpha\sigma}) \\ + 2P_3(k_{1\nu}k_{2\mu}\eta_{\beta\sigma}\eta_{\gamma\alpha}) - 2P_3(k_1\cdot k_2\eta_{\alpha\nu}\eta_{\beta\sigma}\eta_{\gamma\mu})] \end{aligned}$$



About 100 terms in three vertex

Naïve conclusion: Gravity is a nasty mess.

String Ideas Help Tame the Complexity

KLT (1985)

Kawai-Lewellen-Tye string relations in low-energy limit:

gravity ↙ ↙ gauge-theory color ordered

$$M_4^{\text{tree}}(1,2,3,4) = -is_{12} A_4^{\text{tree}}(1,2,3,4)\, A_4^{\text{tree}}(1,2,4,3)\,,$$

$$M_5^{\text{tree}}(1,2,3,4,5) = is_{12}s_{34} A_5^{\text{tree}}(1,2,3,4,5)\, A_5^{\text{tree}}(2,1,4,3,5) + is_{13}s_{24} A_5^{\text{tree}}(1,3,2,4,5)\, A_5^{\text{tree}}(3,1,4,2,5)$$

Generalizes to explicit all-leg form. ZB, Dixon, Perelstein, Rozowsky

1. **Gravity is derivable from gauge theory. Standard Lagrangian methods offers no hint why this is possible.**
2. **It is very generally applicable.**

Duality Between Color and Kinematics

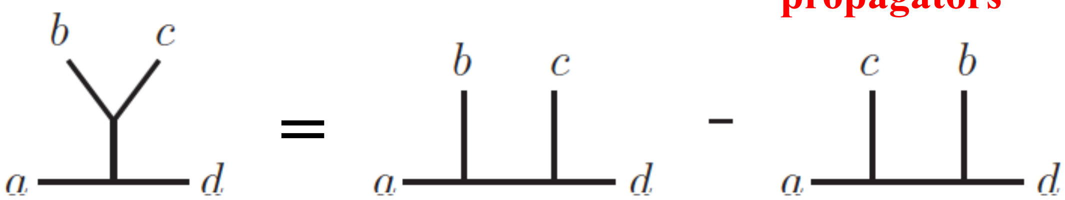
gauge theory scattering amplitudes:

ZB, Carrasco, Johansson

$$\mathcal{A}_m^{\text{tree}} = g^{m-2} \sum_j \frac{c_j n_j}{D_j}$$

color factor → c_j; kinematic numerator → n_j; Feynman propagators → D_j

Color or numerator factors related by Jacobi identities.



Proven at tree level.
Conjectured at loop level.

$$c_i = c_j - c_k \;\Rightarrow\; n_i = n_j - n_k$$

gauge theory → gravity theory

simply take

color factor → kinematic numerator

gravity:

$$\mathcal{M}_m^{\text{tree}} = i\left(\frac{\kappa}{2}\right)^{m-2} \sum_j \frac{\tilde{n}_j n_j}{D_i}$$

$c_k \longrightarrow n_k$

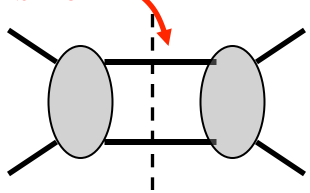
From Tree to Loops: Generalized Unitarity Method

No Feynman rules; no need for virtual particles.

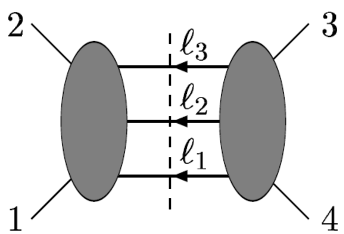
$E^2 = \vec{p}^{\,2} + m^2$ ← **on-shell**

ZB, Dixon, Dunbar and Kosower (1994)

Two-particle cut:



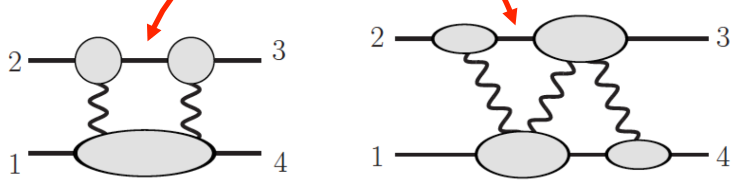
Three-particle cut:



- **Systematic assembly of complete loop amplitudes from tree amplitudes.**
- **Works for any number of particles or loops.**

on-shell

Generalized unitarity as a practical tool for loops.



ZB, Dixon and Kosower;
ZB, Morgan;
Britto, Cachazo, Feng;
Ossala,Pittau,Papadopoulos;
Ellis, Kunszt, Melnikov;
Forde; Badger;
ZB, Carrasco, Johansson, Kosower
and many others

Reproduces Feynman diagrams, except intermediate steps of calculation based on gauge-invariant quantities.

Post Newtonian Approximation

For orbital mechanics:

Expand in G and v^2

$$v^2 \sim \frac{GM}{r} \ll 1$$

virial theorem

In center of mass frame:

$$m = m_A + m_B, \quad \nu = \mu/M,$$

$$\mu = m_A m_B/m, \quad P_R = P \cdot \hat{R}$$

$$\frac{H}{\mu} = \frac{P^2}{2} - \frac{Gm}{R}$$

← Newton

$$+ \frac{1}{c^2}\left\{ -\frac{P^4}{8} + \frac{3\,\nu\,P^4}{8} + \frac{Gm}{R}\left(-\frac{{P_R}^2\,\nu}{2} - \frac{3\,P^2}{2} - \frac{\nu\,P^2}{2} \right) + \frac{G^2 m^2}{2R^2} \right\}$$

$$+ \ldots$$

1PN: Einstein, Infeld, Hoffmann

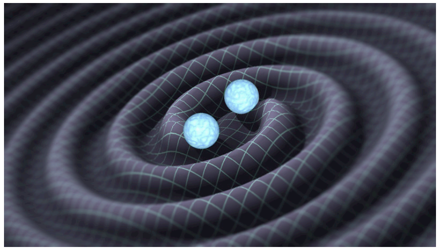
Hamiltonian known to 4PN order.

2PN: Ohta, Okamura, Kimura and Hiida.
3PN: Damour, Jaranowski and Schaefer; L. Blanchet and G. Faye.
4PN: Damour, Jaranowski and Schaefer; Foffa, Porto, Rothstein, Sturani.

Which problem to solve?

ZB, Cheung, Roiban, Shen, Solon, Zeng

Some problems for (analytic) theorists:



1. Spin.
2. Finite size effects.
3. Radiation.
4. High orders in perturbation theory.

Which problem should we solve?

- Needs to be extremely difficult using standard methods.
- Needs to be of direct interest to LIGO theorists.
- Needs to be in a form that can in principle enter LIGO analysis pipeline.

2-body Hamiltonian at 3rd post-Minkowskian order

PN versus PM expansion for conservative two-body dynamics

$$\mathcal{L} = -Mc^2 + \underbrace{\frac{\mu v^2}{2} + \frac{GM\mu}{r}} + \frac{1}{c^2}\left[\dots\right] + \frac{1}{c^4}\left[\dots\right] + \dots$$

From Buonanno Amplitudes 2018

$$E(v) = -\frac{\mu}{2}v^2 + \cdots$$

non-spinning compact objects

		0PN	1PN	2PN	3PN	4PN	5PN	...
0PM:	1	v^2	v^4	v^6	v^8	v^{10}	v^{12}	...
1PM:		$1/r$	v^2/r	v^4/r	v^6/r	v^8/r	v^{10}/r	...
2PM:			$1/r^2$	v^2/r^2	v^4/r^2	v^6/r^2	v^8/r^2	...
3PM:				$1/r^3$	v^2/r^3	v^4/r^3	v^6/r^3	...
4PM:					$1/r^4$	v^2/r^4	v^4/r^4	...
...						...	...	

(credit: Justin Vines)

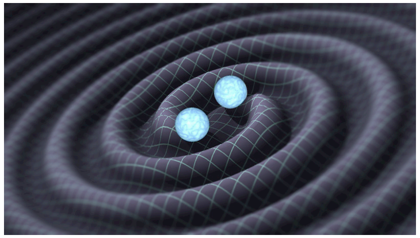
$$1 \to Mc^2, \qquad v^2 \to \frac{v^2}{c^2}, \qquad \frac{1}{r} \to \frac{GM}{rc^2}.$$

current known PN results — current known PM results — overlap between PN & PM results — unknown

- PM results (Westfahl 79, Westfahl & Goller 80, Portilla 79-80, Bel et al. 81, Ledvinka et al. 10, Damour 16-17, Guevara 17, Vines 17, Bini & Damour 17-18, Vines in prep)

Double Copy and Gravitational Radiation

Can we simplify the types of calculations needed for LIGO?



A small industry has developed to study this.

- **Connection to scattering amplitudes.**

  Bjerrum-Bohr, Donoghue, Holstein, Plante, Pierre Vanhove; Luna, Nicholson, O'Connell, White; Guevara; Bjerrum-Bohr, Damgaard, Festuccia, Planté, Vanhove; Cheung, Rothstein, Solon; Damour; Bautista, Guevara; Kosower, Maybee, O'Connell; Plefka, Steinhoff, Wormsbecher; Foffa, Mastrolia, Sturani, Sturm; Guevara, Ochirov, Vines; Chung, Huang, Kim, Lee; etc.

- **Worldline approach for radiation.**

  Goldberger and Ridgway; Goldberger, Li, Prabhu, Thompson; Chester; Shen

- **Removing the dilaton contamination.**

  Luna, Nicholson, O'Connell, White; Johansson, Ochirov; Johansson, Kalin; Henrik Johansson, Gregor Kälin, Mogull.

Key Question: Can we calculate something of direct interest to LIGO/Virgo, clearly *beyond* previous state of the art?

Given BCJ duality and double copy allows us to do supergravity calculations at 5 loops you might expect it can help with gravitation waves.

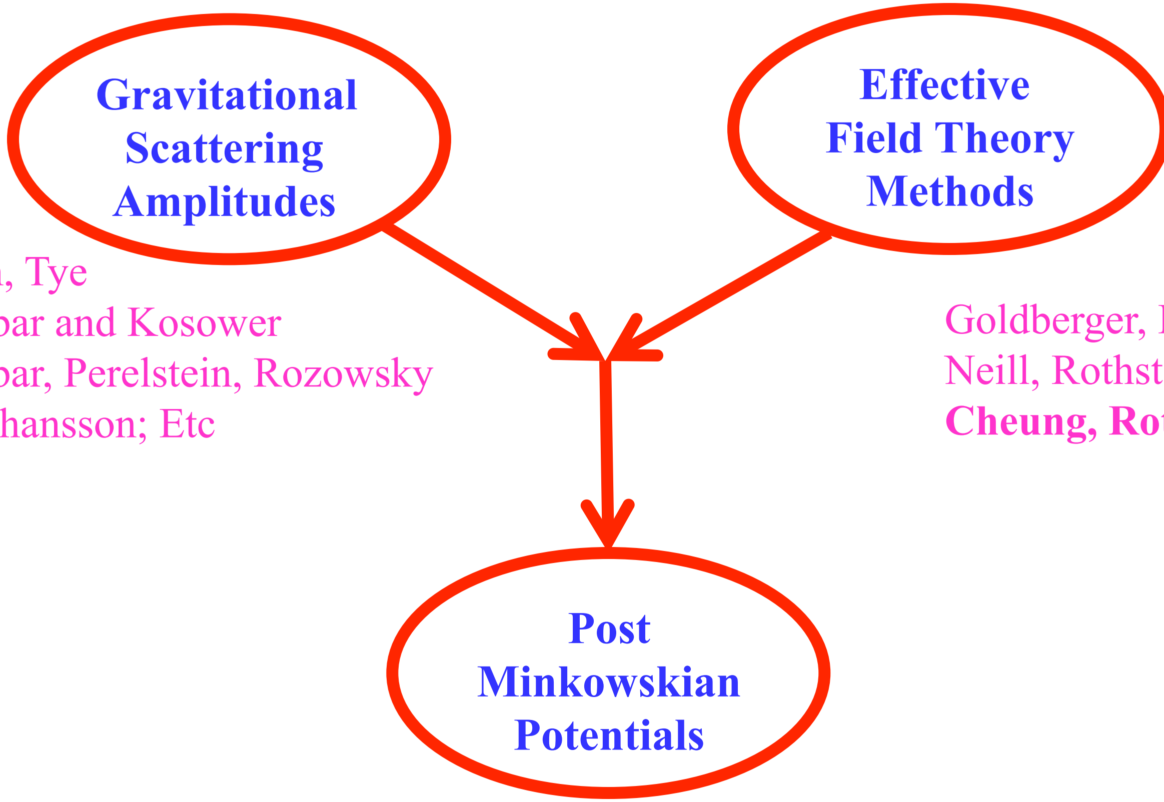
ZB, Carrasco, Chen, Johansson, Roiban, Zeng

Our Approach

ZB, Cheung, Roiban, Shen, Solon, Zeng

Amplitudes community

EFT community



Kawai, Lewellen, Tye
ZB, Dixon, Dunbar and Kosower
ZB, Dixon, Dunbar, Perelstein, Rozowsky
ZB, Carrasco, Johansson; Etc

Goldberger, Rothstein
Neill, Rothstein
Cheung, Rothstein, Solon (2018)

Inefficient: Start with quantum theory and take $\hbar \to 0$

Efficient: Almost magical simplifications for gravity amplitudes.
EFT methods efficiently target pieces we want.

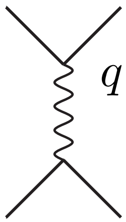
Efficiency wins

2 Body Potentials and Amplitudes

Iwasaki; Gupta, Radford; Donoghue; Holstein, Donoghue; Holstein and A. Ross; Bjerrum-Bohr, Donoghue, Vanhove; Neill, Rothstein; Bjerrum-Bohr, Damgaard, Festuccia, Planté. Vanhove; Chueng, Rothstein, Solon; Chung, Huang, Kim, Lee; etc.

Tree-level: Fourier transform gives classical potential.

$$V(r) = \int \frac{d^3q}{(2\pi)^3} e^{-i\mathbf{q}\cdot\mathbf{r}} A^{\text{tree}}(\mathbf{q})$$



Beyond 1 loop things quickly become less obvious:

- **What you learned in grad school on $\hbar$ counting is wrong. Loops can have classical pieces.**
- **Double counting and iteration.**
- **$1/\hbar^L$ scaling of at L loop.**
- **Cross terms between $1/\hbar$ and $\hbar$.**
- **Which corners of multiloop integrals are classical? How to extract?**

$$e^{iS_{\text{classical}}/\hbar}$$

Piece of loops are classical: Our task is to efficiently extract these pieces.

We harness EFT to clean up confusion

EFT is a Clean Approach

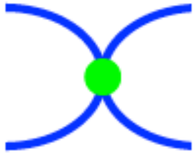
Build EFT from which we can read off potential. Avoids a variety of confusions related to taking classical limit.

Goldberger and Rothstein
Neill, Rothstein
Cheung, Rothstein, Solon (2018)

$$L_{\rm kin} = \int_{\boldsymbol{k}} A^\dagger(-\boldsymbol{k}) \left(i\partial_t + \sqrt{\boldsymbol{k}^2 + m_A^2} \right) A(\boldsymbol{k}) + \int_{\boldsymbol{k}} B^\dagger(-\boldsymbol{k}) \left(i\partial_t + \sqrt{\boldsymbol{k}^2 + m_B^2} \right) B(\boldsymbol{k})$$

***A*, *B* scalars represents spinless black holes**

$$L_{\rm int} = -\int_{\boldsymbol{k},\boldsymbol{k}'} V(\boldsymbol{k},\boldsymbol{k}')\, A^\dagger(\boldsymbol{k}') A(\boldsymbol{k}) B^\dagger(-\boldsymbol{k}') B(-\boldsymbol{k})$$



Match amplitudes of this theory to the full theory in classical limit to extract a potential.

EFT Matching

Cheung, Rothstein, Solon

full general relativity
(complicated)

Amplitude methods
double copy

tree amplitude

generalized
unitarity

$\hbar \to 0$

loop integrand

loop
integration

GR loop amplitude

effective theory
(simpler)

build
ansatz

potential

Feynman
diagrams

loop integrand

loop
integration

EFT loop amplitude

identical physics

GR loop amplitude $=$ EFT loop amplitude

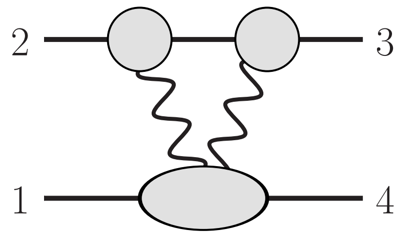
Roundabout, but efficiently determines potential

Full Theory: Unitarity + Double Copy

- **Long range force: Two matter lines must be separated by cut propagators.**
- **Classical potential: 1 matter line per loop is cut (on-shell).**

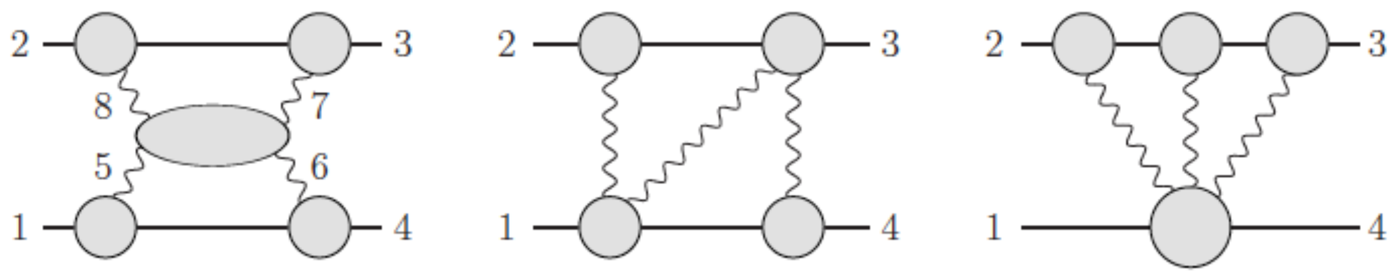
Neill and Rothstein ; Bjerrum-Bohr, Damgaard, Festuccia, Planté, Vanhove; Cheung, Rothstein, Solon

Only independent unitarity cut for 2 PM.



exposed lines on-shell (long range).
Classical pieces simple!

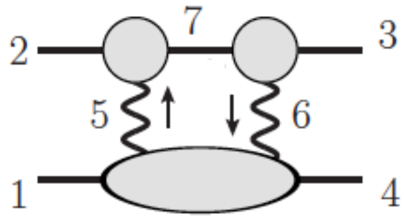
Independent generalized unitarity cuts for 3 PM.



Use our amplitudes tools for this part.

Generalized Unitarity Cuts

Primary means of construction uses BCJ in *D* dimensions, but KLT with helicty should have better scaling at higher loops and gives compact expressions.



2^{nd} post-Minkowkian order

$$C_{\rm GR} = \sum_{h_5,h_6=\pm} M_3^{\rm tree}(3^s,6^{h_6},-7^s)\,M_3^{\rm tree}(7^s,-5^{h_5},2^s)\,M_4^{\rm tree}(1^s,5^{-h_5},-6^{-h_6},4^s)$$
$$= \sum_{h_5,h_6=\pm} it[A_3^{\rm tree}(3^s,6^{h_6},-7^s)\,A_3^{\rm tree}(7^s,-5^{h_5},2^s)\,A_4^{\rm tree}(1^s,5^{-h_5},-6^{-h_6},4^s)]$$
$$\times\,[A_3^{\rm tree}(3^s,6^{h_6},-7^s)\,A_3^{\rm tree}(7^s,-5^{h_5},2^s)\,A_4^{\rm tree}(4^s,5^{-h_5},-6^{-h_6},1^s)]$$

By correlating gluon helicities, removing dilaton is trivial.

$$h^-_{\mu\nu} \rightarrow A^-_\mu A^-_\mu \qquad h^+_{\mu\nu} \rightarrow A^+_\mu A^+_\mu$$

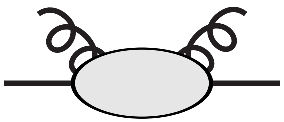
Forbid: $A^+_\mu A^-_\mu$

Problem of computing the generalized cuts in gravity is reduced to multiplying and summing gauge-theory tree amplitudes.

Gauge-Theory Building Blocks for 2 PM Gravity

$$A^{\text{tree}}(1^s, 2^+, 3^+, 4^s) = i\frac{m_1^2[23]}{\langle 23\rangle t_{12}}$$

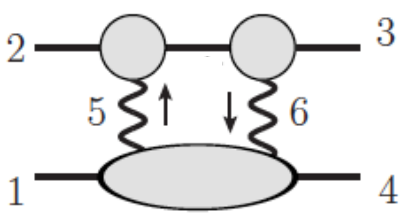
$$A^{\text{tree}}(1^s, 2^+, 3^-, 4^s) = -i\frac{\langle 3|1|2]^2}{\langle 23\rangle [23] t_{12}}$$



color-ordered gauge-theory tree amplitudes

- This is all you need for 2 PM.
- Scaling with number of loops is very good.

$$s_{23} = (p_2 + p_3)^2$$
$$t_{ij} = 2p_i \cdot p_j$$



$$C_{\text{YM}} = 2\left(\frac{\mathcal{E}^2 + \mathcal{O}^2}{s_{23}^2} + m_1^2 m_2^2\right)\frac{1}{t_{15}}$$

sum over helicities

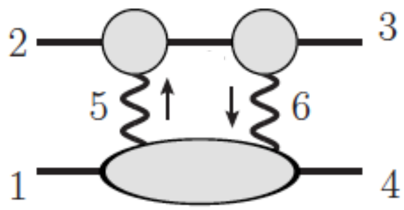
gauge theory

$$\mathcal{E}^2 = \frac{1}{4}s_{23}^2(t_{15} - t_{12})^2$$
$$\mathcal{O}^2 = \mathcal{E}^2 - s_{23}m_2^2(s_{23}m_1^2 + s_{23}t_{15} + t_{15}^2)$$

Simple cut contains all information we need for 2PM Hamiltonian

One loop gravity warmup

$$s_{23} = (p_2 + p_3)^2$$
$$t_{ij} = 2p_i \cdot p_j$$



Apply unitarity and KLT relations.
Import gauge-theory results.

$$C_{\rm GR} = 2\left[\frac{1}{t^4}\left(\mathcal{E}^4 + \mathcal{O}^4 + 6\mathcal{E}^2\mathcal{O}^2\right) + m_1^4 m_2^4\right]\left[\frac{1}{t_{15}} + \frac{1}{t_{45}}\right]$$

- **Same building blocks as gauge theory.**
- **Double copy is visible even though we removed dilaton and axion.**

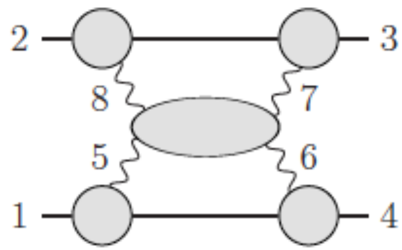
We can extract classical scattering angles or potentials following 2 PM literature.

Damour; Bjerrum-Bohr, Damgaard, Festuccia, Planté, Vanhove; Cheung, Rothstein, Solon; Koemans Collado, Di Vecchia, Russo

Two Loops for 3 PM

$$s_{23} = (p_2 + p_3)^2$$
$$t_{ij} = 2p_i \cdot p_j$$

ZB, Cheung, Shen, Roiban, Solon, Zeng



- Used *D* dimensional BCJ on trees.
- Alternative KLT with helicity in *D* = 4
- Very similar to one loop.

$$C^{\text{H-cut}} = 2i\left[\frac{1}{(p_5 - p_8)^2} + \frac{1}{(p_5 + p_7)^2}\right] \times \left[s_{23}^2 m_1^4 m_2^4 + \frac{1}{s_{23}^6}\sum_{i=1,2}\left(\mathcal{E}_i^4 + \mathcal{O}_i^4 + 6\mathcal{O}_i^2\mathcal{E}_i^2\right)\right]$$

D =4 result

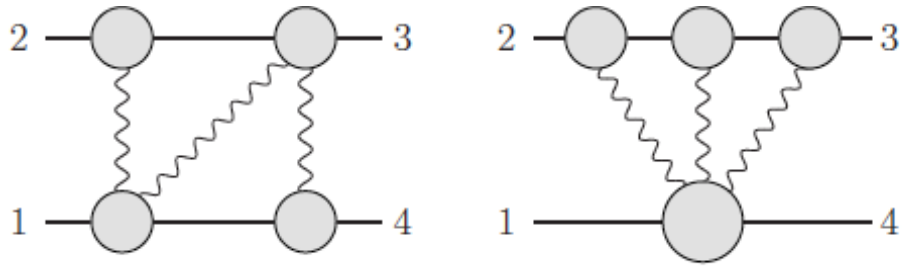
$$\mathcal{E}_1^2 = \frac{1}{4}s_{23}^2(t_{18}t_{25} - t_{12}t_{58})^2, \qquad \mathcal{O}_1^2 = \mathcal{E}_1^2 - m_1^2 m_2^2 s_{23}^2 t_{58}^2,$$
$$\mathcal{E}_2^2 = \frac{1}{4}s_{23}^2(t_{17}t_{25} - t_{12}t_{57} - s_{23}(t_{17} + t_{57}))^2,$$
$$\mathcal{O}_2^2 = \mathcal{E}_2^2 - m_1^2 m_2^2 s_{23}^2 t_{57}^2.$$

Remarkably simple,
given it is two-loop gravity.

- Double copy is visible.
- *D*-dimensional result slightly more complicated involving Gram dets.
- Differences between *D* = 4 and *D* = 4 - 2ε drops out in classical limit.

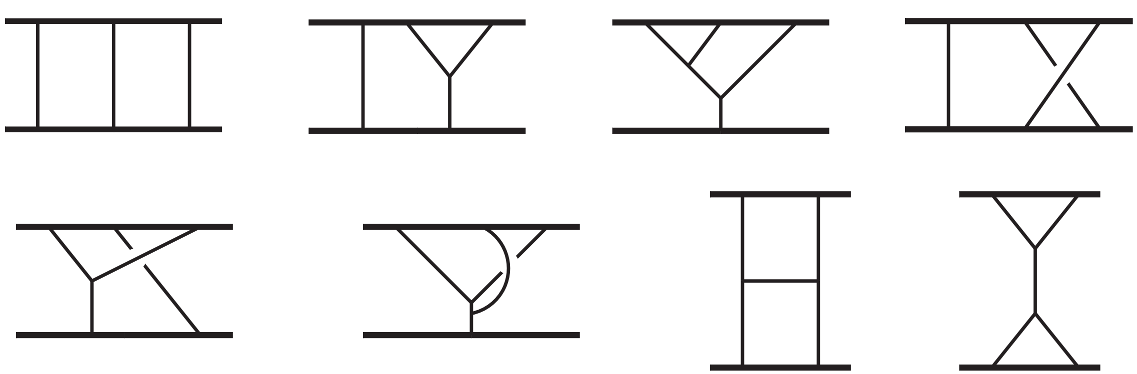
Two Loops and 3 PM

Also need contributions from other cuts.



- **More complicated than previous cut.**
- **Evaluated using *D*-dimensional BCJ double copy, with 4-*D* KLT double copy as check.**
- **To interface easily with EFT approach, merged unitarity cuts into diagrams to get integrand.**

Integrand organized into 8 independent diagrams that may contribute in classical limit:



Integration + Extraction of Potential

To integrate, follow path of Cheung, Rothstein and Solon.

- **Efficiently targets the classical pieces of integrals we want.**
- **Integrals reduce via energy residues to 3 dimensional integrals.**

$$\ell_0 \ll |\vec{\ell}| \ll |\vec{p}| \ll m_i$$

potential classical nonrelativistic

$$\mathcal{I} = \left[\prod_{i=1}^{n_M} \frac{1}{\varepsilon_i^2 - \boldsymbol{k}_i^2 - m_i^2}\right]\left[\prod_{j=1}^{n_G} \frac{1}{\omega_j^2 - \boldsymbol{\ell}_j^2}\right]\mathcal{N}$$

kinematic numerator

matter poles **graviton poles**

In classical potential region don't pick up anti-particle or graviton poles:

$$\mathcal{I} = \left[\prod_{i=1}^{n_M} \frac{1}{\varepsilon_i - \sqrt{\boldsymbol{k}_i^2 - m_i^2}}\right]\tilde{\mathcal{N}} \qquad \tilde{\mathcal{N}} = \left[\prod_{i=1}^{n_M} \frac{1}{\varepsilon_i + \sqrt{\boldsymbol{k}_i^2 + m_i^2}}\right]\left[\prod_{j=1}^{n_G} \frac{1}{\omega_j^2 - \boldsymbol{\ell}_j^2}\right]\mathcal{N}$$

Energy integration: expand and evaluate term by term or even better do by residues. Get 3*D* integrals that are either textbook or use IBP.

Amplitude in Classical Potential Limit

ZB, Cheung, Roiban, Shen, Solon, Zeng (2019)

Classical limit. The O(G^3) or 3PM terms are:

rapidity

$$\mathcal{M}_3 = \frac{\pi G^3 \nu^2 m^4 \log \boldsymbol{q}^2}{6\gamma^2 \xi}\Bigg[3 - 6\nu + 206\nu\sigma - 54\sigma^2 + 108\nu\sigma^2 + 4\nu\sigma^3 - \frac{48\nu\left(3 + 12\sigma^2 - 4\sigma^4\right)\operatorname{arcsinh}\sqrt{\frac{\sigma-1}{2}}}{\sqrt{\sigma^2-1}} - \frac{18\nu\gamma\left(1-2\sigma^2\right)\left(1-5\sigma^2\right)}{(1+\gamma)(1+\sigma)}\Bigg] + \frac{8\pi^3 G^3 \nu^4 m^6}{\gamma^4 \xi}\Big[3\gamma\left(1-2\sigma^2\right)\left(1-5\sigma^2\right)F_1 - 32 m^2 \nu^2 \left(1-2\sigma^2\right)^3 F_2\Big]$$

$$m = m_A + m_B, \qquad \mu = m_A m_B / m, \qquad \nu = \mu/m, \qquad \gamma = E/m,$$
$$\xi = E_1 E_2 / E^2, \qquad E = E_1 + E_2, \qquad \sigma = p_1 \cdot p_2 / m_1 m_2,$$

F_1 and F_2 IR divergent iteration terms that don't affect potential.

Amplitude containing classical potential is remarkably simple!

Conservative 3PM Hamiltonian

ZB, Cheung, Roiban, Shen, Solon, Zeng (2019)

The 3PM Hamiltonian:

$$H(\boldsymbol{p},\boldsymbol{r}) = \sqrt{\boldsymbol{p}^2 + m_1^2} + \sqrt{\boldsymbol{p}^2 + m_2^2} + V(\boldsymbol{p},\boldsymbol{r})$$

$$V(\boldsymbol{p},\boldsymbol{r}) = \sum_{i=1}^{3} c_i(\boldsymbol{p}^2)\left(\frac{G}{|\boldsymbol{r}|}\right)^i ,$$

$$c_1 = \frac{\nu^2 m^2}{\gamma^2\xi}\left(1-2\sigma^2\right), \qquad c_2 = \frac{\nu^2 m^3}{\gamma^2\xi}\left[\frac{3}{4}\left(1-5\sigma^2\right) - \frac{4\nu\sigma\left(1-2\sigma^2\right)}{\gamma\xi} - \frac{\nu^2(1-\xi)\left(1-2\sigma^2\right)^2}{2\gamma^3\xi^2}\right],$$

$$\begin{aligned} c_3 = \frac{\nu^2 m^4}{\gamma^2\xi}\Bigg[&\frac{1}{12}\left(3-6\nu+206\nu\sigma-54\sigma^2+108\nu\sigma^2+4\nu\sigma^3\right) - \frac{4\nu\left(3+12\sigma^2-4\sigma^4\right)\operatorname{arcsinh}\sqrt{\frac{\sigma-1}{2}}}{\sqrt{\sigma^2-1}} \\ &- \frac{3\nu\gamma\left(1-2\sigma^2\right)\left(1-5\sigma^2\right)}{2(1+\gamma)(1+\sigma)} - \frac{3\nu\sigma\left(7-20\sigma^2\right)}{2\gamma\xi} - \frac{\nu^2\left(3+8\gamma-3\xi-15\sigma^2-80\gamma\sigma^2+15\xi\sigma^2\right)\left(1-2\sigma^2\right)}{4\gamma^3\xi^2} \\ &+ \frac{2\nu^3(3-4\xi)\sigma\left(1-2\sigma^2\right)^2}{\gamma^4\xi^3} + \frac{\nu^4(1-2\xi)\left(1-2\sigma^2\right)^3}{2\gamma^6\xi^4}\Bigg], \end{aligned}$$

$$m = m_A + m_B, \qquad \mu = m_A m_B/m, \qquad \nu = \mu/m, \qquad \gamma = E/m,$$

$$\xi = E_1 E_2/E^2, \qquad E = E_1 + E_2, \qquad \sigma = p_1 \cdot p_2/m_1 m_2,$$

Some Subtleties and Features

ZB, Cheung, Roiban, Shen, Solon, Zeng

1. **IR singularities and dimensional regularization.**
   - **Differing integrands constructed in $D = 4$ or $D = 4 - 2\varepsilon$ dimensions.**
   - **No difference in final potential.**
   - **Likely true at all orders, but general proof needed.**
2. **A mass singularity as $m_i \to 0$**
   - **Quantum amplitudes do not have mass singularities but classical amplitude for potential does. Yes, log(m) is definitely there.**
   - **Massless results *cannot* be blindly imported.**
   - **Interesting possibility to resum the logs.**
3. **Integrals done by expanding in velocity and resumming.**
   - **Strong checks from Mellin-Barnes, IBP, differential equations, sector decomposition: no resummation.**

Checks

Primary check:

ZB, Cheung, Roiban, Shen, Solon, Zeng

Compare to 4PN Hamiltonian of Damour, Jaranowski, Schäfer

Need canonical transformation:

$$(\boldsymbol{r},\boldsymbol{p}) \to (\boldsymbol{R},\boldsymbol{P}) = (A\,\boldsymbol{r} + B\,\boldsymbol{p}, C\,\boldsymbol{p} + D\,\boldsymbol{r})$$

$$A = 1 - \frac{Gm\nu}{2|\boldsymbol{r}|} + \cdots, \quad B = \frac{G(1-2/\nu)}{4m|\boldsymbol{r}|}\boldsymbol{p}\cdot\boldsymbol{r} + \cdots$$

$$C = 1 + \frac{Gm\nu}{2|\boldsymbol{r}|} + \cdots, \quad D = -\frac{Gm\nu}{2|\boldsymbol{r}|^3}\boldsymbol{p}\cdot\boldsymbol{r} + \cdots,$$

Our Hamiltonian equivalent to 4PN Hamitonian on overlap.

Additional (somewhat redundant) tests:

1. Classical scattering angle matches 4PN result in overlap Bini and Damour

2. Amplitude using 4PN potentials matches result in overlap.
Damour, Jaranowski, Schaefer

3. In test mass limit, $m_1 \ll m_2$, matches Schwarzschild Hamiltonian.
Wex and Schaefer

4 PN Hamiltonian

Damour, Jaranowski, Schaefer

$$\mathbf{n} = \hat{\mathbf{r}}$$

$$\widehat{H}_{\rm N}(\mathbf{r},\mathbf{p}) = \frac{\mathbf{p}^2}{2} - \frac{1}{r},$$

$$c^2\,\widehat{H}_{\rm 1PN}(\mathbf{r},\mathbf{p}) = \frac{1}{8}(3\nu-1)(\mathbf{p}^2)^2 - \frac{1}{2}\Big\{(3+\nu)\mathbf{p}^2 + \nu(\mathbf{n}\cdot\mathbf{p})^2\Big\}\frac{1}{r} + \frac{1}{2r^2},$$

$$c^4\,\widehat{H}_{\rm 2PN}(\mathbf{r},\mathbf{p}) = \frac{1}{16}\left(1-5\nu+5\nu^2\right)(\mathbf{p}^2)^3 + \frac{1}{8}\Big\{\left(5-20\nu-3\nu^2\right)(\mathbf{p}^2)^2 - 2\nu^2(\mathbf{n}\cdot\mathbf{p})^2\mathbf{p}^2 - 3\nu^2(\mathbf{n}\cdot\mathbf{p})^4\Big\}\frac{1}{r}$$
$$+ \frac{1}{2}\Big\{(5+8\nu)\mathbf{p}^2 + 3\nu(\mathbf{n}\cdot\mathbf{p})^2\Big\}\frac{1}{r^2} - \frac{1}{4}(1+3\nu)\frac{1}{r^3},$$

$$c^6\,\widehat{H}_{\rm 3PN}(\mathbf{r},\mathbf{p}) = \frac{1}{128}\left(-5+35\nu-70\nu^2+35\nu^3\right)(\mathbf{p}^2)^4 + \frac{1}{16}\Big\{\left(-7+42\nu-53\nu^2-5\nu^3\right)(\mathbf{p}^2)^3$$
$$+ (2-3\nu)\nu^2(\mathbf{n}\cdot\mathbf{p})^2(\mathbf{p}^2)^2 + 3(1-\nu)\nu^2(\mathbf{n}\cdot\mathbf{p})^4\mathbf{p}^2 - 5\nu^3(\mathbf{n}\cdot\mathbf{p})^6\Big\}\frac{1}{r}$$
$$+ \Big\{\frac{1}{16}\left(-27+136\nu+109\nu^2\right)(\mathbf{p}^2)^2 + \frac{1}{16}(17+30\nu)\nu(\mathbf{n}\cdot\mathbf{p})^2\mathbf{p}^2 + \frac{1}{12}(5+43\nu)\nu(\mathbf{n}\cdot\mathbf{p})^4\Big\}\frac{1}{r^2}$$
$$+ \bigg\{\bigg(-\frac{25}{8} + \left(\frac{\pi^2}{64} - \frac{335}{48}\right)\nu - \frac{23\nu^2}{8}\bigg)\mathbf{p}^2 + \left(-\frac{85}{16} - \frac{3\pi^2}{64} - \frac{7\nu}{4}\right)\nu(\mathbf{n}\cdot\mathbf{p})^2\bigg\}\frac{1}{r^3} + \bigg\{\frac{1}{8} + \left(\frac{109}{12} - \frac{21}{32}\pi^2\right)\nu\bigg\}\frac{1}{r^4},$$

G^4

4 PN Hamiltonian

Damour, Jaranowski, Schaefer

$$\mathbf{n} = \hat{\mathbf{r}}$$

$$
\begin{aligned}
c^8 \widehat{H}^{\rm local}_{\rm 4PN}(\mathbf{r},\mathbf{p}) =& \left(\frac{7}{256} - \frac{63}{256}\nu + \frac{189}{256}\nu^2 - \frac{105}{128}\nu^3 + \frac{63}{256}\nu^4\right)(\mathbf{p}^2)^5 \\
&+ \Bigg\{\frac{45}{128}(\mathbf{p}^2)^4 - \frac{45}{16}(\mathbf{p}^2)^4\nu + \left(\frac{423}{64}(\mathbf{p}^2)^4 - \frac{3}{32}(\mathbf{n}\cdot\mathbf{p})^2(\mathbf{p}^2)^3 - \frac{9}{64}(\mathbf{n}\cdot\mathbf{p})^4(\mathbf{p}^2)^2\right)\nu^2 \\
&+ \left(-\frac{1013}{256}(\mathbf{p}^2)^4 + \frac{23}{64}(\mathbf{n}\cdot\mathbf{p})^2(\mathbf{p}^2)^3 + \frac{69}{128}(\mathbf{n}\cdot\mathbf{p})^4(\mathbf{p}^2)^2 - \frac{5}{64}(\mathbf{n}\cdot\mathbf{p})^6\mathbf{p}^2 + \frac{35}{256}(\mathbf{n}\cdot\mathbf{p})^8\right)\nu^3 \\
&+ \left(-\frac{35}{128}(\mathbf{p}^2)^4 - \frac{5}{32}(\mathbf{n}\cdot\mathbf{p})^2(\mathbf{p}^2)^3 - \frac{9}{64}(\mathbf{n}\cdot\mathbf{p})^4(\mathbf{p}^2)^2 - \frac{5}{32}(\mathbf{n}\cdot\mathbf{p})^6\mathbf{p}^2 - \frac{35}{128}(\mathbf{n}\cdot\mathbf{p})^8\right)\nu^4\Bigg\}\frac{1}{r} \\
&+ \Bigg\{\frac{13}{8}(\mathbf{p}^2)^3 + \left(-\frac{791}{64}(\mathbf{p}^2)^3 + \frac{49}{16}(\mathbf{n}\cdot\mathbf{p})^2(\mathbf{p}^2)^2 - \frac{889}{192}(\mathbf{n}\cdot\mathbf{p})^4\mathbf{p}^2 + \frac{369}{160}(\mathbf{n}\cdot\mathbf{p})^6\right)\nu \\
&+ \left(\frac{4857}{256}(\mathbf{p}^2)^3 - \frac{545}{64}(\mathbf{n}\cdot\mathbf{p})^2(\mathbf{p}^2)^2 + \frac{9475}{768}(\mathbf{n}\cdot\mathbf{p})^4\mathbf{p}^2 - \frac{1151}{128}(\mathbf{n}\cdot\mathbf{p})^6\right)\nu^2 \\
&+ \left(\frac{2335}{256}(\mathbf{p}^2)^3 + \frac{1135}{256}(\mathbf{n}\cdot\mathbf{p})^2(\mathbf{p}^2)^2 - \frac{1649}{768}(\mathbf{n}\cdot\mathbf{p})^4\mathbf{p}^2 + \frac{10353}{1280}(\mathbf{n}\cdot\mathbf{p})^6\right)\nu^3\Bigg\}\frac{1}{r^2} \\
&+ \Bigg\{\frac{105}{32}(\mathbf{p}^2)^2 + \left(\left(\frac{2749\pi^2}{8192} - \frac{589189}{19200}\right)(\mathbf{p}^2)^2 + \left(\frac{63347}{1600} - \frac{1059\pi^2}{1024}\right)(\mathbf{n}\cdot\mathbf{p})^2\mathbf{p}^2 + \left(\frac{375\pi^2}{8192} - \frac{23533}{1280}\right)(\mathbf{n}\cdot\mathbf{p})^4\right)\nu \\
&+ \left(\left(\frac{18491\pi^2}{16384} - \frac{1189789}{28800}\right)(\mathbf{p}^2)^2 + \left(-\frac{127}{3} - \frac{4035\pi^2}{2048}\right)(\mathbf{n}\cdot\mathbf{p})^2\mathbf{p}^2 + \left(\frac{57563}{1920} - \frac{38655\pi^2}{16384}\right)(\mathbf{n}\cdot\mathbf{p})^4\right)\nu^2 \\
&+ \left(-\frac{553}{128}(\mathbf{p}^2)^2 - \frac{225}{64}(\mathbf{n}\cdot\mathbf{p})^2\mathbf{p}^2 - \frac{381}{128}(\mathbf{n}\cdot\mathbf{p})^4\right)\nu^3\Bigg\}\frac{1}{r^3} \\
&+ \Bigg\{\frac{105}{32}\mathbf{p}^2 + \left(\left(\frac{185761}{19200} - \frac{21837\pi^2}{8192}\right)\mathbf{p}^2 + \left(\frac{3401779}{57600} - \frac{28691\pi^2}{24576}\right)(\mathbf{n}\cdot\mathbf{p})^2\right)\nu \\
&+ \left(\left(\frac{672811}{19200} - \frac{158177\pi^2}{49152}\right)\mathbf{p}^2 + \left(\frac{110099\pi^2}{49152} - \frac{21827}{3840}\right)(\mathbf{n}\cdot\mathbf{p})^2\right)\nu^2\Bigg\}\frac{1}{r^4} \quad \longleftarrow G^4 \\
&+ \Bigg\{-\frac{1}{16} + \left(\frac{6237\pi^2}{1024} - \frac{169199}{2400}\right)\nu + \left(\frac{7403\pi^2}{3072} - \frac{1256}{45}\right)\nu^2\Bigg\}\frac{1}{r^5}. \quad \longleftarrow G^5
\end{aligned}
$$

After canonical transformation we match all but G^4 and G^5 terms

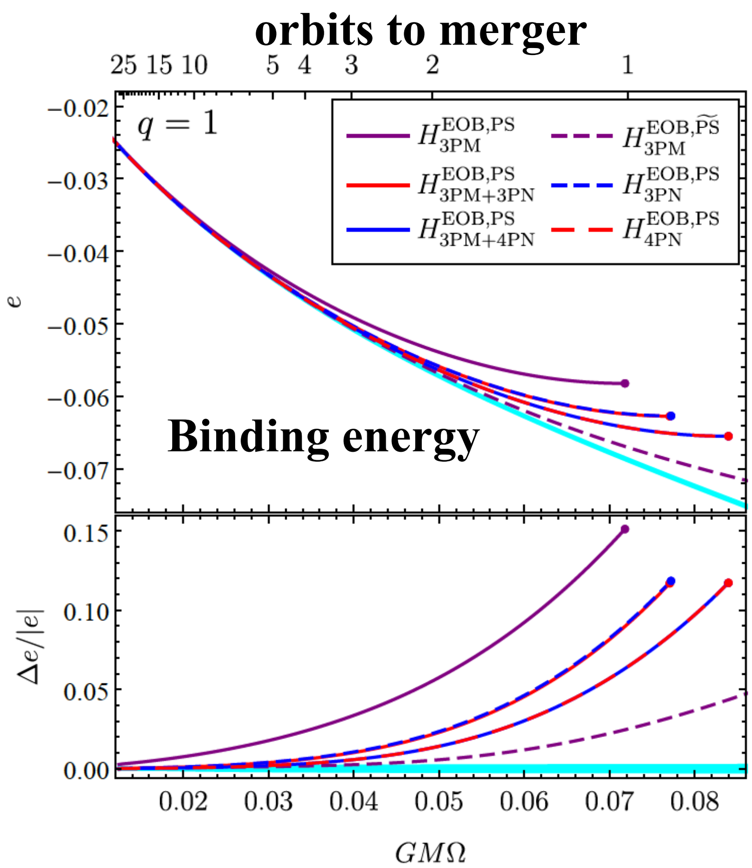
Mess is partly due to their gauge choice.

Ours is all orders in *p* at G^3

Tests of Our 3PM Hamiltonian for LIGO

Antonelli, Buonanno, Steinhoff, van de Meent, and Vines, arXiv:1901.07102

(8 days after our paper!)



Fed into EOB models.

Test against numerical relativity.

Note: Not conclusive, e. g. radiation not taken into accounted

Winning curve is based on feeding 3PM through machinery.

numerical relativity taken as truth

"This rather encouraging result motivates a more comprehensive study…"

3PM + 4PN fed into EOB ➔ Most advanced 2 body Hamiltonian

Outlook

- **Methods far from exhausted.**
- **Methods scale well to higher orders.**
- **Started working on 4PM. Methods certainly look up to the task.**

Obvious topics to investigate:

- **Higher orders. Resummation in *G*.**
- **Radiation.**
- **Spin.**
- **Finite size effects.**

The future looks bright.

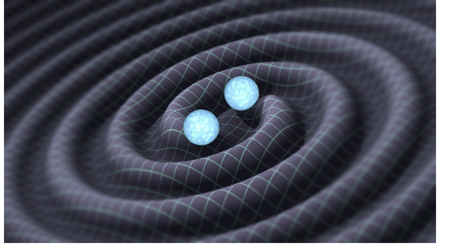
Summary

- String theory ideas gives us new ways to think about problems of current interest in gravity.
- Double-copy idea is a powerful way to think about perturbative gravity. Unified framework for gravity and gauge theory.
- Combining with EFT methods gives a powerful new tool for gravitational wave physics.
- Obtained the 3PM conservative 2-body potential. State of the art.
- Methods nowhere close to exhausted.
- Higher orders in G, resummations in G, spin, finite-size effects, radiation obvious next step to investigate.

Expect many more advances in coming years, not only for gravitational wave physics, but more generally for understanding gravity and its relation to the other forces via double copy.

Extra Slides

Outline

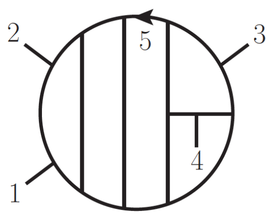


Will show you how ideas from string theory allow us to push the state of the art in calculations.

In amplitudes community we have great tools for gravity.

Some amplitude tools that we will use:

- **Generalized unitarity.** ZB, Dixon, Dunbar, Kosower; Britto, Cachazo, Feng; ZB, Carrasco, Johansson, Kosower.
- **Duality between color and kinematics.** ZB, Carrasco, Johansson
- **Double copy.** Kawai, Lewellen, Tye; ZB, Carrasco, Johansson



Able to compute supergravity at 5 loops

ZB, Carrasco, Chen, Edison, Johansson, Parra-Martinez, Roiban, Zeng (2018)

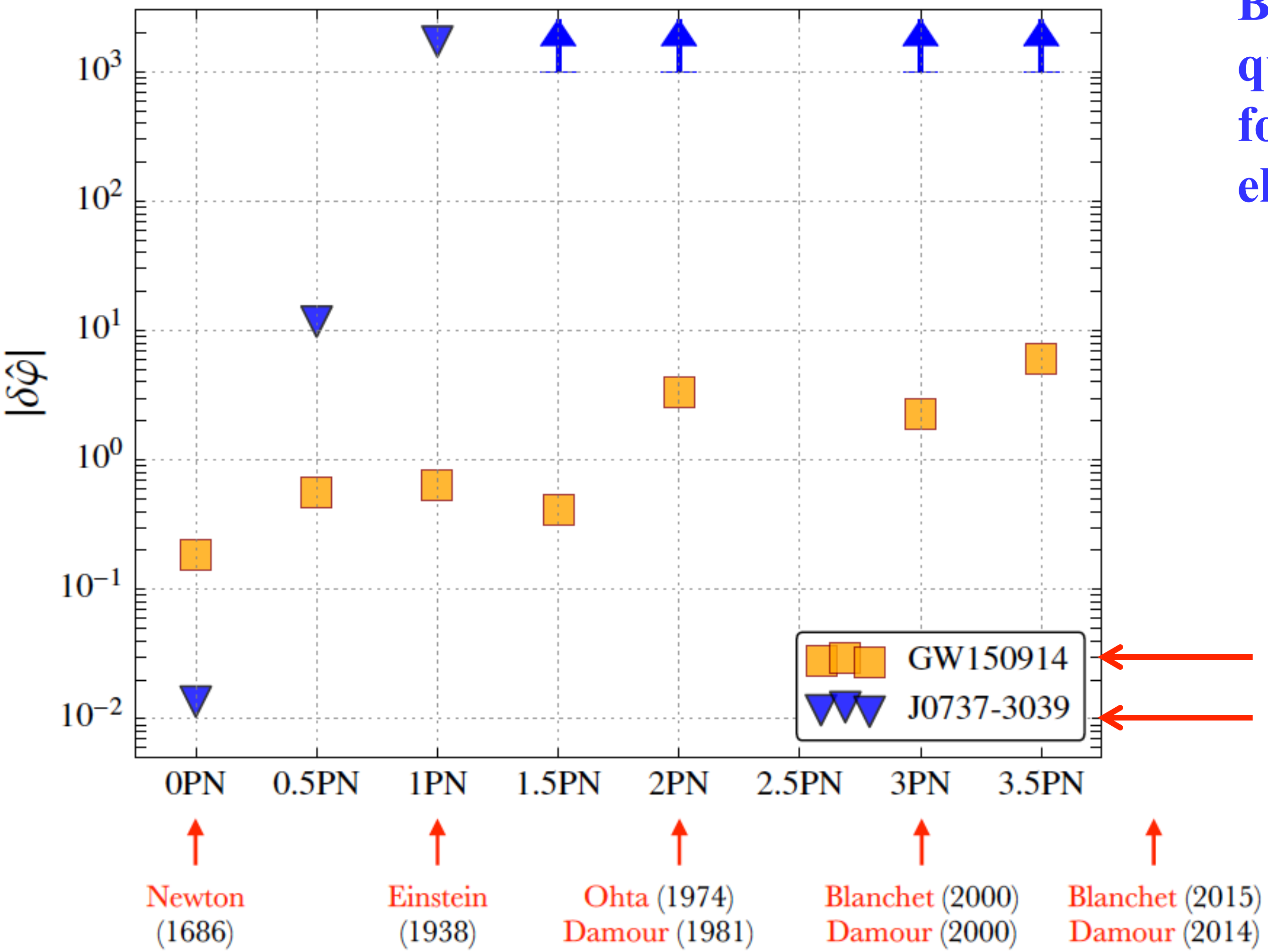
Combine with EFT approach of Cheung, Rothstein and Solon.

See Clifford Cheung's Amplitudes 2018 talk

Idea is to go beyond previous state of the art for a quantity of interest to LIGO/Virgo theorists.

Importance of higher orders for LIGO

LIGO/Virgo Collaboration arXiv:1602.03841



Binary pulsar confirms quadrupole radiation formula and not much else.

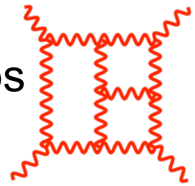
LIGO

Binary pulsar

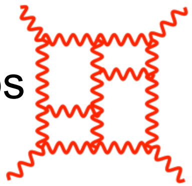
LIGO/Virgo sensitive to high PN orders.

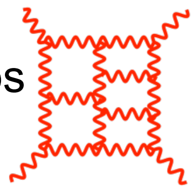
Feynman Diagrams for Gravity

Spectacularly poor scaling in GR

3 loops  $\sim 10^{20}$ TERMS

No surprise it has never been calculated via Feynman diagrams.

4 loops  $\sim 10^{26}$ TERMS

5 loops  $\sim 10^{31}$ TERMS

More terms than atoms in your brain!

- Such calculations seemed utterly hopeless!
- Seemed destined for dustbin of undecidable questions.

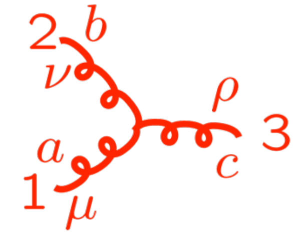
Modern methods make such calculations routine, but challenging.

Duality Between Color and Kinematics

ZB, Carrasco, Johansson (2007)

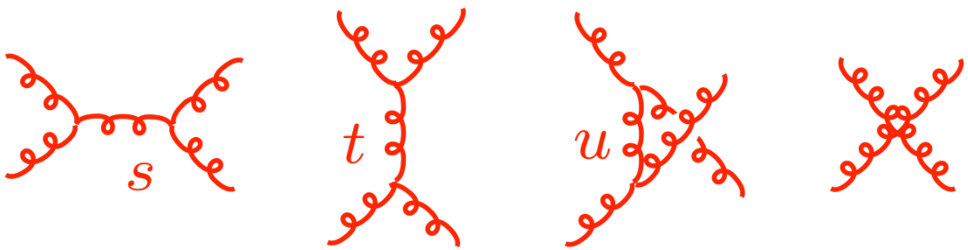
coupling constant → color factor → momentum dependent kinematic factor

$$-gf^{abc}(\eta_{\mu\nu}(k_1 - k_2)_\rho + \text{cyclic})$$



Color factors based on a Lie algebra: $[T^a, T^b] = if^{abc}T^c$

Jacobi Identity $f^{a_1 a_2 b} f^{b a_4 a_3} + f^{a_4 a_2 b} f^{b a_3 a_1} + f^{a_4 a_1 b} f^{b a_2 a_3} = 0$



Use 1 = *s/s* = *t/t* = *u/u* to assign 4-point diagram to others.

$s = (k_1 + k_2)^2 \quad t = (k_1 + k_4)^2$

$u = (k_1 + k_3)^2$

$$\mathcal{A}_4^{\text{tree}} = g^2\left(\frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u}\right)$$

Color factors satisfy Jacobi identity:

Numerator factors satisfy similar identity:

$$c_u = c_s - c_t$$
$$n_u = n_s - n_t$$

Proven at tree level

ZB, Carrasco, Johansson; Kiermaier; Bjerrum-Bohr, Damgaard, Sondergaard, Vanhove; Cachazo, etc

Duality Between Color and Kinematics

ZB, Carrasco, Johansson (BCJ)

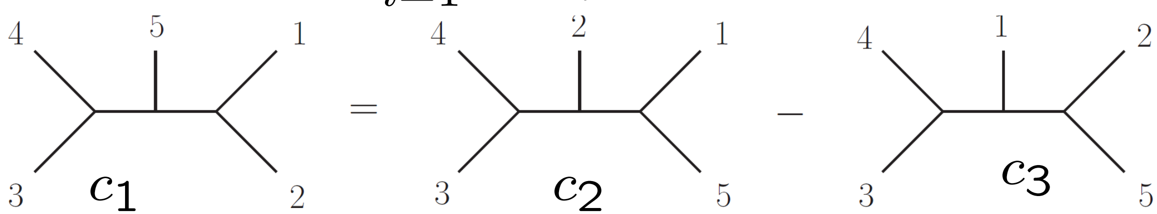
Consider five-point tree amplitude:

$$\mathcal{A}_5^{\text{tree}} = \sum_{i=1}^{15} \frac{c_i n_i}{\prod_{\alpha_i} p_{\alpha_i}^2}$$

color factor

kinematic numerator factor

Feynman propagators



$$c_1 = f^{a_3 a_4 b} f^{b a_5 c} f^{c a_1 a_2} \qquad c_2 = f^{a_3 a_4 b} f^{b a_2 c} f^{c a_1 a_5} \qquad c_3 = f^{a_3 a_4 b} f^{b a_1 c} f^{c a_2 a_5}$$

$$n_i \sim k_4 \cdot k_5 \, k_2 \cdot \varepsilon_1 \, \varepsilon_2 \cdot \varepsilon_3 \, \varepsilon_4 \cdot \varepsilon_5 + \cdots$$

$$c_1 + c_2 + c_3 = 0 \;\Leftrightarrow\; n_1 + n_2 + n_3 = 0$$

Claim: We can always find a rearrangement so color and kinematics satisfy the *same* algebraic constraint equations.

Progress on unraveling relations.

BCJ, Bjerrum-Bohr, Feng, Damgaard, Vanhove, ; Mafra, Stieberger, Schlotterer;
Tye and Zhang; Feng, Huang, Jia; Chen, Du, Feng; Du, Feng, Fu; Naculich, Nastase, Schnitzer
O'Connell and Montiero; Bjerrum-Bohr, Damgaard, O'Connell and Montiero; O'Connell, Montiero, White;
Du, Feng and Teng, Song and Schlotterer, etc.

Higher-Point Gravity and Gauge Theory

ZB, Carrasco, Johansson

color factor

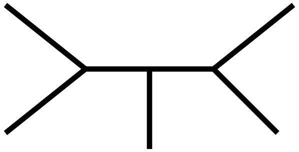
kinematic numerator factor

gauge theory: $$\mathcal{A}_n^{\text{tree}} = i g^{n-2} \sum_i \frac{c_i\, n_i}{D_i}$$

Feynman propagators

Einstein gravity: $$\mathcal{M}_n^{\text{tree}} = i \kappa^{n-2} \sum_i \frac{n_i^2}{D_i}$$

$$n_i \sim k_4 \cdot k_5\, k_2 \cdot \varepsilon_1\, \varepsilon_2 \cdot \varepsilon_3\, \varepsilon_4 \cdot \varepsilon_5 + \cdots$$



sum over diagrams with only 3 vertices

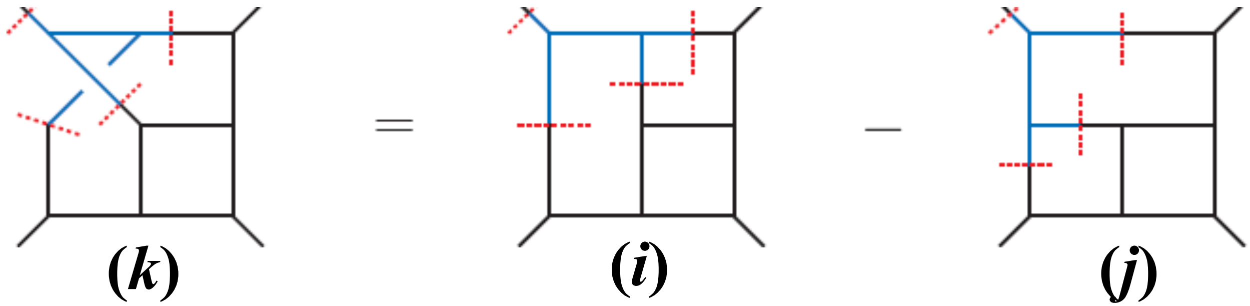
Gravity and QCD kinematic numerators are the same!

Cries out for a unified description of gravity with gauge theory, presumably along the lines of string theory.

BCJ

Gravity Loop Integrands from Gauge Theory

Ideas conjectured to generalize to loops:



$$c_k = c_i - c_j$$
$$n_k = n_i - n_j$$

color factor → c_j; kinematic numerator → n_j

If you have a set of duality satisfying numerators.

To get:

gauge theory → gravity theory

simply take

color factor → kinematic numerator

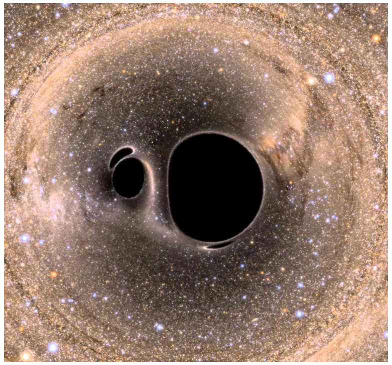
$$c_k \longrightarrow n_k$$

Gravity loop integrands follow from gauge theory!

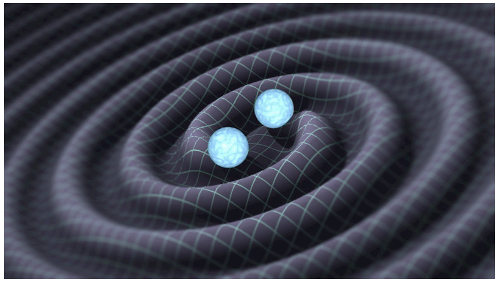
Double Copy for Classical Solutions

Goal is to formulate gravity solutions directly in terms of gauge theory

Variety of special cases:



- **Schwarzschild and Kerr black holes.**
- **Taub-NUT space.**
- **Solutions with cosmological constant.**
- **Radiation from accelerating black hole.**
- **Maximally symmetric space times.**
- **Plane wave background.**
- **Gravitational radiation.**



Luna, Monteiro, O'Connell and White;
Luna, Monteiro, Nicholsen, O'Connell and White;
Ridgway and Wise; Carrillo González, Penco, Trodden;
Adamo, Casali, Mason, Nekovar;
Goldberger and Ridgway; Chen;
Luna, Monteiro, Nicholson, Ochirov;
Bjerrum-Bohr, Donoghue, Vanhove;
O'Connell, Westerberg, White; Kosower, Maybee, O'Connell, etc

Still no general understanding.
But plenty of examples.

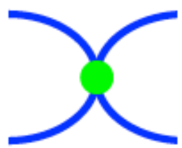
Feynman diagrams for EFT

- **EFT scattering amplitudes easy to compute using Feynman diagrams.**
- **No need for advanced methods.**

Newton's constant

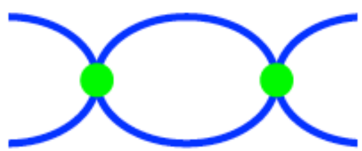
$$A_{\mathrm{EFT}} = \sum_{i=1}^{\infty} G^i A^{(i)}_{\mathrm{EFT}}$$

$=$

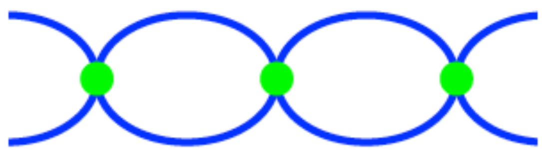


vertices contain all powers of *G*

$+$



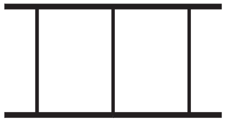
$+$



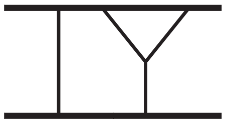
$+ \cdots$

Match to Full Theory

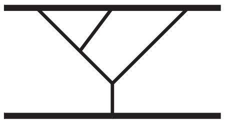
Two-Loop Diagram Numerators



$$(t_{12}^2 - 2m_1^2 m_2^2)^3$$



$$2m_2^3 t_{47}^2 (t_{12}^2 - 2m_1^2 m_2^2)$$



$$\begin{aligned}&2m_2^4(s_{23}^4 + s_{23}^3(2t_{12} + 2t_{15} - t_{47} - 2t_{67}) - 2m_1^2 m_2^2 (s_{23} - t_{67})^2 + (t_{15}t_{56} + (t_{12} - t_{47})t_{67})^2\\&+ s_{23}^2(t_{12}^2 + t_{15}^2 + t_{47}^2 - t_{47}t_{56} + t_{12}(4t_{15} - 2t_{47} + t_{56} - 4t_{67}) + t_{15}(-2t_{47} + t_{56} - 2t_{67})\\&+ 2t_{47}t_{67} + t_{67}^2) + s_{23}(t_{15}(t_{56}^2 + 2(-2t_{12} + t_{47})t_{67} - t_{56}t_{67})\\&+ t_{67}(-2t_{12}^2 + t_{47}(-2t_{47} + t_{56} - t_{67}) + t_{12}(4t_{47} - t_{56} + 2t_{67}))))\end{aligned}$$

etc. Remaining 5 diagrams somewhat more complicated but not a big deal.

- **Simple compared to the usual Feynman diagram explosion.**
- **Higher-loop integrand constructions definitely possible!**