Generalized Particles & Strings from Combinatorial Geometry

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with Nima Arkani-Hamed, Yuntao Bai, Gongwang Yan arXiv: 1711.09102

to appear with Nima Arkani-Hamed, Giulio Savlatori, Hugh Thomas

with Nima Arkani-Hamed, Thomas Lam

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Strings 2019, Brussel

July 12, 2019

Motivations

Search for "holographic" S-matrix theory: fascinating geometric structures underlying scattering amplitudes, in some auxiliary space

- $\mathcal{M}_{g,n}$: perturbative string amps = correlators of worldsheet CFT
- (ambi-)twistor strings & scattering equations, same worldsheet but for particles, without stringy excitations [Witten] [Cachazo, SH, Yuan] [Mason, Skinner; ...]
- Generalized $G_+(k,n)$: amplituhedron for all-loop S-matrix in planar $\mathcal{N}=4$ SYM [Arkani-Hamed, Trnka] [+ Bourjaily, Cachazo, Goncharov, Postnikov; ...]

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Also w. factorizing structure: cluster polytopes (generalized associahedra)

Quiver + mutations $\rightarrow$ infinite in general, but finite for Dynkin diagrams $\rightarrow$ finite-type cluster algebra [Fomin, Zelevinski], each with a "factorizing" polytope

Scattering amplitudes as differential forms → geometries directly in kinematic space

A new picture for amplituhedron in momentum-twistor space: $\mathcal{N}=4$ SYM amps as its "volume" form [Arkani-Hamed, Thomas, Trnka]; also in 4d momentum space [SH, Zhang]

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Geometry for ϕ^3: kinematic associahedron in Mandelstam space [Arkani-Hamed, Bai, SH, Yan]

- "volume"/canonical form → (something new for) bi-adjoint ϕ^3 tree amps
- connected to worldsheet by scattering equations → CHY formula [Cachazo, SH, Yuan]
- general scattering forms, *e.g.* for gluons and pions: "geometrizing" colors & related to color-kinematics duality [Bern, Carrasco, Johansson]

Natural Q: What about loops? Other types of cluster polytopes & beyond?

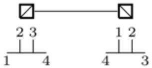
What is the origin of (generalized) particles, strings & CHY (in ϕ^3 toy model)?

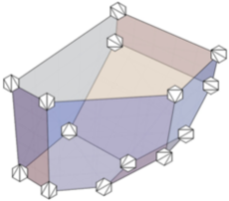
Key: combinatorics → geometries → physics

Outline

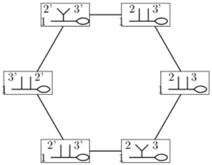
Feynman diagrams form polytopes

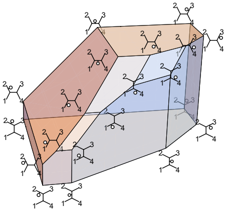
Associahedron $\mathcal{A}_{n-3}$: n-pt bi-adjoint ϕ^3 trees (dual to triangulations of n-gon)





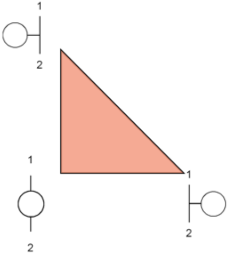
Cyclohedron $\mathcal{B}/\mathcal{C}_{n-1}$: n-pt tadpole diagrams (cent. sym. triangulations of $2n$-gon)

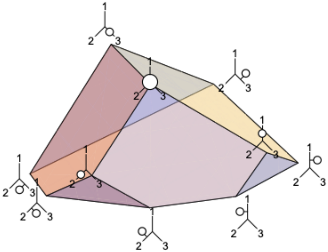




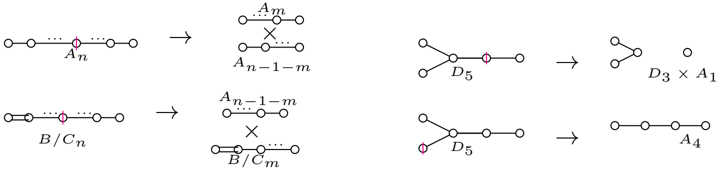
What about $\mathcal{D}_n$ (last family with an n)? too many facets/variables? "cut in half" $\rightarrow$

$\overline{\mathcal{D}}_n$ polytope : n-pt one-loop ϕ^3 (including tadpoles *etc.*) [Arkani-Hamed, SH, Salvatori, Thomas]





Encodes "combinatorial factorization": each facet is product of lower-dim polytopes

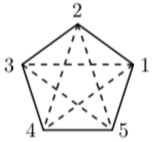


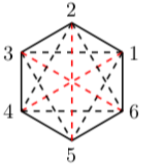
Planar basis of (tree) kinematic space

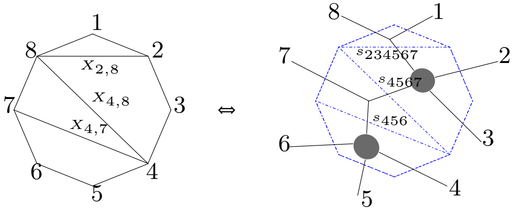
$\mathcal{K}_n$: spanned by Mandelstam variables $s_{i\,j} := (p_i+p_j)^2$'s subject to momentum conservation, $\sum_{j\neq i} s_{i\,j} = 0 \implies \dim \mathcal{K}_n = \binom{n}{2} - n = \frac{n(n-3)}{2}$ (for $D \geq n-1$)

Planar basis: $\frac{n(n-3)}{2}$ $X_{a,b} := (p_a + \cdots + p_{b-1})^2 \leftrightarrow$ facets of $\mathcal{A}_{n-3}$
cubic trees $\leftrightarrow$ vertices of $\mathcal{A}_{n-3}$ (trees with d propagators $\leftrightarrow$ co-dim d faces)

e.g. $\{X_{13} = s, X_{24} = t\}$ for $n = 4$
$\{X_{13} = s_{12},\ X_{24} = s_{23}, \cdots, X_{52} = s_{51}\}$ for $n = 5$
$\{X_{13}\ \cdots,\ X_{62},\ X_{14},\ X_{25},\ X_{36}\}$ for $n = 6$



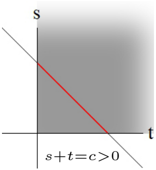




Kinematic (tree) associahedra

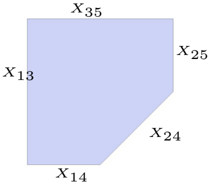
Top-dim cone Δ_n : $X_{a,b} \geq 0$ & $(n-3)$-dim plane H_n: for all non-adjacent $i < j \neq n$ impose $-s_{i,j} = X_{i,j} + X_{i+1,j+1} - X_{i,j+1} - X_{i+1,j} = c_{i,j} > 0$ (positive const.)

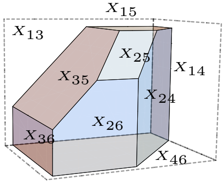
$\Longrightarrow \quad \Delta_n \ \cap \ H_n = \mathcal{A}_{n-3}$ [Arkani-Hamed, SH, Bai, Yan] $\to$ a discrete version of wave eq in $1+1$ dim (factorization from causal diamonds *etc.*) [Nima's talk at **Amplitudes 2019**]



$$e.g.\ \mathcal{A}_1 = \{s > 0, t > 0\} \cap \{-u = \text{const} > 0\}$$

$$\mathcal{A}_2 = \{X_{13}, \cdots, X_{25} > 0\} \cap \{-s_{13} = c_{13}, -s_{14} = c_{14}, -s_{24} = c_{24}\}$$





Canonical form & ϕ^3 amps

Unique form for any polytope (& beyond) $\mathcal{P}$: $\Omega^{(d)}(\mathcal{P})$ has only simple pole on $\partial\mathcal{P}$, with $\mathrm{Res} = \Omega^{(d-1)}(\partial\mathcal{P})$ (recursive def.); canonical function $\underline{\Omega}_X(\mathcal{P}) \equiv \Omega(\mathcal{P})/(d^d X)$

Key: $\Omega(\mathcal{A}_{n-3})$ = pullback of scattering form to $H_n \propto$ planar ϕ^3 tree

e.g. $\Omega(\mathcal{A}_1) = (\frac{ds}{s} - \frac{dt}{t})|_{-u=c>0} = (\frac{1}{s} + \frac{1}{t})\, ds$

$\Omega(\mathcal{A}_2) = (d\log X_{13} \wedge d\log X_{14} - \cdots)|_{H_5} = \left(\frac{1}{X_{13}X_{14}} + \cdots + \frac{1}{X_{25}X_{35}}\right)\, d^2 X$

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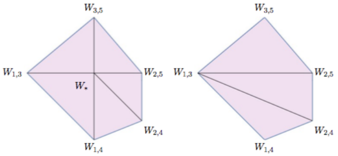
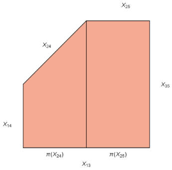
Geometric picture: FD expansion = a particular triangulation of $\mathcal{A}^*_{n-3}$

other triangulations $\to$ new formulas & efficient recursions for $m_n^{\phi^3}$





Hidden symmetry of ϕ^3 amps (invisible in FD's), analog of dual conformal symmetry of $\mathcal{N} = 4$ SYM (but no SUSY/integrability), becomes manifest by geometry!

B/C & D: one-loop amps [Arkani-Hamed, SH, Salvatori, Thomas]

All finite-type cluster polytopes have ABHY realizations [Thomas et al]; for rank d with N facets (cluster variables) $X_\alpha \geq 0$, $N-d$ conditions like $X+X-X-X=c \implies$

d-dim polytope, with boundaries “factorizing” into lower ABHY polytopes!

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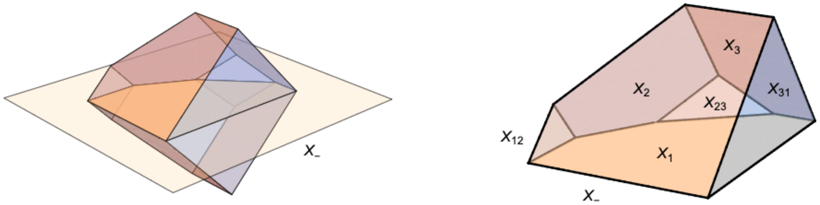
$\mathcal{B}_{n-1}/\mathcal{C}_{n-1}$: $N = n(n{-}1)$ facets $\sim \mathcal{A} \times \mathcal{B}/\mathcal{C} \to \underline{\Omega}(\mathcal{B}/\mathcal{C})$ = sum of tadpole diagrams

$\mathcal{D}_n$: $N = n^2$ facets $\partial\mathcal{D} \sim \mathcal{D} \times \mathcal{A} + \mathcal{A} \times \mathcal{D} + \mathcal{A}$ (fact. + forward limit of $(n{+}2)$-pt tree)
(after slicing along "tadpole plane") $\to \underline{\Omega}(\overline{\mathcal{D}}_n) =$ 1-loop ϕ^3 integrand



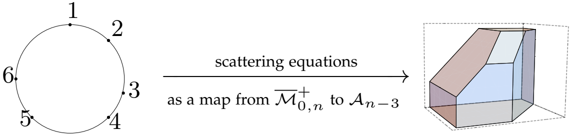
Triangulations $\to$ new recursion for 1-loop amps $\to$ expose hidden sym. of loop ϕ^3

Outline

Scattering equations & push-forward

An well-known associahedron: compactification of moduli space of open-string worldsheet $\mathcal{M}^+_{0,n} := \{z_1 < z_2 < \cdots < z_n\}/\mathrm{SL}(2,\mathbb{R})$ [Deligne, Mumford] (more later)

Pullback *scattering equations* on H_n: a diffeomorphism from $\overline{\mathcal{M}}^+_{0,n}$ to $\mathcal{A}_{n-3}$:



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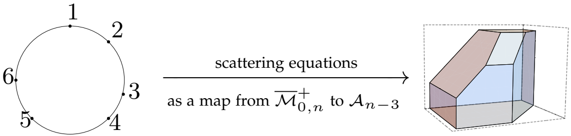

Diffeomorphism $A \to B \implies$ pushforward $\Omega(A) \to \Omega(B)$ [Arkani-Hamed, Bai, Lam]

$$y = f(x) \qquad \Longrightarrow \qquad \Omega(B)_y = \sum_{x=f^{-1}(y)} \Omega(A)_x$$

$m_n^{\phi^3} = \underline{\Omega}(\mathcal{A}_{n-3})$ as pushforward $\sum_{\text{sol.}} \Omega(\overline{\mathcal{M}}^+_{0,n}) \to$ geometric origin of CHY [ABHY]

Is this special to strings, or is it general for polytopes & canonical forms?

Stringy canonical forms [Arkani-Hamed, SH, Lam]

Any polytope $\mathcal{P}$ $\rightarrow$ integrals as α'-deformation of canonical form $\Omega(\mathcal{P})$

- New way of computing $\Omega(\mathcal{P})$ in the $\alpha' \to 0$ limit & finite-α' extension of it
- $\Omega(\mathcal{P})$ obtained as a push-forward using the SE map that appear in $\alpha' \to \infty$!

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Consider integral over $\mathbb{R}_+^d = \{0 < x_i < \infty | i = 1, \cdots d\}$ with regulators

$$\mathcal{I}_P(\mathbf{X}, c) := (\alpha')^d \int_0^\infty \frac{dx_1}{x_1} \cdots \frac{dx_d}{x_d} x_1^{\alpha' X_1} \cdots x_d^{\alpha' X_d} \, P(\mathbf{x})^{-\alpha' c},$$

with $X_i > 0, c > 0$ & *positive* polynomial $P(\mathbf{x})$, *e.g.* $\int_0^\infty \frac{dx}{x} x^X (1+x)^{-c}$

Key: consider Newton polytope of P [Arkani-Hamed, Bai, Lam]:

$P(\mathbf{x}) := \sum_\alpha p_\alpha \mathbf{x}^{\mathbf{n}_\alpha} \rightarrow N(P)$ is the convex hull of (exponent vectors) $\mathbf{n}_\alpha \in \mathbb{Z}^d$

e.g. $N(1+2x) = [0,1]$, $N(1+x+3xy) = \mathbf{conv}[(0,0),(1,0),(1,1)]$,
$N(1+2xy+y^2+xz^3+\cdots) = \mathbf{conv}[(0,0,0),(1,1,0),(0,2,0),(1,0,3),\cdots]$

Newton polytope & SE map

Theorem [Arkani-Hamed, SH, Lam] (1). $\mathcal{I}_P$ converges iff $\mathbf{X}$ is inside top-dim $cN(P)$,

$$\lim_{\alpha' \to 0} \mathcal{I}_P = \underline{\Omega}_{\mathbf{X}}(cN(P)),$$

3 (2). The scattering-eq map is a *diffeomorphism* from $\mathbb{R}^d_+$ to (interior of) $cN(P)$

$$\text{SE}: \quad d\log\left(\prod_i x_i^{X_i} P^{-c}\right) = 0 \implies \text{map}: \quad X_i = x_i \frac{c}{P}\frac{\partial P}{\partial x_i},$$

$\implies$ Pushforward of $\omega := \prod_{i=1}^d \frac{dx_i}{x_i}$, by summing over all solutions of SE

$$\sum_{\text{sol.}} \omega = \Omega(cN(P)) = \underline{\Omega}(cN(P))\, d^d\mathbf{X}$$

e.g. for $\int_0^\infty \frac{dx}{x} x^X (1+x)^{-c}$, $cN(P) = [0, c] \to$ leading order $\frac{1}{X} + \frac{1}{c-X}$

SE map $X = c\frac{x}{1+x} \implies$ pushforward $\frac{dx}{x}|_{x=\frac{X}{c-X}} = dX(\frac{1}{X} + \frac{1}{c-X})$

Prototype of stringy canonical form & the phenomenon $\alpha' \to 0$ vs. $\alpha' \to \infty$!

Minkowski sum

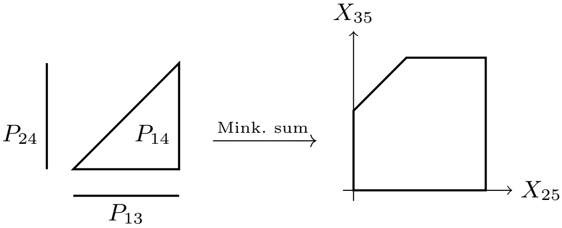
Trivial to generalize to multiple polynomials (first consider rational c_I)

$$\mathcal{I}_{\mathcal{P}}(\mathbf{X},\mathbf{c}) := (\alpha')^d \int_0^\infty \prod_{i=1}^d \frac{dx_i}{x_i}\, x_i^{\alpha' X_i} \prod_I P_I(x)^{-\alpha' c_I}\,.$$

$N(\prod_I P_I(x)^{-\alpha' c_I})$ is the Minkowski sum $c_1 N(P_1) \oplus c_2 N(P_2) \oplus \cdots$ (recall $c_1 A \oplus c_2 B := \{c_1 \mathbf{a} + c_2 \mathbf{b} | \mathbf{a} \in A, \mathbf{b} \in B\}$).

$\mathcal{I}_{\mathbf{P}}$ converges when $\mathbf{X}$ is inside $\mathcal{P} := \oplus_I c_I N(P_I)$, & leading order$= \underline{\Omega}_{\mathbf{X}}(\mathcal{P})$.

$$e.g.\quad \mathcal{I}_{\text{pentagon}} = \int_0^\infty \frac{dx}{x}\frac{dy}{y} x^{X_{2,5}} y^{X_{3,5}} (1+x)^{-c_{1,3}} (1+y)^{-c_{2,4}} (1+x+xy)^{-c_{1,4}}$$



$$\lim_{\alpha'\to 0} \mathcal{I}_{\text{pentagon}} = \underline{\Omega}(\mathcal{A}_2) = m_5^{\phi^3}$$

Scattering equations: $\alpha' \to 0$ vs. $\alpha' \to \infty$

General SE: for any $\mathcal{I}_\mathcal{P}$, scattering-eq map is a *diffeomorphism* to Minkowski-sum $\mathcal{P}$

$$\text{SE}: \quad d\log\left(\prod_i x_i^{X_i} \prod_I P_I^{-c_I}\right) = 0 \implies \text{map}: \quad \mathbf{X} = \sum_I c_I \frac{\partial \log P_I}{\partial \log \mathbf{x}},$$

The $\alpha' \to 0$ limit of $\mathcal{I}_\mathcal{P} = \underline{\Omega}(\mathcal{P})$ = pushforward using SE from $\alpha' \to \infty$:

$$\lim_{\alpha' \to 0} d^d\mathbf{X}\, \mathcal{I}_\mathcal{P} = \sum_{\text{sol.}} \omega \quad \Leftrightarrow \quad \int \omega \prod \delta(\mathbf{X} - \sum_I c_I \frac{\partial \log P_I}{\partial \log \mathbf{x}}) = \underline{\Omega}(\mathcal{P}).$$

For any polytope, low-energy limit of stringy canonical form agrees with pushforward /CHY formula from saddle points in the high-energy limit [Gross, Mende]

This has nothing to do with actual strings *per se*, rather a general phenomenon.

Applying stringy canonical form to ABHY $\mathcal{A}_{n-3}$ → discover open-string integral of $\Omega(\overline{\mathcal{M}}^+_{0,n})$ & "Koba-Nielsen" factor as regulator:

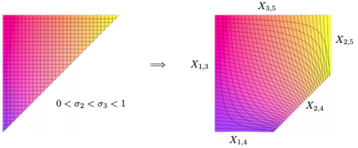
$$\mathcal{I}_n^{\rm disk}(\{X\}) := (\alpha')^{n-3} \int_{\overline{\mathcal{M}}^+_{0,n}} \Omega(\overline{\mathcal{M}}^+_{0,n}) \prod_{a<b} |z_a - z_b|^{\alpha' s_{a,b}}$$

The field-theory limit of $\mathcal{I}_n^{\rm disk} = \phi^3$ tree amps, $\underline{\Omega}(\mathcal{A}_{n-3})$; also computed by

pushforward using CHY scattering equations (from Gross-Mende limit $\alpha' \to \infty$)

$$\sum_{b \neq a} \frac{s_{a,b}}{z_a - z_b} = 0\,, \quad a = 1, 2, \cdots, n\,,$$

e.g. SE map for $\mathcal{I}_{\rm pentagon}$ → ABHY $\mathcal{A}_2 \implies \sum_{2\text{ sol.}} \frac{dx}{x}\frac{dy}{y} = \underline{\Omega}(\mathcal{A}_2)$



Similarly $\Omega_{\alpha'}$ for ABHY $\mathcal{B}/\mathcal{C}$ & $\mathcal{D}$ → α'-deformation of tadpoles & 1-loop ϕ^3 amps!

Outline

1. Tree & loops in ϕ^3 from cluster polytopes
2. Stringy canonical forms & scattering equations
3. Binary realization & generalized string integrals

Binary realization of associahedra

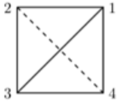
Polytopal realization of"factorizing" combinatorics (highly non-trivial), but not fully rigid→ binary realization, *i.e.* with 0 & 1? → generalized string amps

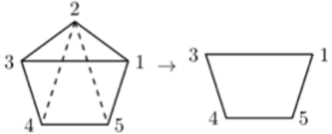
type $\mathcal{A}$: natural to consider u eqs, 1 for each diagonal $u_{i,j}$ of n-gon

$$1-u_{i,j}=\prod_{(k,l)\ \text{cross}\ (i,j)} u_{k,l} \quad \text{or} \quad u_{i,j}+\prod_{\substack{k\in[i+1,j)\\ l\in[j+1,i)}} u_{k,l}=1\,.$$

e.g. $n=4$, $1-u_{1,3}=u_{2,4}$ (1 eq); $n=5$, $1-u_{1,3}=u_{2,4}u_{2,5}$ & cyclic (3 independent)

note $u_{1,3}\to 0$, $u_{2,4},u_{2,5}\to 1$ (decouple from u eqs) $\to u_{1,4}+u_{3,5}=1$ ($n=4$)

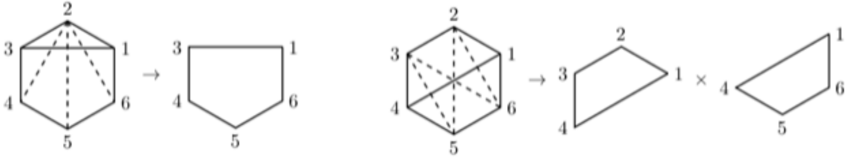




U space & positive part [Arkani-Hamed, SH, Lam, Thomas]

Define U space as solution space of u eqs with $u \neq 0, \infty$ (open set)

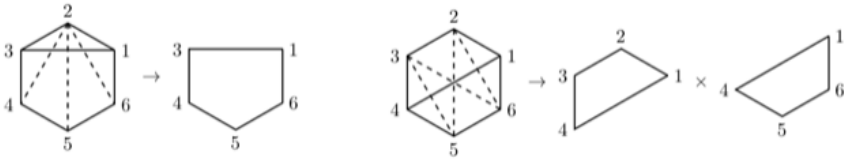
- U_n space is $n{-}3$ dim (only $\frac{n(n-3)}{2} - (n{-}3)$ of u eqs are independent)
- U has same boundary structures as $\mathcal{A}_{n-3}$ assoc. (purely algebraically, defined over any $\mathbb{F}$): any $u_{i,j} \to 0$, all incompatible $u_{k,l} \to 1 \to U_L \times U_R$



U space & positive part [Arkani-Hamed, SH, Lam, Thomas]

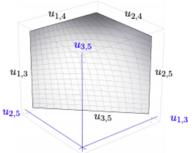
Define U space as solution space of u eqs with $u \neq 0, \infty$ (open set)

- U_n space is $n-3$ dim (only $\frac{n(n-3)}{2} - (n-3)$ of u eqs are independent)
- U has same boundary structures as $\mathcal{A}_{n-3}$ assoc. (purely algebraically, defined over any $\mathbb{F}$): any $u_{i,j} \to 0$, all incompatible $u_{k,l} \to 1 \to U_L \times U_R$



$u > 0 \implies 0 < u < 1$ defines positive part of U_n: U_n^+

U_n^+ is a (curvy) $\mathcal{A}_{n-3}$, *e.g.* U_5^+ is a curvy pentagon



Cross-ratios & moduli space

The u eqs $\implies$ $\binom{n}{4}$ eqs $\prod u + \prod u = 1$ (for ordered a, b, c, d):

$$[a, b|c, d] + [b, c|d, a] = 1\,, \qquad [a, b|c, d] := \prod_{i\in[a,b),\ j\in[c,d)} u_{i,j}\,,$$

(special case: $u_{i,j} = [i, i{+}1|j, j{+}1]$) also $[a, b|c, e][a, b|e, d] = [a, b|c, d]$

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$\to$ exactly constraints satisfied by cross ratios of n points on $\mathbb{P}^1$

$$[a, b|c, d] = \frac{(ad)(bc)}{(ac)(bd)} = \frac{(z_a - z_d)(z_b - z_c)}{(z_a - z_c)(z_b - z_d)}\,.$$

$\implies$ U space is an *invariant way* to parametrize $\mathcal{M}_{0,n}$ (dihedral coordinates of [Brwon])

- For $U_n^+ \sim \mathcal{M}_{0,n}^+$, we have $0 < [a, b|c, d] < 1 \implies n$ points are ordered
- $U(\mathbb{R})$ has $(n{-}1)!/2$ connected components (each one is an U_n^+ for that *ordering*)
- For $U(\mathbb{C})$, monomial transform. $\to$ S_n automorphism (permutations of n points)

General U space & (finite-type) cluster algebra [Arkani-Hamed, SH, Lam, Thomas]

Generalizes to all finite-type cluster algebra: needs compatibility degree $\alpha||\beta = 0, 1$ for $\mathcal{A}$, now also 2 for $\mathcal{B}, \mathcal{C}, \mathcal{D}$ (& 3 for exceptional cases)

$$1 - u_\alpha = \prod_\beta u_\beta^{\alpha||\beta}, \quad \forall\alpha \qquad (N \text{ eqs for } N \text{ variables})$$

$\Longrightarrow$ d-dim U space: "algebraic cluster polytopes" $\rightarrow$ applications in cluster algebra

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$\implies$ d-dim U space: "algebraic cluster polytopes" $\to$ applications in cluster algebra

- u are related to special X var. (same num. as A var.) [Fomin, Zelevinski]
- $\{u, u^{-1}\}$ generate cluster algebra mod torus action
- $\prod u + \prod u = 1 \leftrightarrow$ *exchange relation* (u eqs $\leftrightarrow$ primitive mutations [Yang, Zelevinski])

$u > 0$ again cut out a curvy cluster polytope: *e.g.* $U^+(\mathcal{B}), U^+(\mathcal{C})$ are cyclohedra

$U^+(\mathcal{D}_n)$: curvy polytope with facets $\sim \ \mathcal{A}_m \times \mathcal{D}_{n-1-m} \ + \ \mathcal{A}_{n-1}$ *etc.*

Again $U(\mathbb{R})$ tiled by different "orderings", *e.g.* $\mathcal{B}_2/\mathcal{C}_2$: 4 hexagons + 12 pentagons

Generalized string integrals on U space

The most natural integral over U^+ with regulators at all boundaries $u_I \to 0$:

$$\mathcal{I}^{U^+}(\{X\}) := (\alpha')^d \int_{U^+} \Omega^{(d)}(U^+) \prod_{I=1}^{N} u_I^{\alpha' X_I} \,,$$

For $\mathcal{A}_{n-3} \to \mathcal{I}_n^{\rm disk}$; all generalized open-string integrals reminiscent string amps:

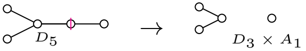
- Stringy canonical form for ABHY: $\lim_{\alpha' \to 0}$ = pushforward = $\underline{\Omega}$ of ABHY
- *Meromorphic* with poles of the form $X_I = 0, -1, -2, \cdots$
- Channel-duality & exponential soft at UV
- Magic factorization at *massless* poles, $X_I = 0$, for **finite** α'!

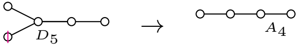
For $\mathcal{I}_n^{\text{disk}}$, self-factorization at massless poles: as $X_{i,j} \to 0$, $u_{i,j} \to 0 \implies$ by u eqs the Koba-Nielsen factor $\prod_{i,j} u_{i,j}^{\alpha' X_{i,j}}$ factorizes into L and R part:

$$\text{Res}_{X_{i,j}=0}\, \mathcal{I}_n^{\text{disk}} = \int_{\partial_{ij}\mathcal{A}_{n-3}} \partial_{ij}\left(\Omega \prod u^X\right) = \mathcal{I}_L \times \mathcal{I}_R\,,$$

e.g. at $X_{1,3} \to 0$, $\mathcal{I}_5^{\text{disk}} \;\to\; \mathcal{I}_4^{\text{disk}} = \int_0^1 d\log\frac{u}{1-u} u^{X_{1m4}}(1-u)^{X_{3,5}}$ (Veneziano amp)

General integrals " factorize" at $X=0$ for finite α' *e.g.* $\mathcal{I}_{\mathcal{D}_n} \to \mathcal{I}_{\mathcal{A}} \times \mathcal{I}_{\mathcal{D}}$ & $\mathcal{I}_{\mathcal{A}_{n-1}}$





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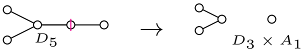

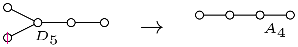

Natural to integrate $\Omega(U_\alpha^+)$ in U_β^+ (for $\mathcal{A} \to Z(\alpha|\beta)$ [Carrasco, Mafra, Schlotterer])

closed-string integrals for pair of orderings in $U(\mathbb{C})$ (intersection num. [Mizera])

For $\mathcal{A}$, basis integrals for *any* massless string tree amps (gluons/gravitons,...) [Schlotterer et al.] [SH, Teng, Zhang] Q: physical meaning of gen. string integrals for other finite types?

Arithemetic geometry for U space

Point count in U_{F_p} for a prime $p \implies$ topological properties of U [Weil conjectures...]

For $\mathcal{A}_{n-3}$, we know it is hyperplane arr. $\implies$ polynomial count
$N(p) = (p-2)(p-3)\cdots(p-n+2) \implies$ (twisted) cohomology of $\mathcal{M}_{0,n}(\mathbb{C}/\mathbb{R})$! [Zaslavsky]

- $|N(-1)| =$ num. of connected components/orderings: $\frac{(n-1)!}{2}$
- $|N(0)| =$ num. of independent $d\log$ top forms: $(n-2)!$ [Kleiss, Kuijff] (generally $|H^k|$)
- $|N(1)| = \chi = |H^{n-3}_{\text{twisted}}(U)| =$ num. of saddle points: $(n-3)!$ [BCJ/CHY]

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Also polynomials: $(p-n-1)^n$ for $\mathcal{B}_n$ & $(p-n-1)(p-3)(p-5)\cdots(p-2n+1)$ for $\mathcal{C}_n \implies$

	orderings	KK	BCJ/solutions
$\mathcal{B}_n$	$(n+2)^n$	$(n+1)^n$	n^n
$\mathcal{C}_n$	$(2n)!!\frac{n+2}{2}$	$(2n-1)!!(n+1)$	$\frac{(2n)!!}{2}$

Beyond $\mathcal{ABC}$: *quasi-polynomials*? *e.g.* 25 regions for $\mathcal{G}_2$, 547 regions for $\mathcal{D}_4$ ...

Summary & outlook

- For any polytope, stringy canonical form $\Omega_{\alpha'}$ provides α'-deformation scattering equations for $\alpha' \to \infty$ (Gross-Mende) & $\alpha' \to 0$ (pushforward/CHY)
- ABHY realization: explains factorizing & polytopal for ϕ^3 tree & 1-loop amps
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- ABHY realization: explains factorizing & polytopal for ϕ^3 tree & 1-loop amps
- $\Omega_{\alpha'}$ for ABHY = generalized string integrals $\leftrightarrow$ binary realization

- All-loop scattering forms & polytopes for ϕ^3? gluons & gravitons?
- Connections to $\mathcal{N} = 4$ amplitudes [Arkani-Hamed, Lam, Spradlin]? Twistor strings?
- A unified geometric picture for amps & beyond: AdS? cosmology?

Thank you!