Quantum Complexity of Time Evolution with Chaotic Hamiltonians

Onkar Parrikar

Department of Physics and Astronomy
University of Pennsylvania.

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Penn
University of Pennsylvania

It from Qubit
Simons Collaboration on
Quantum Fields, Gravity and Information

Based on

- V. Balasubramanian, M. DeCross, A. Kar & OP, arXiv:1905.05765 [hep-th].

Introduction

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- For this purpose, identifying universal probes of gravitational dynamics is of significant interest.
- Several criteria, such as spectral properties, chaotic dynamics and entanglement structure have already shed light on this question.

[Heemskerk, Penedones, Polchinski, Sully '09..., Maldacena, Shenker, Stanford '15, Kitaev...,

Faulkner, Guica, Hartman, Myers, Van Raamsdonk '13, Lewkowycz, OP '18...]

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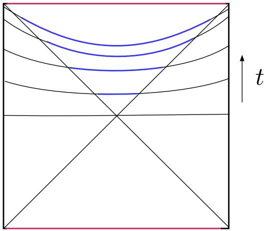

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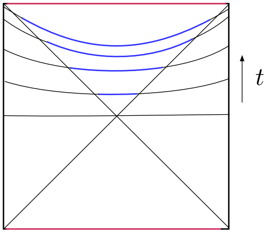

$$\frac{d\mathcal{C}}{dt} = ST, \quad \mathcal{C} = \frac{1}{\ell G_N}\text{Vol.}$$

- This phenomenon was conjectured to be dual to the growth of complexity of the dual CFT state [Stanford, Susskind '14]. (See also [Brown, Roberts, Swingle, Susskind, Zhao '15...].)

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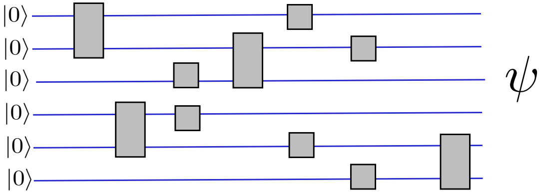

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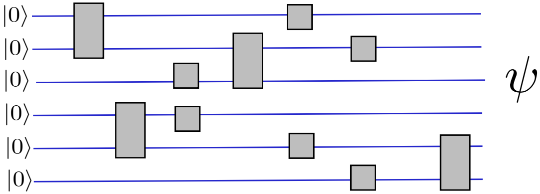

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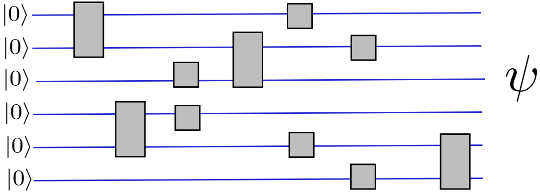

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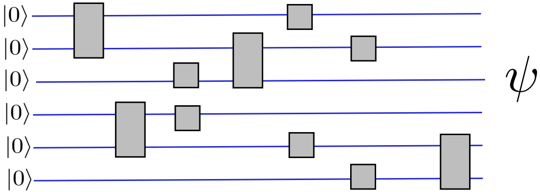
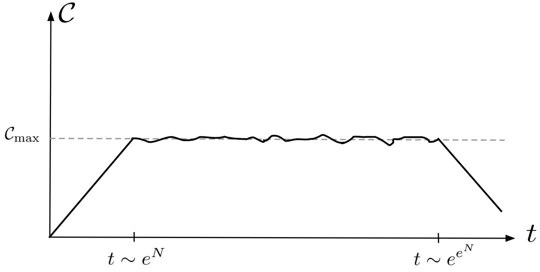


- Complexity is a fine-grained probe of entanglement structure.
- In order to make contact with gravity, we need to generalize this to more general quantum systems, in particular quantum field theories.
- Some progress towards this has been made... [Jefferson, Myers '17, Chapman, Heller, Marrochio, Pastawski '17, Caputa, Magan '18, Belin, Lewkowycz, Sarosi '18...]

Time evolution of Complexity

- Gravity plus general quantum considerations suggest that the complexity in holographic theories should grow linearly in time till $t \sim e^N$, and saturate thereafter till $t \sim e^{e^N}$ [Brown, Susskind].



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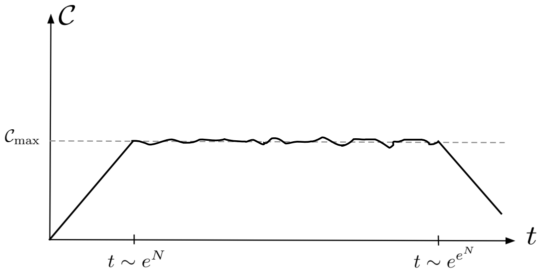

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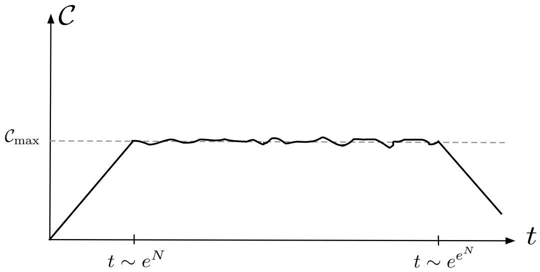

- Our aim here is to take steps towards proving this conjecture for large N, chaotic systems.
- We will work with the Sachdev-Ye-Kitaev model as a concrete example.

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- In the SYK model, we can take the k-local operators as being simple:

$$T_a = \psi_a, \;\; T_{a_1 a_2} = i\psi_{a_1}\psi_{a_2}, \cdots, T_{a_1 \ldots a_k} \propto \psi_{a_1} \cdots \psi_{a_k}$$

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- **Note**: We will always pick k large enough so that the Hamiltonian is built from local generators.

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- We take the metric on $\mathcal{U}$ to be the right-invariant metric which follows from this bilinear form on the Lie-algebra.

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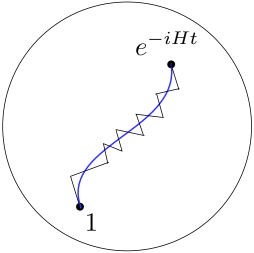

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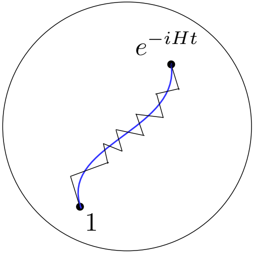

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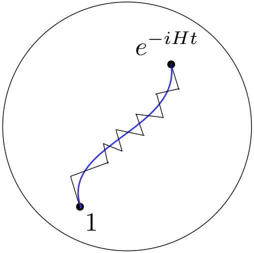

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- When μ is taken to be exponentially large, then geodesic complexity has been argued to be polynomially equivalent to the circuit complexity [Nielsen, Dowling, Gu, Doherty '06].

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- We are interested in the geodesic complexity of the time evolution operator e^{-iHt} in the SYK model.
- Parametrizing the geodesic in terms of the velocity:

$$U(s) = \mathcal{P} \exp\left(-i\int_0^s ds' \sum_m V_m(s')\, T_m\right), \quad \cdots s \in [0,1]$$

the geodesic equation becomes

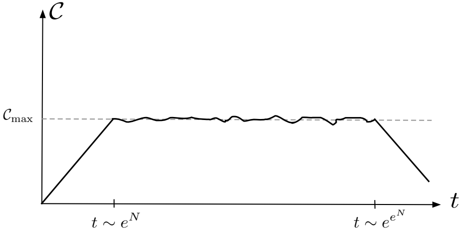
$$i\frac{d\mathbf{V}_L}{ds} = \mu\, [\mathbf{V}_L, \mathbf{V}_{NL}]_L\,,$$

$$i\frac{d\mathbf{V}_{NL}}{ds} = \frac{\mu}{1+\mu}\, [\mathbf{V}_L, \mathbf{V}_{NL}]_{NL}\,,$$

$$U(1) = e^{-iHt}.$$

where $\mathbf{V}_L$ is the projection of the velocity along the local directions, and $\mathbf{V}_{NL}$ is the projection along the non-local directions.

To do list



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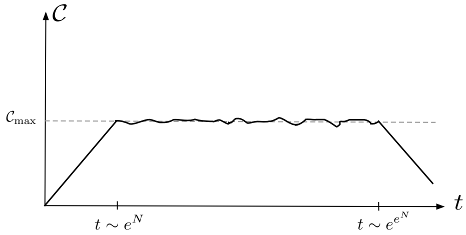

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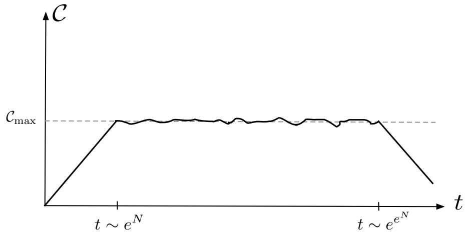

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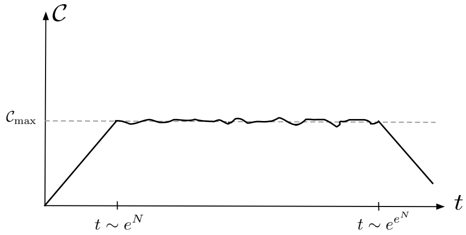

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- Show that the geodesic is a global minimum till $t \sim e^{\alpha S}$, after which other geodesics take over and lead to saturation of complexity.

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where

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- **Note**: Since the linear geodesic only lies along the local directions, its length is independent of the cost factor μ.

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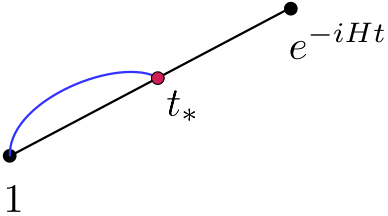

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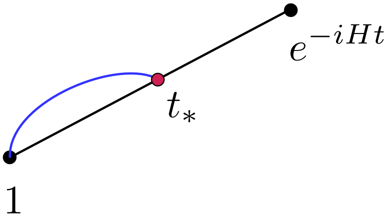

- The original geodesic stops being minimizing past the first conjugate point (i.e., it is a saddle point thereafter).

Jacobi equation

- The linearized geodesic equation around the linear geodesic is called the Jacobi equation. In terms of the velocity, it takes the form

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i\frac{d\delta \mathbf{V}_L}{ds} = \mu t\,[H, \delta \mathbf{V}_{NL}]_L
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- We need to show that this equation has no solutions till exponential time.

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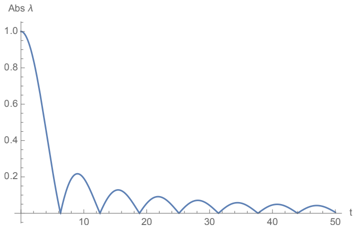
- We can obtain the spectrum:

$$\mathbb{Y}_{(0,t)}(|m\rangle\langle n|) = \lambda_{mn}|m\rangle\langle n|, \quad \lambda_{mn} = \frac{e^{i(E_m-E_n)t}-1}{(E_m-E_n)t},$$

where $|m\rangle, |n\rangle$ etc. are energy eigenstates.

Conjugate points at $\mu = 0$



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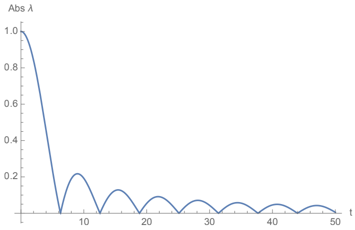

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- So we will encounter our first conjugate point at $t_c = \frac{2\pi}{(E_{\max} - E_{\min})}$, which in the SYK model, happens at a time of $O(1/N)$.

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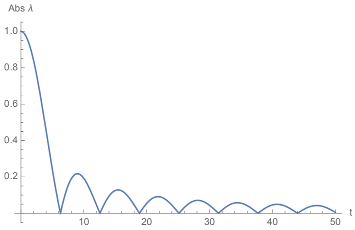

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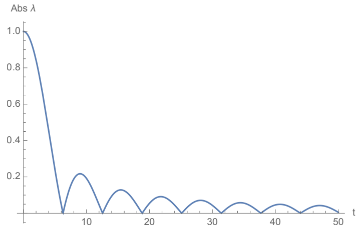

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- We wish to track these conjugate points/zero modes as μ becomes large, and show that they move off to large times.

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- However, the problem simplifies greatly in large N chaotic systems.

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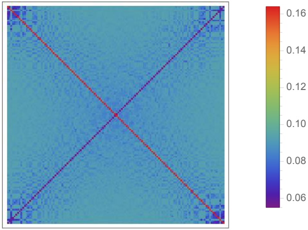
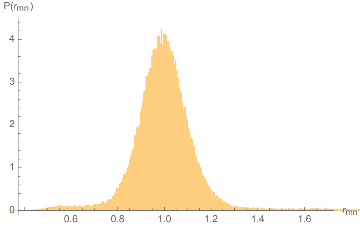
- We can test this in the SYK model. We can numerically compute

$$R_{mn} = || \, |m\rangle\langle n|_L ||^2 := \text{poly}(S) \, e^{-2S} r_{mn}.$$





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- With this observation in hand, the Jacobi equations simplify greatly and we can show that at finite μ, with $\mu t << e^S$:

$$\mathbb{Y}_{(\mu,t)}(|m\rangle\langle n|) \simeq \lambda_{mn}|m\rangle\langle n|, \quad \lambda_{mn} = \frac{e^{\frac{i(E_m-E_n)t}{1+\mu}} - 1}{\frac{(E_m-E_n)t}{1+\mu}}.$$

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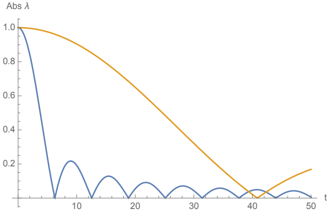

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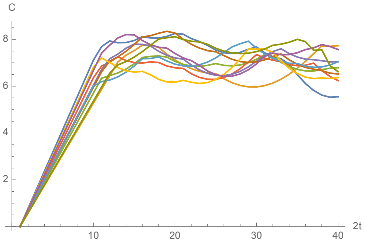

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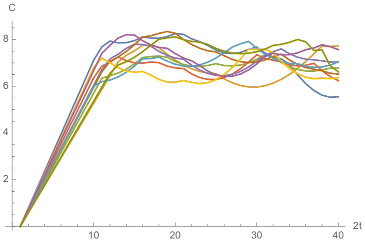

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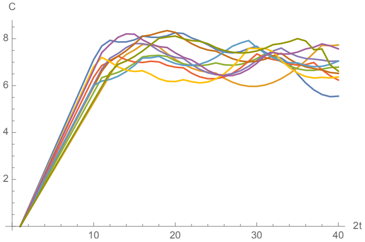
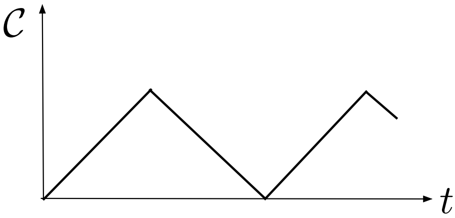


- This shows linear growth initially followed by a plateau. But the plateau starts at $t \sim \frac{2\pi}{(E_{\max} - E_{\min})}$.
- At finite μ, we expect all but the linear geodesic to move into the non-local directions.

Global minimality at large μ

- However, it is easy to find integrable Hamiltonians where one can explicitly demonstrate the existence of non-trivial geodesics which entirely lie within the local directions. These compete with the linear geodesic and lead to linear growth terminating at small times.



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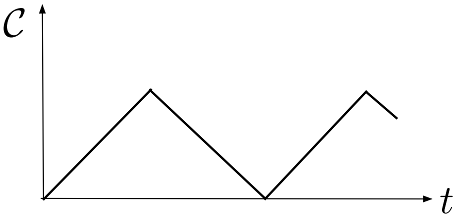

- For chaotic Hamiltonians such geodesics do not exist – it is possible to argue that any non-trivial geodesic other than the linear one must necessarily move into the hard directions.

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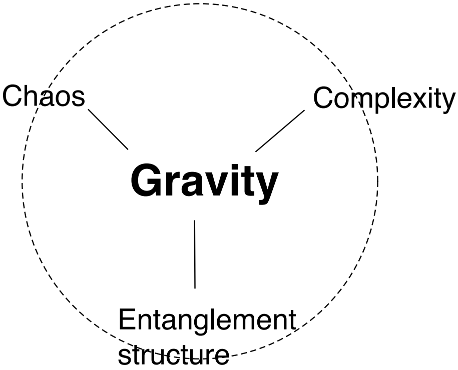
- We argued that there is always a geodesic whose length grows linearly in time.
- We showed that in large N chaotic systems, this geodesic is a local minimum till exponential time.
- It would be interesting if we can prove the global minimality of this geodesic till exponential time, in particular by using universal properties of chaotic systems, such as spectral statistics or the eigenstate thermalization hypothesis [Deutsch, Srednicki, Rigol et al...].

Outlook

This may eventually lead to a deeper understanding of the relations between complexity, chaos, entanglement structure and emergence of gravitational dynamics in the AdS/CFT correspondence.



Appendix: $N = 2$, $\mathcal{U} = SU(2)$ at finite μ

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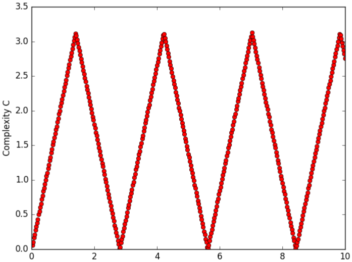
- The complexity of e^{-iHt} can be obtained with a combination of analytic and numerical methods:



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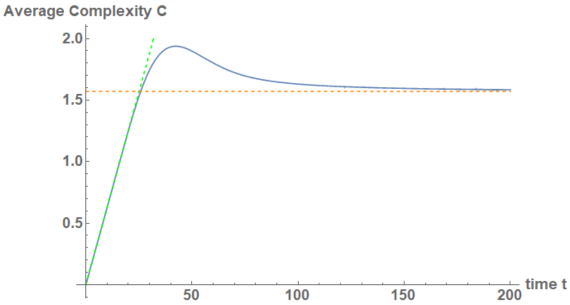
- If we average over J_1, J_2 with Gaussian distributions, then the averaged complexity develops a plateau:



- This saturation is an effect of disorder averaging, but at large N we expect the complexity in even a single instance of the SYK model to saturate.