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AdS From Optimization of Path-Integrals in CFTs

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① Motivations

Motivation 1

What is the basic mechanism of AdS/CFT ?

[After 20 years from Maldacena's discovery]

⇒ One intriguing idea is the conjectured interpretation of AdS/CFT as tensor networks (TNs).

[Swingle 2009,....]

``Emergent space from Quantum Entanglement''

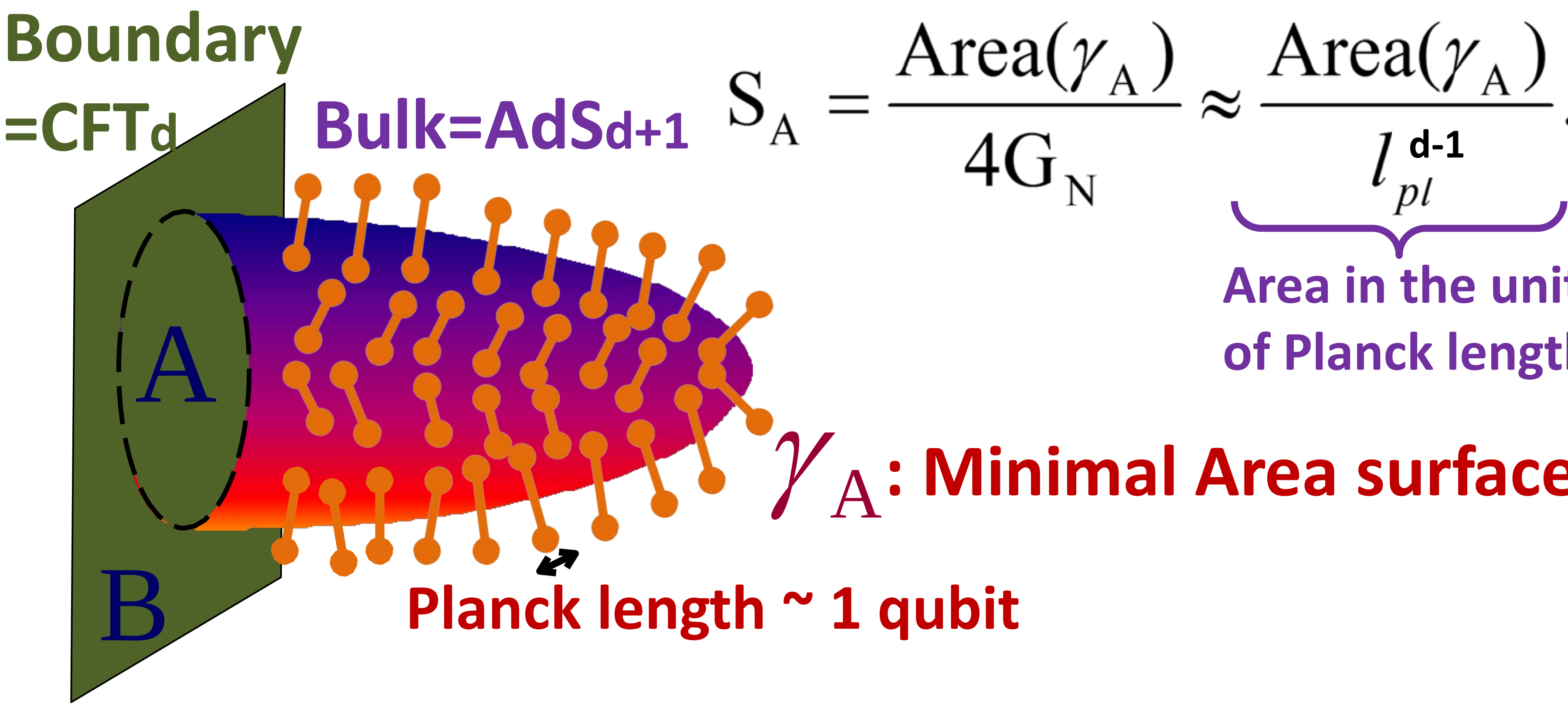
Tensor network = Network of Quantum entanglement

=``Geometry'' of Wave-functional in QFTs

In holography, the entanglement is computed as the area of minimal surface [RT 06, HRT 07]

$$S_A = \frac{\mathrm{Area}(\gamma_A)}{4G_N} \approx \frac{\mathrm{Area}(\gamma_A)}{l_{pl}^{d-1}}.$$

Area in the unit of Planck length



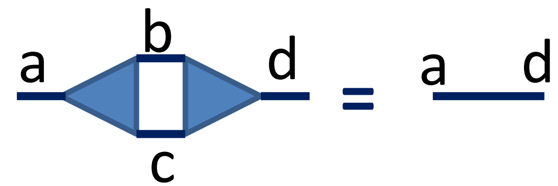
Spacetime in gravity = Collections of bits of entanglement

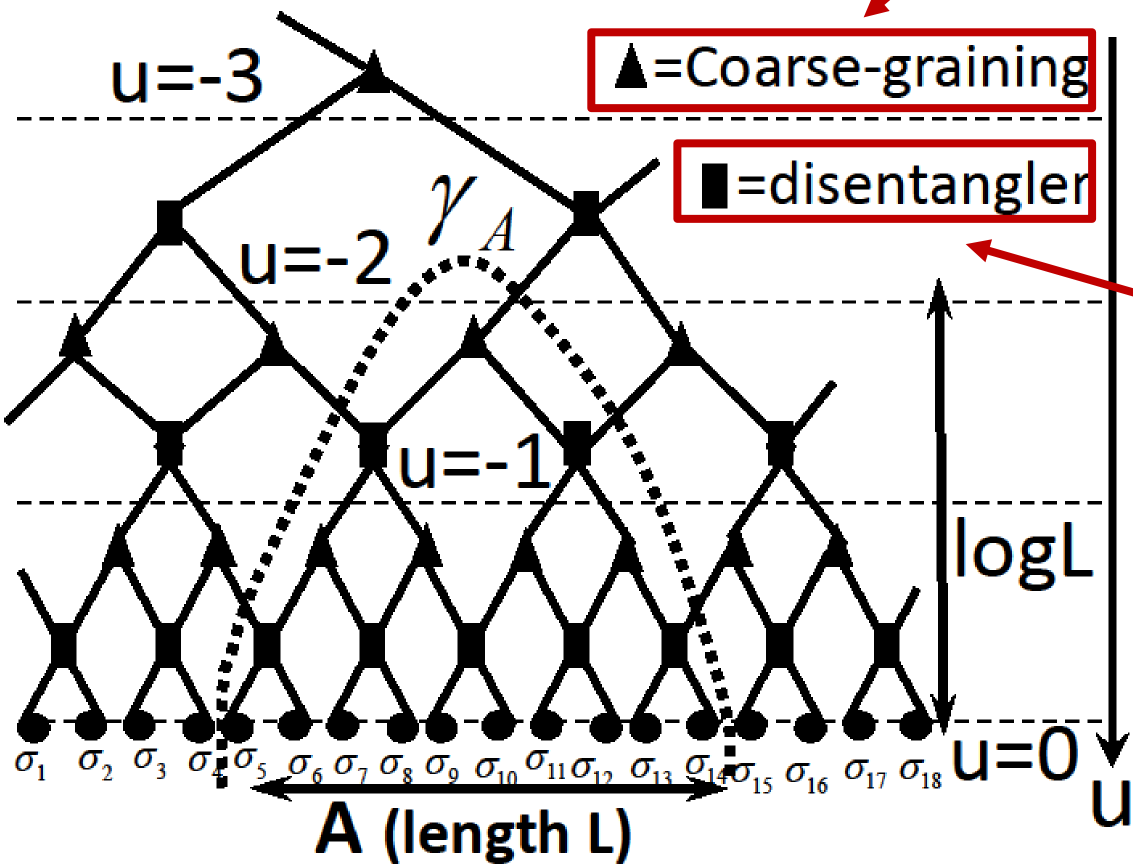
⇒ Emergent space via tensor network ?

MERA [Vidal 05, …] [TN for AdS/CFT: Swingle 09,…]

Coarse-graining = Isometry

$$[T]^{\dagger}_{abc}[T]_{bcd} = \delta_{ad}$$





Disentangler
= Unitary trf.

$S_A \leq \mathrm{Min}[\#\,\mathrm{links}]$

$\propto \log L$

$\Rightarrow$ agrees with results in 2d CFT !

A Basic Key Idea: Tensor Network of MERA
= a time slice of AdS space

Questions [see e.g. Beny 2011, Bao et.al. 2015, Czech et.al. 2015]

(a) Special Conformal invariance ?

(b) Non-isotropic tensor → ∃ causal structure in MERA ?

(c) Why the EE bound is saturated ?

(d) How to derive Einstein eq. ? (Sub AdS Scale Locality)

Recent developments in lattice models

・Improved TN models:

⇒(a),(b),(c) [Perfect TN: Pastawski-Yoshida-Harlow-Preskill 15]
[Random TN: Hayden-Nezami-Qi-Thomas-Walter-Yang 16]

・Another Interpretation :

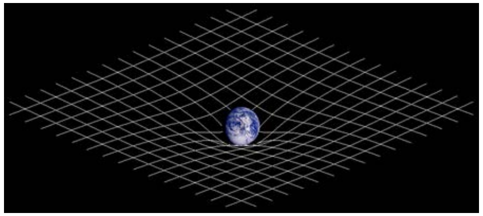
⇒(a),(b) [Kinematic Space: Czech, Lamprou, McCandlish, Sully 15]

Some of these problems may be due to lattice artifacts.

Moreover, we want to eventually understand the genuine AdS/CFT in the continuum limit.

⇒ We propose a new alternative approach **based on path-integrals**, related to a continuum limit of TNs.

Our guiding principle 1



Eliminating unnecessary tensors in TN for a given state

= Creating the most efficient TN (= **Optimization of TN**)

⟺ Solving the dynamics of Gravity (Einstein eq. etc.)

Motivation 2

How can we define **complexity** in CFTs ?

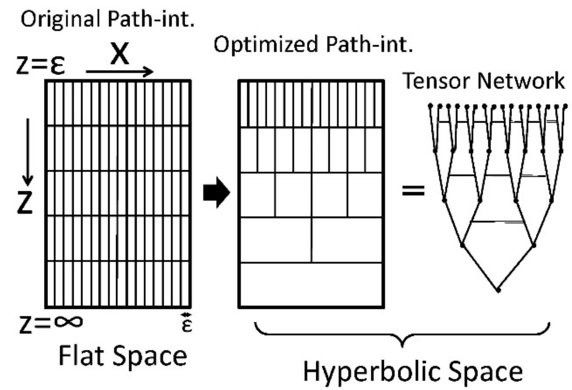
Computational Complexity of a quantum state
= Min [# of Quantum Gates]
= Min [# of Tensors] in TN

Recently, holographic formulas of complexity have been proposed. [Refer to Brown's ,Myers's talk]

(i) Complexity = Max. volume in AdS [Stanford-Susskind 14]
(ii) Complexity= Gravity action in WDW patch of AdS [Brown-Roberts-Susskind-Swingle-Zhao 15, Lehner-Myers et.al. 16]
For CFTs, Complexity ~ Info.Metric [Miyaji-Numasawa-Shiba -Watanabe-TT 15]

This motivates us to consider its QFT counterpart

➡ We introduce **``Path-integral Complexity''** .



Our guiding principle 2

- Lattice structure (= arrangement of tensors) in TN

⟺ Background metric g_{ab} in Euclid path-integral

- Optimization of TN for a state Ψ

⟺ Minimizing Path-integral Complexity $I_{\Psi}[g_{ab}]$

w.r.t the metric g_{ab}

Contents

② AdS from Optimization of Path-Integrals

(2-1) Formulation

A Basic Rule: Simplify a path-integral s.t. it produces the correct UV wave functional.

Consider 2D CFTs for simplicity. **(z=- Euclidean time, x=space)**

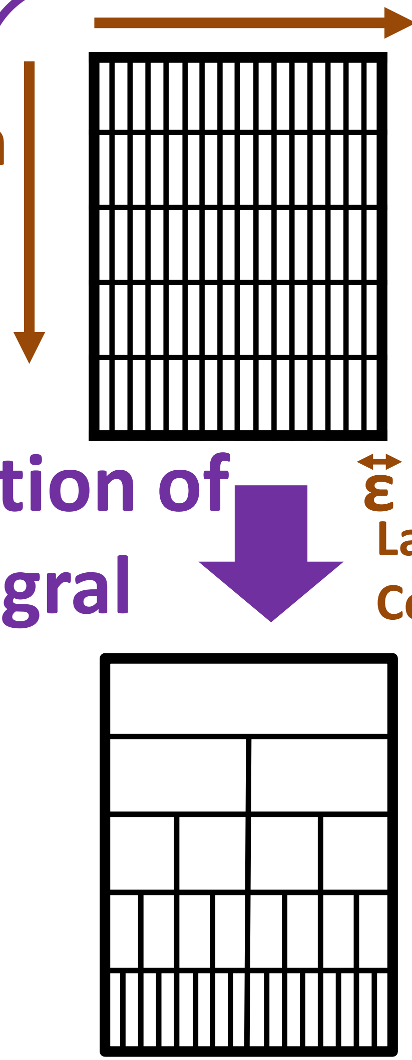
Deformation of discretizations in path-integral
= Curved metric such that one cell (bit) = unit length.

$$\Rightarrow \quad ds^2 = e^{2\phi(x,z)}(dx^2 + dz^2).$$

Note: The original flat metric is given by (ε is UV cutoff):

$$ds^2 = \varepsilon^{-2} \cdot (dx^2 + dz^2).$$

Optimization of Path-Integral [Miyaji-Watanabe-TT 16]



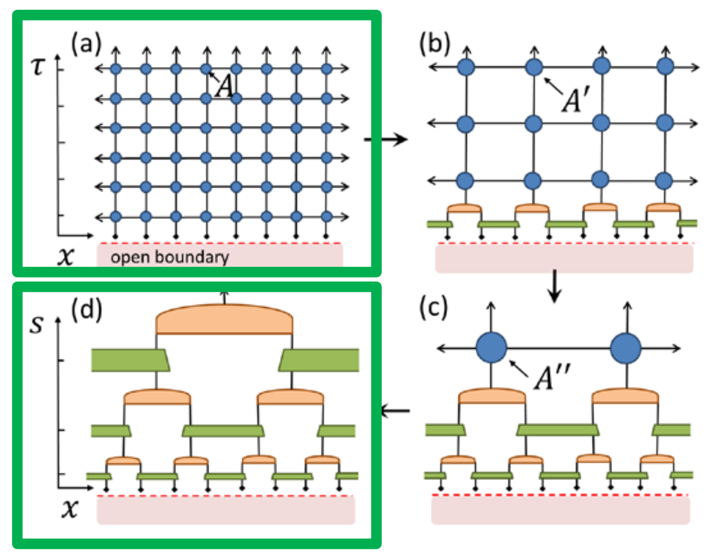
Space (x)

Euclidean Time (-z)

Optimization of Path-integral

$\overleftrightarrow{\varepsilon}$ Lattice Constant

Tensor Network Renormalization (TNR) = Optimization of TN [Evenbly-Vidal 14, 15]



Hyperbolic Space = Time slice of AdS3

The wave functional for CFT vacuum is given by

$g_{ab}(x,z)$: background metric

$$\Psi_{UV}^{g}[\Phi(x)] = \int \prod_{\substack{0<z<\infty \\ -\infty<x<\infty}} D\Phi(x,z)\, e^{-S_{CFT}(\Phi)} \cdot \delta\big(\Phi(x) - \Phi(x, z=0)\big)$$

In CFTs, owing to the Weyl invariance, we have

$$\Psi_{UV}^{g_{ab}=e^{2\phi}\delta_{ab}}[\Phi(x)] = \exp\big(I[\phi(x,z)]\big) \cdot \Psi_{UV}^{\mathrm{Flat}}[\Phi(x)] \,.$$

Optimized wf. **Original wf.**

<u>**Our Proposal**</u> (Optimization of Path-integral for CFTs):

Minimize $I[\phi(x,z)]$ w.r.t $\phi(x,z)$

with the boundary condition $e^{2\phi}\big|_{z=\varepsilon} = \varepsilon^{-2}$.

<u>A Reason for Minimization</u>

The normalization N estimates repetitions of same operations of path-integration. → **Minimize this !**

⇒ Our Complexity Formula:

$$C_\Psi = \underset{\phi(x,z)}{\mathrm{Min}}[I_\Psi[\phi(x,z)]]$$

$$C_\Psi \equiv \text{computational complexity of the quantum state } |\Psi\rangle$$

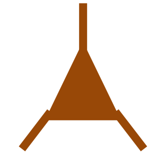
(2-2) Liouville Action as Complexity in 2D CFTs

[Caputa-Kundu-Miyaji-Watanabe-TT 17]

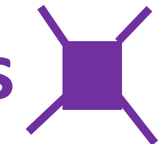
$$I[\phi]=\mathrm{Log}\left[\frac{\Psi_{g=e^{2\phi}\delta_{ab}}}{\Psi_{g=\delta_{ab}}}\right]=S_L[\phi],$$

Liouville Action

# of Isometries
[Czech 17]



# of Unitaries



$$S_L[\phi]=\frac{c}{24\pi}\int dxdz\left[(\partial_x\phi)^2+(\partial_z\phi)^2+e^{2\phi}\right]$$

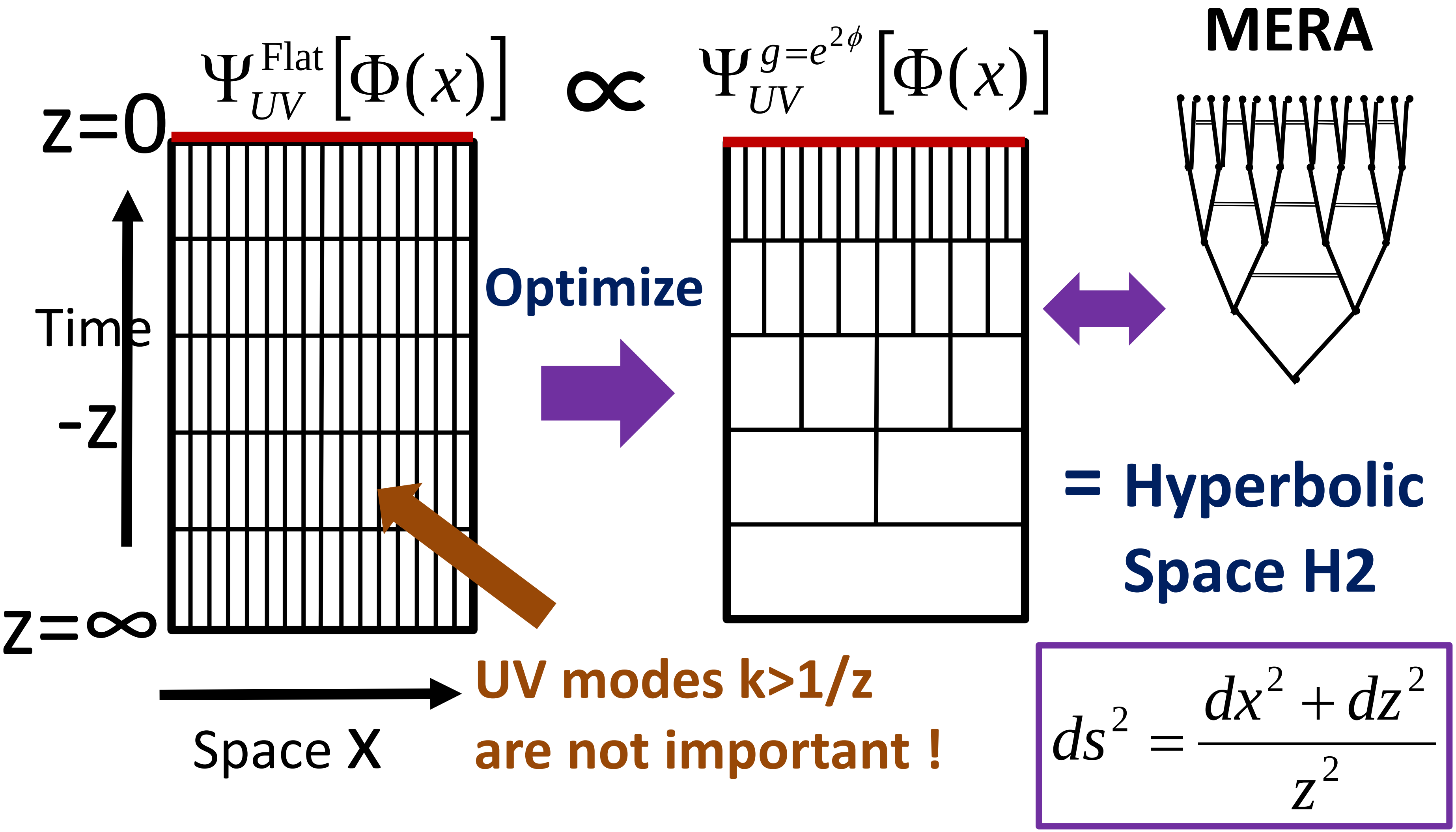
$$=\frac{c}{24\pi}\int dxdz\left[(\partial_x\phi)^2+\left(\partial_z\phi+e^{\phi}\right)^2\right]+(\text{surface term})$$

$$\Rightarrow \text{Minimum}: \ e^{2\phi}=\frac{1}{z^2}.$$

Hyperbolic plane (H_2)
= Time slice of AdS3

$$ds^2=(dx^2+dz^2)/z^2.$$

A Sketch: Optimization of Path-Integral



(2-3) Thermofield Double of 2D CFT

The TFD state at T=1/β is described as the path-integral

$$\Psi_g[\Phi_1(x),\Phi_2(x)]$$

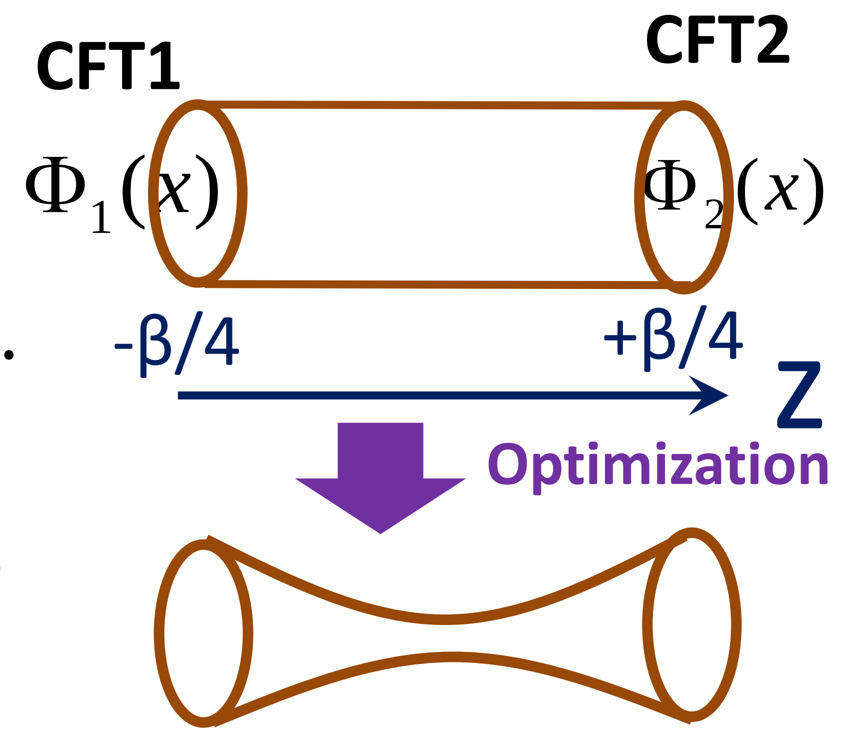
$$=\int \prod_{\substack{-\beta/4<z<\beta/4 \\ -\infty<x<\infty}} D\Phi(x,z)\, e^{-S_{CFT}(\Phi)} \cdot \delta\left(\Phi_1(x)-\Phi(x,z=\frac{\beta}{4})\right)\delta\left(\Phi_2(x)-\Phi(x,z=-\frac{\beta}{4})\right) \propto e^{S_L[\phi]}.$$

Minimization of $S_L[\phi(x,z)]$

$$\Rightarrow \quad e^{2\phi(z)} = \frac{4\pi^2}{\beta^2}\cdot\frac{1}{\cos^2(2\pi z/\beta)}.$$

= Time slice of BTZ black hole.

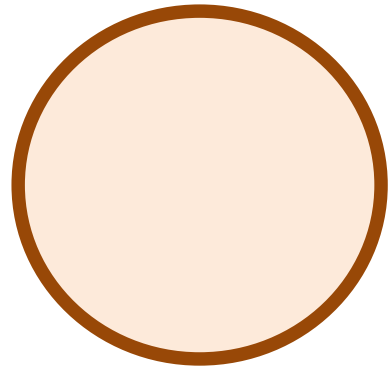
(i.e. Einstein-Rosen Bridge).



(2-4) Primary States and Back-reactions

Vacuum state on a circle

We optimize the path-integral on a disk with the unit radius.



The solution of Liouville equation

$$ds^2 = \frac{4dwd\overline{w}}{(1-|w|^2)^2}.$$

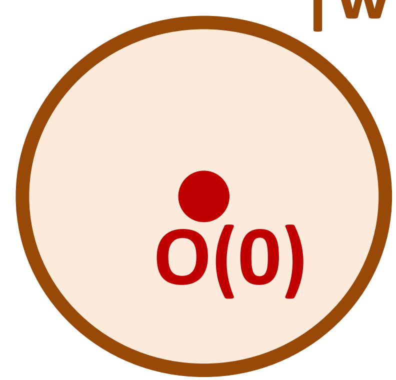
= Hyperbolic Disk (=Time slice of Global AdS3)

Primary state on a circle

We insert an operator $O(w,\overline{w})$ at w=0.

It has conformal dim. $h_L=h_R=h$.

$$\Rightarrow O(x) \sim e^{-2h\cdot\phi}.$$



Thus we minimize $\Psi_{g=e^{2\phi}} / \Psi_{g=1} \propto e^{S_L[\phi]} \cdot e^{-2h\phi(0)}$.

$$\Rightarrow \quad \boxed{\partial_w \partial_{\bar{w}} \phi \frac{1}{4} e^{2\phi} + \frac{6\pi h}{c} \delta^2(w) = 0.}$$

Solution: $ds^2 = \dfrac{4 d\zeta d\bar{\zeta}}{(1-|\zeta|^2)^2}, \qquad \zeta \equiv w^a = re^{i\theta}$

$\Rightarrow$ Deficit angle: $\theta \sim \theta + 2\pi a. \quad (a \equiv 1 - 12h/c)$.

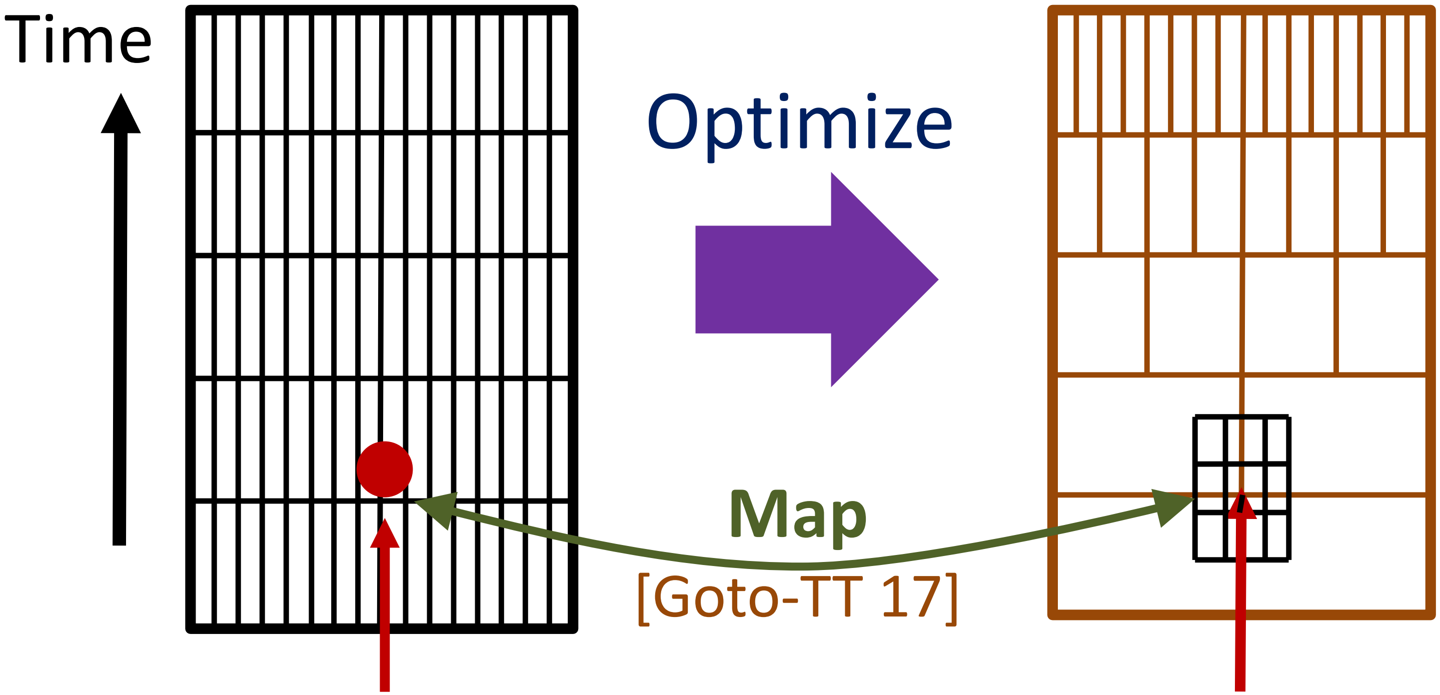
Note: the AdS/CFT predicts $a = \sqrt{1 - 24h/c}$.

Interestingly, if we consider **the quantum Liouville CFT**,

then $\quad h = \dfrac{\gamma\alpha}{4}(Q - \alpha\gamma/2.), \quad c = 1 + 3Q^2, \quad (Q \equiv 2/\gamma + \gamma)$.

$\Rightarrow$ We get $a = \sqrt{1 - 24h/c}$.

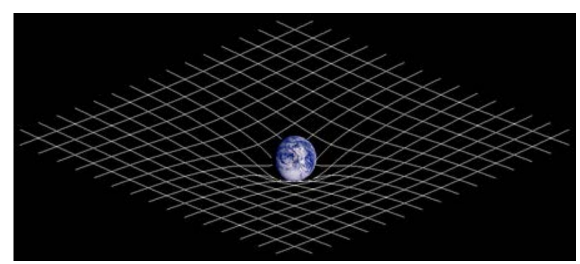
Heuristic Summary



Local excitation (energy source)

A fine graining is needed
⇒ The metric gets larger !

This provides the back-reaction mechanism as in general relativity !



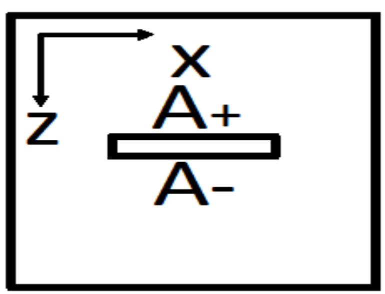
③ Entanglement Wedge and Entropy

(3-1) Entanglement Wedge from Optimization

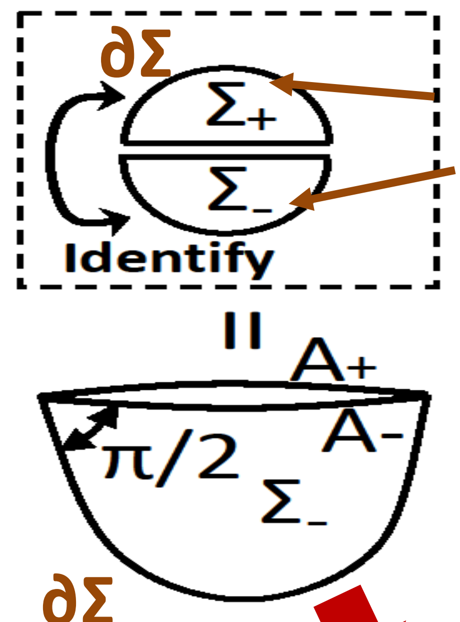
Consider an optimization of reduced density matrix ρ_A.

We decompose the geometry into two halves. A=an interval

$\rho_A =$



Optimize (Squeezing)



Boundary ∂Σ Satisfies K=0 ⇒ Geodesic

$$S_L = 2\times\frac{c}{24\pi}\int_\Sigma dxdz\left[(\partial_x\phi)^2+(\partial_z\phi)^2+e^{2\phi}\right]$$

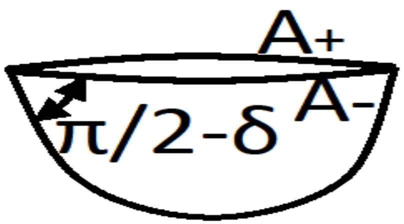
$$+2\times\frac{c}{12\pi}\int_{\partial\Sigma} ds\left[K_0\cdot\phi\right].$$

$$\Rightarrow \text{Bdy condition}: e^{\phi}K = K_0 + \partial_n\phi = 0.$$

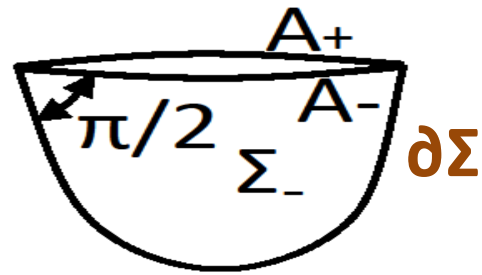
Entanglement wedge (EW) !

(3-2) Hol. Ent. Entropy from Optimization

$$\rho_A^n =$$



Deficit angle deformation



$$\delta = \pi(1-n)$$

Replica Method

$$\mu_B = \pi(1-n)$$

$$S_L^{(n)}[\phi] = \frac{c}{6\pi}\int_\Sigma dxdz\left[(\partial_x\phi)^2 + (\partial_z\phi)^2 + e^{2\phi}\right] + \frac{c}{3\pi}\int_{\partial\Sigma} ds\left[K\phi + \mu_B e^{\phi}\right].$$

$$S_A = -\frac{\partial}{\partial n} S_L^{(n)}\bigg|_{n=1} = \frac{c}{6}\int_{\partial\Sigma} ds\, e^{\phi}.$$

Reproduce the correct HEE !

④ NAdS2/CFT1 (SYK model)

At first sight, one may be confused as the action

$$ds_{1d}^2 = e^{2\phi} dz^2, \qquad S_{1d} = \mu \int dz\, e^{\phi} \sim O(\varepsilon^{-1}).$$

gets trivial when minimized.

⇒We need to add the conformal sym. breaking effect !

[Sachdev-Ye-Kitaev model, Polchinski-Rosenhaus 16, Maldacena-Stanford 16]

$$\Psi_{\phi} = e^{cS_{1d}} \cdot \Psi_{\phi=0}, \qquad S_{1d} = \int dz[(\partial_z \phi)^2 + \mu\, e^{\phi}].$$

↑ **Schwarz Derivative term**

⇒ $$ds_{1d}^2 = \frac{dz^2}{z^2}.$$

⑤ Higher Dimensional CFTs

For simplicity, we focus on the optimization of Weyl rescaling mode: $ds^2 = e^{2\phi(x,z)}\left(dz^2 + d\vec{x}^2\right)$

We argue that the complexity functional is given by

$$I_d[\phi] = N\int dz dx^{d-1}\left[e^{d\phi} + e^{(d-2)\phi}\left(\partial_z\phi\right)^2 + e^{(d-2)\phi}\left(\partial_x\phi\right)^2 + \frac{e^{(d-2)\phi}\cdot R_0}{(d-1)(d-2)}\right]$$

$$+2N\int_{bdy} dx^{d-1}\left[\frac{e^{(d-2)\phi}\cdot K_0}{(d-1)(d-2)} + \frac{\mu_B}{d-1}e^{(d-1)\phi}\right]. \qquad \left(N \equiv \frac{(d-1)R_{AdS}^{d-1}}{16\pi G_N}\right).$$

$$\text{Note:} \quad \lim_{d\to 2}\left[I_d[\phi] - I_d[0]\right] = S_L[\phi] - S_L[0]$$

Properties

- Time slices of pure AdS_{d+1} are reproduced for CFT vacua.
- When A= a round ball, EW and HEE are reproduced.
- The first order mass deformation (=AdS BH deformation) of the metric is correctly reproduced.

[cf. AdS3/CFT2, ∃ h/c corrections.]

Evaluations of Path-integral Complexity in various dim.

2d CFT (1) Poincare AdS3: $C=\frac{c}{12\pi}\cdot\frac{L}{\varepsilon}.$

(2) global AdS3: $C=\frac{c}{6}\cdot\left[\frac{1}{\varepsilon}-1\right].$

(3) BTZ(TFD): $C=\frac{c}{3}\left[\frac{1}{\varepsilon}-\frac{\pi^2}{2\beta}\right].$

3d CFT global AdS4: $C=4\pi N\left[\frac{1}{\varepsilon^2}+\frac{1}{2}+\log\left(\frac{2}{\varepsilon}\right)\right].$

4d CFT global AdS5: $C=2\pi^2 N\left[\frac{2}{3\varepsilon^3}+\frac{1}{\varepsilon}-\frac{5}{12}\right].$

→ C ~ Volume law divergence + subleading terms.

[Holographic complexity: Carmi-Myers-Rath 16, Reynolds-Ross 16]

A Connection to ``Complexity = Action'' proposal

Consider a Pure AdSd+1 and Pick up the following patch:

$$ds^2 = R_{AdS}^2\left(-dt^2 + \cos^2 t \cdot e^{2\phi(x)} \cdot h_{ab}dx^a dx^b\right).$$

This agrees with Wheeler-DeWitt patch

in [Brown-Roberts-Susskind-Swingle-Zhao 15]

if $e^{2\phi(x)}h_{ab}dx^a dx^b$ is given by Hd.

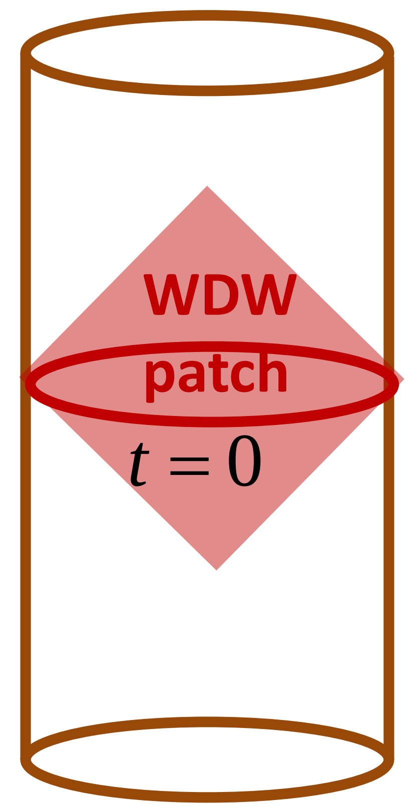
In this setup, we can show

$$I_{WDW} = \frac{1}{16\pi G_N}\int_{WDW} dx^d dt \sqrt{-g}\left[R - 2\Lambda\right] + (\text{bdy. term})$$

$$= (d-2)\cdot n_d \cdot I_d[\phi] + (\text{IR surface term}).$$

Our complexity

$$\left(n_d \equiv \frac{\sqrt{\pi}\,\Gamma\left((d-1)/2\right)}{\Gamma\left(d/2\right)}\right)$$



⑥ Conclusions

- We introduced ``path-integral complexity'' of a given state, which measures complexity of the corresponding TN.

- In 2D CFT, it is given by the Liouville action, which is supported by the Weyl anomaly and the TNR complexity.

  An optimization of path-integral of a CFT state

  = Minimizing the complexity

  ⇔ a time slice of its gravity dual in AdS/CFT

- We gave generalizations to higher dim. and CFT1.

Future Problems

Time dependent b.g. (gtt: covariant formulation) ?

Sub AdS locality ? , dS/CFT version ?, ...

Thank you very much !

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