Sphere Packing and Quantum Gravity

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D.M. + M. Paulos: 1803.10233

Quantum Gravity and the Bootstrap

Interested in understanding the landscape of consistent theories of quantum gravity.

A theory of quantum gravity in AdS is consistent $\Leftrightarrow$ The dual CFT satisfies bootstrap constraints.

Probe the boundary of the landscape using the bootstrap.

General expectation: UV consistency requires other states besides gravitons in the spectrum (black holes, KK modes, string modes).

Concrete goal for today:

Look for an upper bound on the mass of the lightest non-graviton state.

c.f. WGC [Arkani-Hamed, Motl, Nicolis, Vafa '06]

Does pure gravity exist as a fully consistent quantum theory?

← only gravitons and black holes in the spectrum

The Main Result

The task is particularly sharp in AdS_3/CFT_2, where gravitons = Virasoro descendants of the vacuum. [Witten '07; Maloney, Witten '07]

We want a universal upper bound on Δ of the lightest non-vacuum Virasoro primary at large central charge c.

Modular invariance and unitarity imply such a bound with $\Delta \lesssim \frac{c}{6}$. [Hellerman '09]

Our main new result:

Theorem: Every unitary 2D CFT with $c \geq 12$ contains a Virasoro primary (other than identity) with

$$\Delta < \frac{c}{8} + \frac{1}{2}.$$

The proof uses mainly the technique of analytic functionals, developed recently in the context of the correlator bootstrap. [DM '16; DM, Paulos '18]

Along the way will uncover a very close connection to the recent solution of the sphere packing problem in dimensions 8 and 24.

[Cohn, Elkies '01; Viazovska '16; Cohn, Kumar, Miller, Radchenko, Viazovska '16]

The Main Result

Theorem: Every unitary 2D CFT with $c \geq 12$ contains a Virasoro primary (other than identity) with

$$\Delta < \frac{c}{8} + \frac{1}{2}\,.$$

Stronger bound at large central charge

$$\Delta < \frac{2c}{17} + O(1)$$

$$\sim \frac{c}{8.5}$$

Road Map

1. Virasoro Modular Bootstrap
    - AdS_3/CFT_2 and the modular bootstrap
    - Analytic functionals review
    - Proof of the main theorem

2. Sphere Packing Problem
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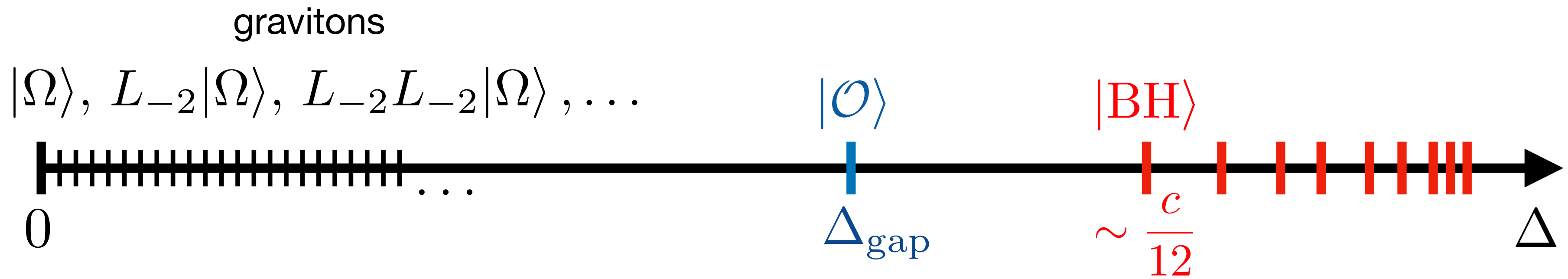
AdS$_3$/CFT$_2$ and the Modular Bootstrap

weakly-coupled gravity $\Leftrightarrow$ $\ell_{\text{AdS}} \gg \ell_{\text{Planck}}$ $\Leftrightarrow$ $c \gg 1$



Torus partition function at zero angular potential

$$q = e^{2\pi i \tau}$$

$$Z(\tau) = \sum_{\text{states}} q^{\Delta - \frac{c}{12}} = \sum_{\text{primaries}} \chi_\Delta(\tau) \qquad \chi_\Delta(\tau) = \frac{q^{\Delta - \frac{c-1}{12}}}{\eta(\tau)^2}$$

Modular invariance $\quad S: \quad Z(\tau) = Z(-1/\tau)$

$$\sum_{\text{primaries}} [\chi_\Delta(\tau) - \chi_\Delta(-1/\tau)] = 0$$

impossible to satisfy with vacuum module alone

Working with full-fledged CFTs, not chiral CFTs!

$$\Delta \notin \mathbb{Z}$$

$$Z(\tau) \neq Z(\tau + 1)$$

in general

Functional Bootstrap [Rattazzi, Rychkov, Tonni, Vichi '08]

Upper bounds on $\Delta_{\rm gap}$ can be found as follows:

$$\Phi_{\rm vac}(\tau) + \sum_{\substack{\rm primaries \\ \Delta>0}} \Phi_\Delta(\tau) = 0 \qquad \Phi_\Delta(\tau) = \chi_\Delta(\tau) - \chi_\Delta(-1/\tau)$$

If there exists a linear functional ω acting on functions of τ such that:

$$\omega[\Phi_{\rm vac}] > 0$$

$$\omega[\Phi_\Delta] \geq 0 \quad \text{for all} \quad \Delta \geq \Delta_*$$

then $\Delta_{\rm gap} < \Delta_*$.

Central question: for given central charge, what is the best (minimal) upper bound $\Delta_V(c)$, and what is the corresponding ω ?

Expectation: $\Delta_V(c) \sim \mu\, c$ as $c \to \infty$

What is the value of μ ?

$\mu < \frac{1}{12}$ would prove that semi-classical pure gravity is not consistent as a quantum theory.

Functional Bootstrap: Previous Results

Ansatz: $\omega = \sum_{n=0}^{N} \alpha_n \, \partial_\tau^{2n+1}|_{\tau=i}$ optimize over α_n

Analytics: $N = 1$ $\Delta_V(c) < \frac{c}{6} + O(1)$ as $c \to \infty$ **[Hellerman '09]**

no asymptotic improvement for any finite fixed N. **[Friedan, Keller '13]**

Numerics: Indicates that the true asymptotic bound is stronger, i.e. need to take $N \to \infty$ at fixed central charge.

Conjectures based on finite-c numerics:

$\Delta_V(c) < \frac{c}{9} + O(1)$ as $c \to \infty$ **[Collier, Lin, Yin '16]**

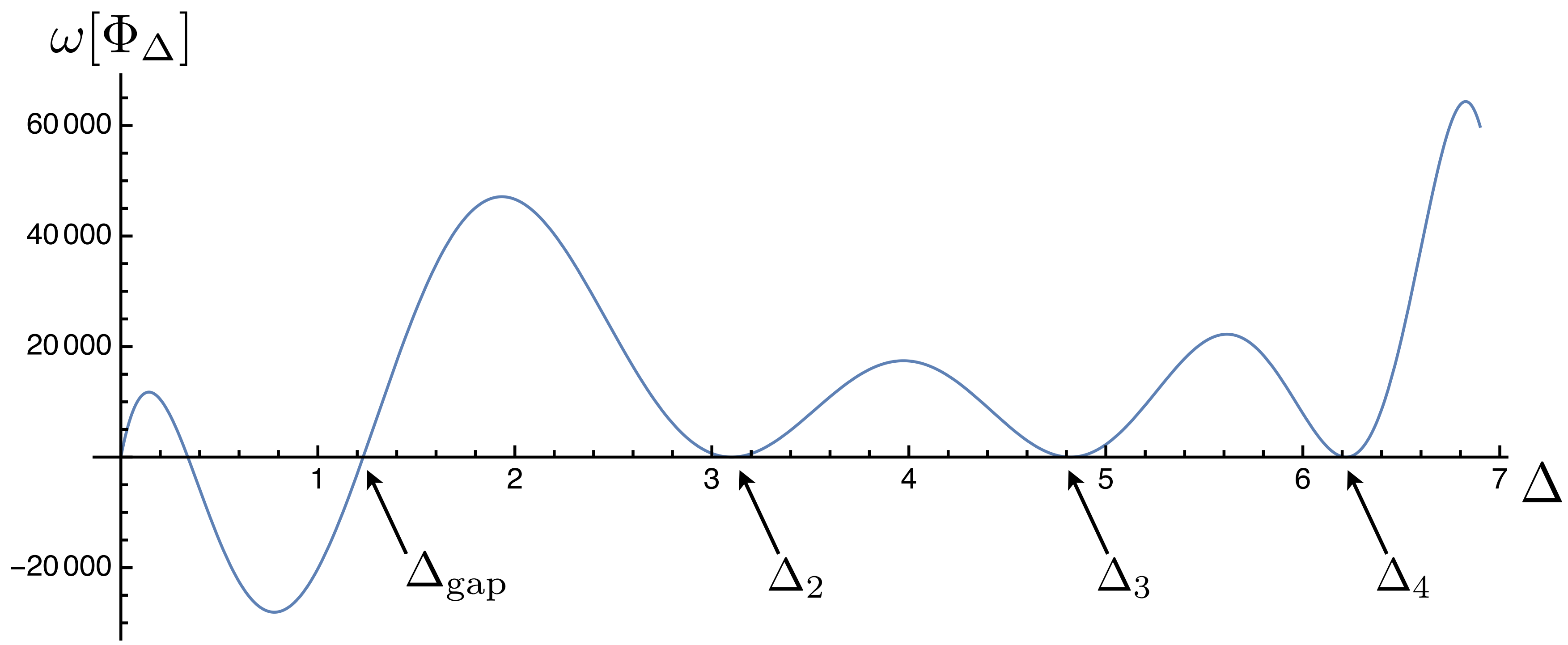
$\Delta_V(c) \approx \frac{c}{9.08}$ as $c \to \infty$ **[Afkhami-Jeddi, Hartman, Tajdini '19]**

A different construction of ω is needed to make analytic progress.

The Optimal Functional

The solution of the bootstrap with the maximal $\Delta_{\text{gap}} = \Delta_V(c)$ comes together with the optimal (aka extremal) functional ω .

The optimal functional must vanish on the optimal spectrum and is non-negative above Δ_{gap} .



The only analytic construction of the optimal functional known so far is for the four-point function bootstrap on a line.

Nevertheless, this will be enough to prove our main theorem.

Optimal Bound for the 1D Bootstrap [DM '16; DM, Paulos '18]

Put four conformal primaries on a line: $\langle\sigma(x_1)\sigma(x_2)\sigma(x_3)\sigma(x_4)\rangle$

The crossing equation is $\sum_{\text{primaries}} f^2\left[G^{(s)}_\Delta(z) - G^{(t)}_\Delta(z)\right] = 0$ $\quad z =$ cross-ratio

$sl(2,\mathbb{R})$ conformal blocks

The solution with maximal gap is the fermionic mean-field theory.

Spectrum: $2\Delta_\sigma + 1,\ 2\Delta_\sigma + 3, \ldots$

Theorem: The OPE of two identical primaries σ in a unitary CFT always contains a non-identity conformal primary of dimensions

$$\Delta \leq 2\Delta_\sigma + 1$$

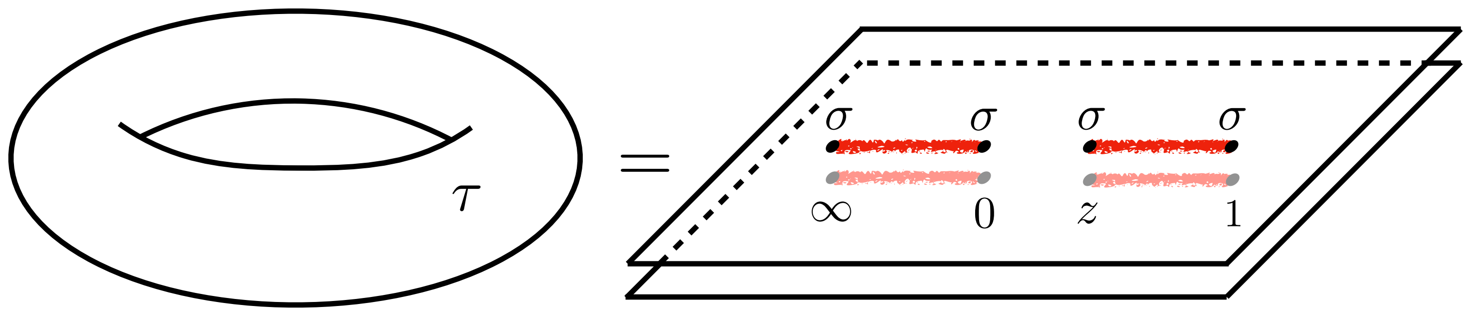
Proof: Construct the optimal functional. Natural ansatz:

kernel is uniquely fixed from self-consistency

$$\omega[G^{(s)}_\Delta(z) - G^{(t)}_\Delta(z)] = \sin^2\left[\frac{\pi}{2}(\Delta - 2\Delta_\sigma - 1)\right]\int_0^1 dz\, Q_{\Delta_\sigma}(z) G^{(s)}_\Delta(z)$$

dDisc c.f. [Hartman, Jain, Kundu '15; Caron-Huot '17]

Back to the Torus: The Pillow Map



The torus is a double cover of the four-punctured sphere.

$$z = \frac{\theta_2(\tau)^4}{\theta_3(\tau)^4}$$

$$Z_{\mathrm{A}}(\tau) \sim \langle \sigma(0)\sigma(z)\sigma(1)\sigma(\infty)\rangle_{\mathrm{A}\times\mathrm{A}/\mathbb{Z}_2}$$

twist-operator: $\Delta_\sigma = \frac{c}{8}$

$\tau \leftrightarrow -1/\tau$ maps to $z \leftrightarrow 1 - z$

The analytic functional ω for the 1D bootstrap can be immediately applied to the modular bootstrap!

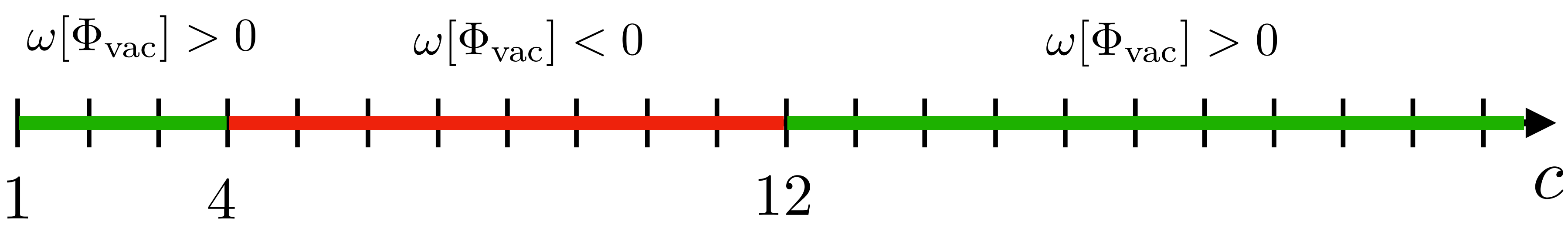
Naive conclusion from the previous slide: $\Delta_V(c) = \frac{2\Delta_\sigma + 1}{2} = \frac{c}{8} + \frac{1}{2}$

Subtlety: Virasoro characters $\neq$ $sl(2,\mathbb{R})$ conformal blocks.

Need to check $\omega[\Phi_{\mathrm{vac}}] \geq 0$

Modular Bootstrap Conclusions [Hartman, DM, Rastelli '19]

Surprise: $\omega[\Phi_{\rm vac}]$ changes sign precisely at $c=4$ and $c=12$!



$c\in(1,4)\cup(12,\infty)$ $\quad \Delta_V(c)<\frac{c}{8}+\frac{1}{2}$ $\quad \omega$ valid but suboptimal

$c\in(4,12)$ $\quad \Delta_V(c)>\frac{c}{8}+\frac{1}{2}$ $\quad \omega$ invalid

At $c=4$ and $c=12$, $\frac{c}{8}+\frac{1}{2}$ is the optimal bound!

$\Delta_V(4)=1$ spectrum $\Delta=1,2,3,\dots$ $\quad Z_4(\tau)=\frac{E_4(\tau)}{\eta(\tau)^8}$ $\quad$ 8 free fermions with a GSO projection

$\Delta_V(12)=2$ spectrum $\Delta=2,3,4,\dots$ $\quad Z_{12}(\tau)=j(\tau)-744$ $\quad$ chiral half of the monster CFT

These two cases will map to the solution of the sphere packing problem in $d=8$ and $d=24$.

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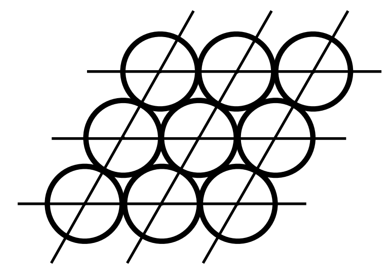
Sphere Packing Problem

Statement: Find the densest arrangement of identical non-overlapping spheres in $\mathbb{R}^d$.

Deep problem, connections to number theory, cryptography, etc.

$d = 1$ trivial

$d = 2$ the honeycomb lattice [Toth '40]



$d = 3$ Kepler's conjecture: FCC lattice. Proved by [Hales '98] .
Computer-assisted proof took 11 years to verify.

$d \geq 4$ open, with the exception of:

$d = 8$ E_8 lattice is optimal $|x|^2 = 0, 2, 4, 6, \ldots$

[Viazovska '16]

self-dual lattices, spectrum:

$d = 24$ Leech lattice is optimal $|x|^2 = 0, \cancel{2}, 4, 6, \ldots$

[Cohn, Kumar, Miller, Radchenko, Viazovska '16]

No requirement to be a lattice in general! Efficient packings in large d highly irregular.

[Torquato, Stillinger '05]

The Sphere Packing Bootstrap [Cohn, Elkies '01]

Idea: Prove a universal upper bound on the density of any packing in $\mathbb{R}^d$ and show that this bound is saturated by the E_8 and Leech lattice in $d = 8, 24$.

Argument to derive the bound:

- Define the partition function of a sphere packing: $Z(\tau) = \sum_{(ij)} \frac{e^{i\pi|x_i - x_j|^2 \tau}}{\eta(\tau)^d}$

- The Poisson summation formula implies $Z(\tau)$ satisfies a modular bootstrap-like identity under $\tau \leftrightarrow -1/\tau$.

- The terms in the sum are characters of $U(1)^c$ with central charge $c = \frac{d}{2}$.

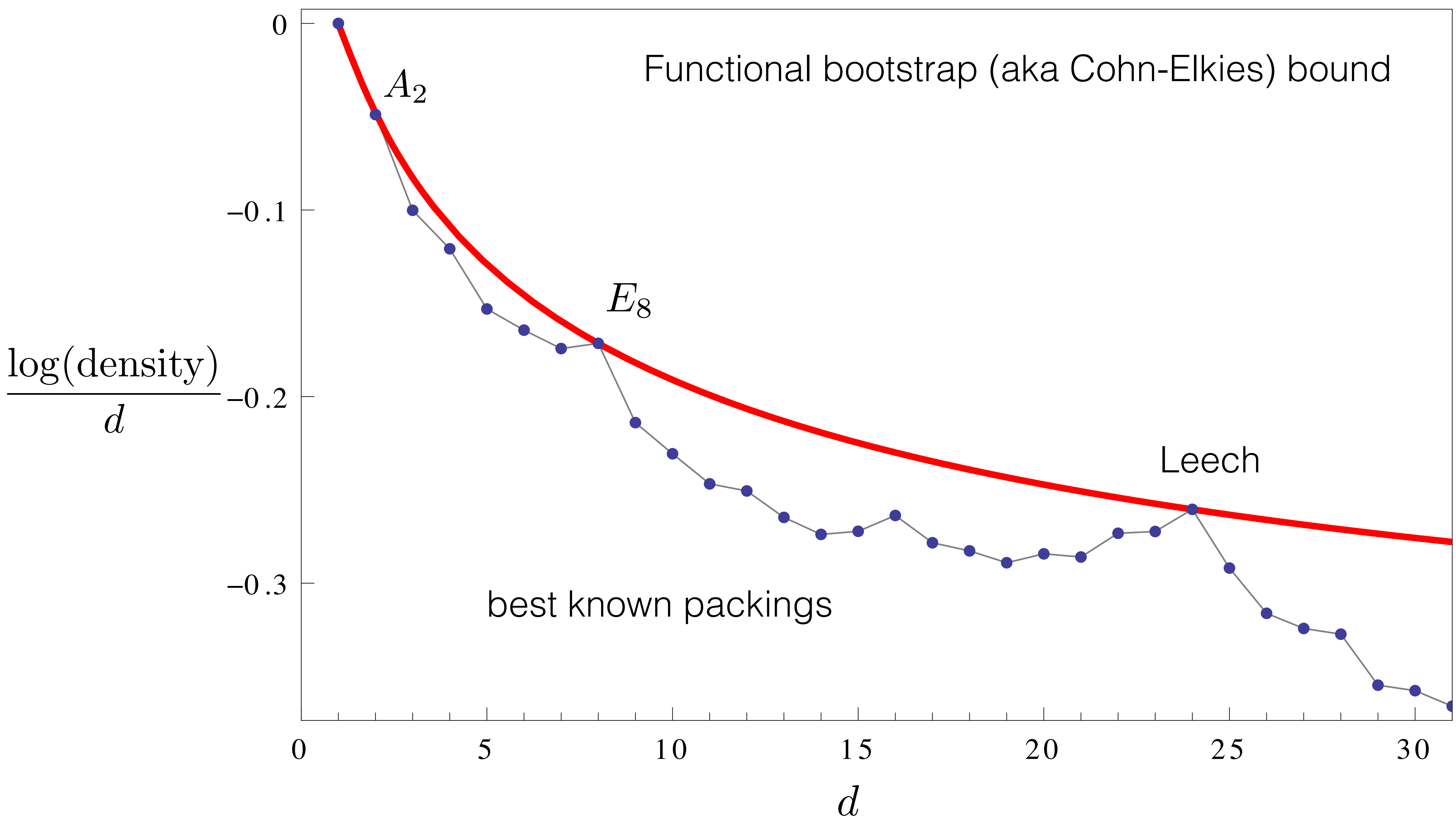
$$\Delta_{ij} = \frac{|x_i - x_j|^2}{2}$$

Δ_{gap} "=" shortest distance between sphere centers

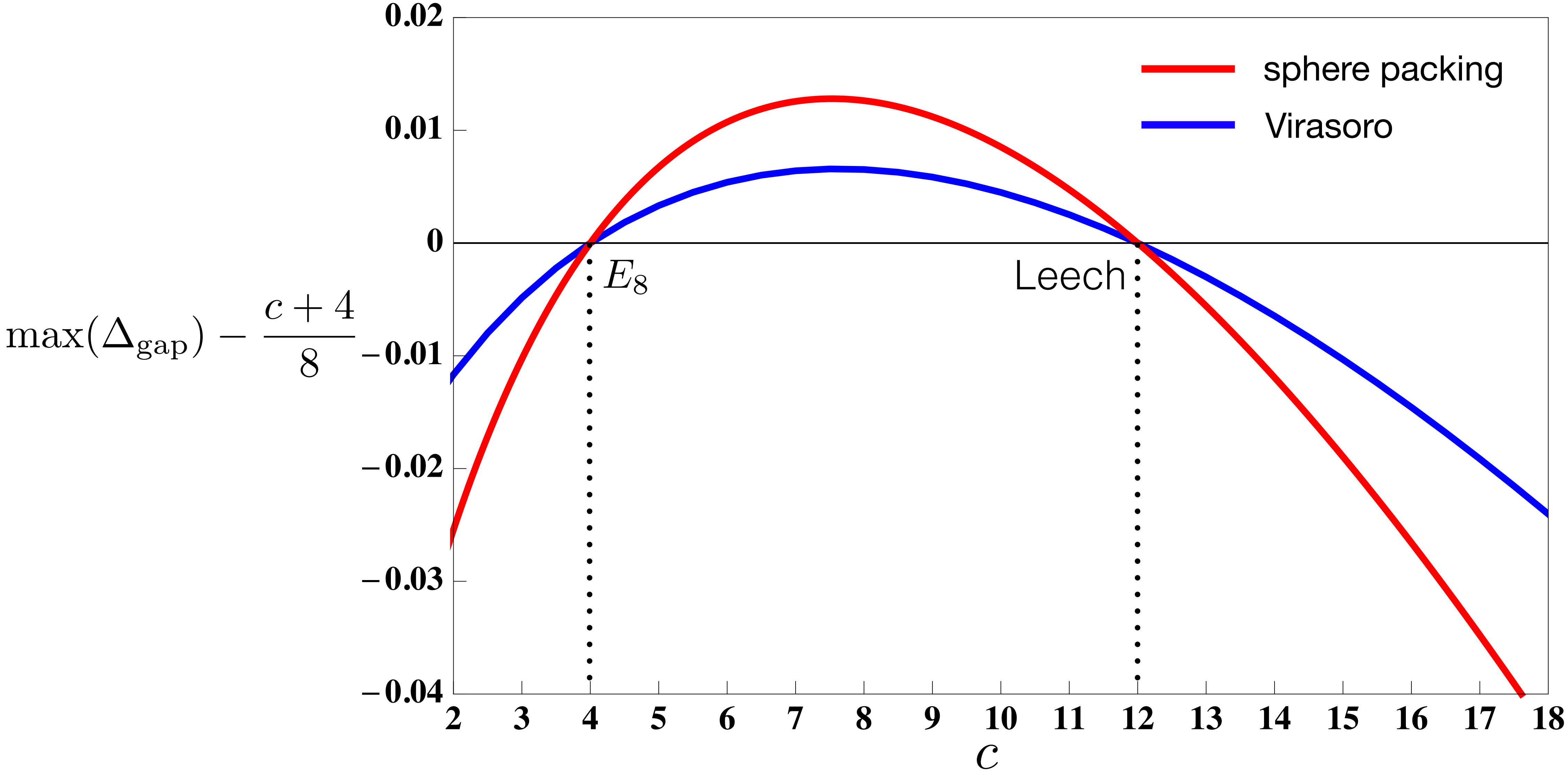
- Use functional bootstrap to derive an upper bound on Δ_{gap}

  $\Longrightarrow$ upper bound on the sphere packing density

Conclusion: Modular bootstrap in the presence of $U(1)^c$ symmetry constrains the sphere-packing density in $d = 2c$ dimensions!



The Last Step: Using the Functional Again



The same optimal functionals which proved $\Delta_V(4) = 1$ and $\Delta_V(12) = 2$ apply also to the sphere packing bootstrap.

$\Rightarrow$ E_8 and Leech lattice are optimal in 8 and 24 dimensions.

What I have described is a condensed version of Viazovska's solution.

Summary

The first non-identity primary in a unitary 2D CFT satisfies $\Delta_{\text{gap}} < \frac{c}{8} + \frac{1}{2}$ provided $c > 12$.

The result can be strenghtened to $\Delta_{\text{gap}} < \frac{2c}{17} + O(1)$ at large central charge.

Via AdS/CFT, this gives a rigorous constraint on the spectrum of black hole microstates in any 3D theory of quantum gravity in AdS.

The bounds were derived from unitarity and modular invariance using analytic functionals.

A very similar bound constrains the density of sphere packings in $\mathbb{R}^d$.

In this context, the analytic functionals were discovered independently by Viazovska, leading to the solution of the sphere-packing problem in 8 and 24 dimensions.

Open questions

What is the true asymptotics of the Virasoro modular bootstrap bound at large c?

Can pure gravity be ruled out, perhaps with some extra assumptions?

[Benjamin, Ooguri, Shao, Wang '19]

What is the asymptotics of the Cohn-Elkies sphere packing bound in large dimension? Is it better than the best bound currently known? $\Delta \sim c/9.795$

[Kabatiansky, Levenshtein '78]

Combine our technique with the complex tauberian theorems to get more detailed information about the spectrum? **[Mukhametzhanov, Zhiboedov '19]**

How deep is the analogy between CFTs and sphere packings?

I explained that the simplest constraint agrees on the two sides. A variety of other constraints exists:

modular bootstrap with spin, four-point function crossing, higher genus, … $\overset{?}{\sim}$ n-point correlations between spheres, …

Hints:

Black holes in quantum gravity exhibit chaos. **[Susskind, Shenker, Stanford, Maldacena, Kitaev, Hayden, Preskill, …]** $\sim$ Efficient packings in a large number of dimensions are highly disordered. **[Torquato, Stillinger '05]**

Large scaling dimensions (UV) $\sim$ Large distances in the packing (IR)

Thank you!

Dictionary

	3D quantum gravity	**sphere packing**
parameter	central charge c	dimension of space $d = 2c$
symmetry	Virasoro^2	$U(1)^c \times U(1)^c$
partition function	$Z(\tau) = \sum_{\text{primaries}} \frac{e^{2\pi i \tau (\Delta - \frac{c-1}{12})}}{\eta(\tau)^2}$	$Z(\tau) = \sum_{\substack{\text{pairs of} \\ \text{spheres}}} \frac{e^{\pi i \tau \lvert x_i - x_j \rvert^2}}{\eta(\tau)^d}$
scaling dimension	Δ	distance in $\mathbb{R}^d$ $\quad r = \sqrt{2\Delta}$
optimal bounds	$c = 4$: $\quad \Delta_{\text{gap}} \leq 1$ $c = 12$: $\quad \Delta_{\text{gap}} \leq 2$	E_8 lattice optimal in $d = 8$ Leech lattice optimal in $d = 24$