Review talk: What's up with the SYK model?

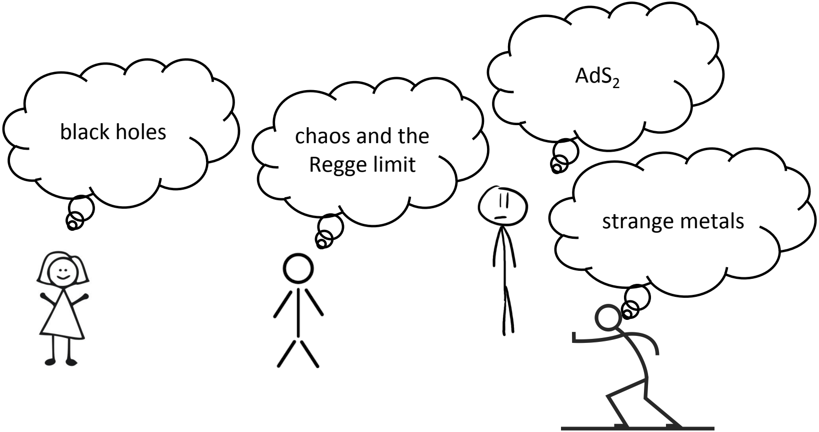
Douglas Stanford

IAS

June 26, 2017

The SYK model is a strongly interacting quantum system that is solvable at large N.

The SYK model is a strongly interacting quantum system that is solvable at large N.



Plan of this talk

1. The SYK model and its large N solution
2. Connection to AdS_2 and assorted comments
3. Generalizations of the SYK model

Introduction to the SYK model

The Sachdev-Ye-Kitaev model

Majorana fermions in QM are matrices ψ_a satisfying

$$\{\psi_a, \psi_b\} = \delta_{ab}, \qquad a, b = 1, ..., N$$

The Sachdev-Ye-Kitaev model

Majorana fermions in QM are matrices ψ_a satisfying

$$\{\psi_a, \psi_b\} = \delta_{ab}, \qquad a, b = 1, ..., N$$

A general Hamiltonian would be

$$H_{\text{general}} = i m_{ab} \psi_a \psi_b + j_{abcd} \psi_a \psi_b \psi_c \psi_d + \cdots$$

The Sachdev-Ye-Kitaev model

Majorana fermions in QM are matrices ψ_a satisfying

$$\{\psi_a, \psi_b\} = \delta_{ab}, \qquad a, b = 1, ..., N$$

A general Hamiltonian would be

$$H_{\text{general}} = im_{ab}\psi_a\psi_b + j_{abcd}\psi_a\psi_b\psi_c\psi_d + \cdots$$

The SYK Hamiltonian is

$$H_{SYK_4} = j_{abcd}\,\psi_a\psi_b\psi_c\psi_d \qquad\qquad \langle j_{abcd}^2\rangle = \frac{J^2}{N^3}$$

The Sachdev-Ye-Kitaev model

Majorana fermions in QM are matrices ψ_a satisfying

$$\{\psi_a, \psi_b\} = \delta_{ab}, \qquad a, b = 1, ..., N$$

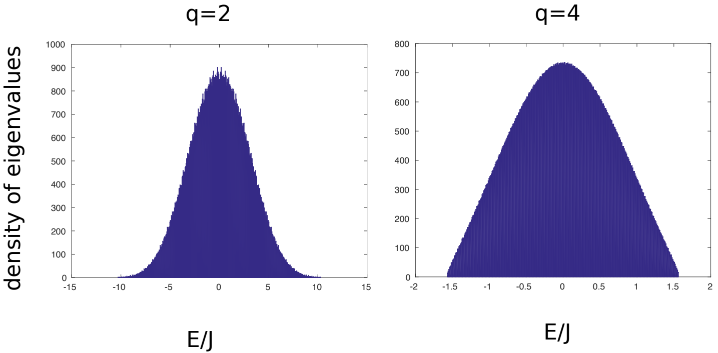
A general Hamiltonian would be

$$H_{\text{general}} = i m_{ab} \psi_a \psi_b + j_{abcd} \psi_a \psi_b \psi_c \psi_d + \cdots$$

The SYK Hamiltonian is

$$H_{SYK_4} = j_{abcd}\, \psi_a \psi_b \psi_c \psi_d \qquad\qquad \langle j_{abcd}^2 \rangle = \frac{J^2}{N^3}$$

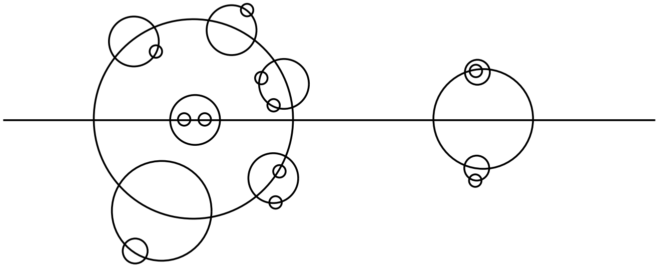
- Dimensionless coupling is βJ. Interesting behavior at $\beta J \gg 1$.
- Can also consider a version with fermions interacting in groups of q, instead of four. $q \to \infty$ and $q \to 2$ are simpler limits.
- System “self-averages” provided $q > 2$.



One realization of disorder, $N = 34$ fermions.

Feynman diagrams

Typical diagram for $G(\tau) = \langle \psi_a(\tau)\psi_a(0) \rangle$ at large N:



Self-consistency equation for sum of diagrams:

$$G(\omega) = \frac{1}{-i\omega - \Sigma(\omega)}, \qquad \Sigma(\tau) = J^2 G(\tau)^3.$$

[Kitaev]

IR equations

In the IR limit $\tau J \gg 1$, drop the "$-i\omega$" to simplify

$$G(\omega) = \frac{1}{-i\omega - \Sigma(\omega)} \approx \frac{1}{-\Sigma(\omega)}, \qquad \Sigma(\tau) = J^2 G(\tau)^{q-1}.$$

Exact solution to IR equations on the line:

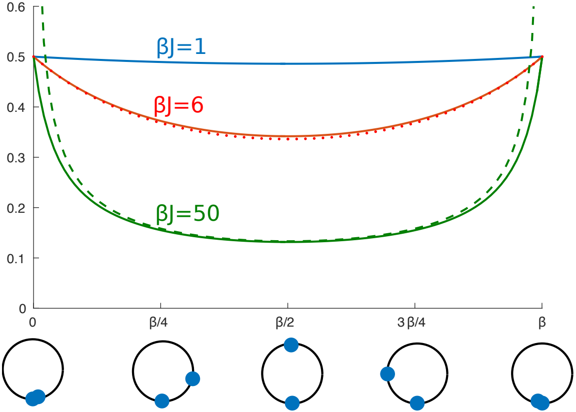
$$G(\tau) \propto \frac{\mathrm{sgn}(\tau)}{|\tau|^{2\Delta}}, \qquad \Delta = \frac{1}{q},$$

and on the circle (finite temp):

$$G(\tau) \propto \frac{\mathrm{sgn}(\tau)}{\sin^{2\Delta}(\frac{\pi\tau}{\beta})}.$$

[Sachdev,Ye][Parcollet,Georges]
$SL(2,R)$ covariant under $x \equiv \tan\frac{\pi\tau}{\beta} \to \frac{ax+b}{cx+d}$.

Plots of $G(\tau) = \langle \psi(\tau)\psi(0)\rangle_\beta$

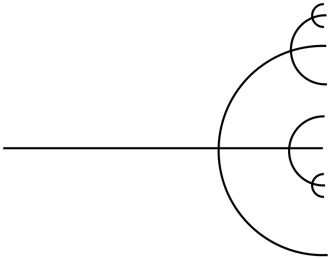


The decay of the two point function

In real time, we have

$$G(t) \propto \frac{1}{\sinh^{2\Delta} \frac{\pi t}{\beta}}$$

which gives exponential decay. What is happening is ψ is leaking into the space of more complicated operators, $\psi \to \psi\psi\psi\psi$...



Systematic approach to SYK at large N

The large N action

The path integral for fixed disorder is

$$Z(\beta) = \int D\psi e^{-\int_0^\beta i\dot{\psi}(\tau)\psi(\tau) + j_{abcd}\psi_a(\tau)\psi_b(\tau)\psi_c(\tau)\psi_d(\tau)}.$$

Averaging over j_{abcd} with Gaussian measure gives nonlocal-in-time theory. Can introduce new fields G, Σ to simplify. Σ is a Lagrange multiplier that sets $G(\tau_1, \tau_2) = \frac{1}{N}\sum_a \psi_a(\tau_1)\psi_a(\tau_2)$.

The large N action

The path integral for fixed disorder is

$$Z(\beta) = \int D\psi e^{-\int_0^\beta i\dot{\psi}(\tau)\psi(\tau) + j_{abcd}\psi_a(\tau)\psi_b(\tau)\psi_c(\tau)\psi_d(\tau)}.$$

Averaging over j_{abcd} with Gaussian measure gives nonlocal-in-time theory. Can introduce new fields G, Σ to simplify. Σ is a Lagrange multiplier that sets $G(\tau_1, \tau_2) = \frac{1}{N}\sum_a \psi_a(\tau_1)\psi_a(\tau_2)$. After integrating out the fermions,

$$\langle Z(\beta)\rangle_J = \int DG\, D\Sigma\, e^{-N\, I(G,\Sigma)}$$

$$I(G,\Sigma) = -\frac{1}{2}\log\det(\partial_\tau - \Sigma) + \frac{1}{2}\int_0^\beta d\tau_1 d\tau_2 \Big[\Sigma(\tau_1,\tau_2)G(\tau_1,\tau_2) - \frac{J^2}{q}G(\tau_1,\tau_2)^q\Big]$$

The large N action

The path integral for fixed disorder is

$$Z(\beta) = \int D\psi e^{-\int_0^\beta i\dot{\psi}(\tau)\psi(\tau)+j_{abcd}\psi_a(\tau)\psi_b(\tau)\psi_c(\tau)\psi_d(\tau)}.$$

Averaging over j_{abcd} with Gaussian measure gives nonlocal-in-time theory. Can introduce new fields G, Σ to simplify. Σ is a Lagrange multiplier that sets $G(\tau_1, \tau_2) = \frac{1}{N}\sum_a \psi_a(\tau_1)\psi_a(\tau_2)$. After integrating out the fermions,

$$\langle Z(\beta)\rangle_J = \int DG\, D\Sigma\, e^{-N\, I(G,\Sigma)}$$

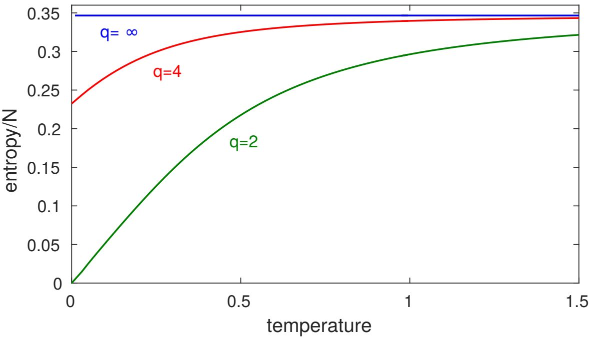
$$\begin{aligned} I(G,\Sigma) = &-\frac{1}{2}\log\det(\partial_\tau - \Sigma) \\ &+ \frac{1}{2}\int_0^\beta d\tau_1 d\tau_2 \Big[\Sigma(\tau_1,\tau_2)G(\tau_1,\tau_2) - \frac{J^2}{q}G(\tau_1,\tau_2)^q\Big] \end{aligned}$$

Saddle point eqs: $G = [\partial_\tau - \Sigma]^{-1}$, $\Sigma(\tau_1,\tau_2) = J^2 G(\tau_1,\tau_2)^{q-1}$.

[Parcollet,Georges,Sachdev][Kitaev]

The large N action: entropy

To get large N thermodynamics, plug G_*, Σ_* back into the action, $Z(\beta) \approx e^{-N\, I(G_*, \Sigma_*)}$.



$$\rho(E) \propto e^S$$

[Parcollet,Georges,Sachdev].

Procedurally similar to how we compute entropy using gravity.

The large N action: emergent conformal symmetry

In the IR limit, we drop the ∂_τ term in the effective action, so it is

$$I = -\frac{1}{2}\log\det(\Sigma) + \frac{1}{2}\int_0^\beta d\tau_1 d\tau_2 \Big[\Sigma(\tau_1,\tau_2)G(\tau_1,\tau_2) - \frac{J^2}{q}G(\tau_1,\tau_2)^q\Big].$$

This is reparametrization invariant, under [Kitaev]

$$G(\tau_1,\tau_2) \to \big(\phi'(\tau_1)\phi'(\tau_2)\big)^{1/q}\, G\big(\phi(\tau_1),\phi(\tau_2)\big)$$
$$\Sigma(\tau_1,\tau_2) \to \big(\phi'(\tau_1)\phi'(\tau_2)\big)^{1-1/q}\, \Sigma\big(\phi(\tau_1),\phi(\tau_2)\big).$$

So in the strict IR limit, the theory has $\mathit{diff}(S^1)$ symmetry.

The large N action: emergent conformal symmetry

In the IR limit, we drop the ∂_τ term in the effective action, so it is

$$I = -\frac{1}{2}\log\det(\Sigma) + \frac{1}{2}\int_0^\beta d\tau_1 d\tau_2 \Big[\Sigma(\tau_1,\tau_2)G(\tau_1,\tau_2) - \frac{J^2}{q}G(\tau_1,\tau_2)^q\Big].$$

This is reparametrization invariant, under [Kitaev]

$$\begin{aligned} G(\tau_1,\tau_2) &\to \big(\phi'(\tau_1)\phi'(\tau_2)\big)^{1/q}\, G\big(\phi(\tau_1),\phi(\tau_2)\big) \\ \Sigma(\tau_1,\tau_2) &\to \big(\phi'(\tau_1)\phi'(\tau_2)\big)^{1-1/q}\, \Sigma\big(\phi(\tau_1),\phi(\tau_2)\big). \end{aligned}$$

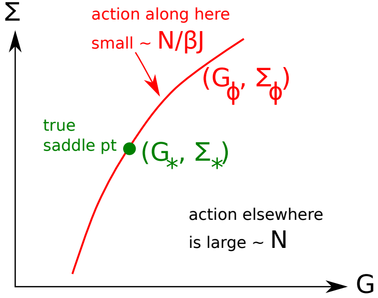
So in the strict IR limit, the theory has $diff(S^1)$ symmetry. But our solution $(\sin\frac{\pi\tau}{\beta})^{-2\Delta}$ only has $SL(2,R)$. Expanding about this saddle, we expect Nambu-Goldstone bosons living in the space

$$\text{space of NG bosons} = \frac{\text{full group}}{\text{preserved subgroup}} = \frac{diff(S^1)}{SL(2,R)}$$

Integration over these zero modes leads to divergences.

The large N action: integration space

Beyond the strict IR limit, the zero modes get lifted slightly

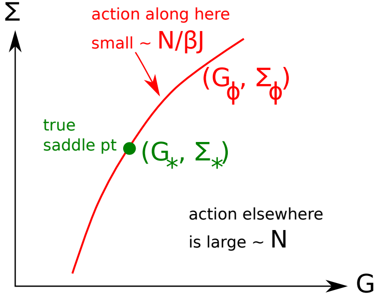


The soft directions are parametrized by $\phi \in \text{diff}(S^1)/SL(2,R)$

$$G_\phi \equiv \big(\phi'(\tau_1)\phi'(\tau_2)\big)^\Delta \, G_*\big(\phi(\tau_1),\phi(\tau_2)\big).$$

The large N action: integration space

Beyond the strict IR limit, the zero modes get lifted slightly



The soft directions are parametrized by $\phi \in \mathrm{diff}(S^1)/SL(2,R)$

$$G_\phi \equiv \left(\phi'(\tau_1)\phi'(\tau_2)\right)^\Delta G_*\left(\phi(\tau_1), \phi(\tau_2)\right).$$

$SL(2,R)$ acts as $x \equiv \tan\frac{\phi}{2} \to \frac{ax+b}{cx+d}$. EFT suggests the action [Kitaev][Maldacena, DS]:

$$I_{Sch} = -\frac{N\alpha}{J}\int_0^\beta d\tau\, \mathrm{Sch}(\tan\phi/2, \tau), \qquad \mathrm{Sch}(x,\tau) \equiv \left(\frac{x''}{x'}\right)' - \frac{1}{2}\frac{x''^2}{x'^2}.$$

The large N action: mini summary

1. Can rewrite SYK in terms of bilocal fields G, Σ

$$\langle Z \rangle_J = \int DG\, D\Sigma\, e^{-N\, I(G,\Sigma)}$$

2. In IR, $I(G, \Sigma)$ has spontaneously broken conformal symmetry. Dominant fluctuations are reparametrizations of the saddle

$$G_\phi \equiv \left(\phi'(\tau_1)\phi'(\tau_2)\right)^\Delta G_*\left(\phi(\tau_1), \phi(\tau_2)\right).$$

3. Leading action for ϕ is the "Schwarzian theory"

$$I_{Sch} = -\frac{N\alpha}{J}\int_0^\beta d\tau\, \mathrm{Sch}(\tan\phi/2, \tau) = \frac{N\alpha}{2J}\int_0^\beta \left(\frac{\phi''^2}{\phi'^2} - \phi'^2\right),$$

breaks the physical conformal symmetry.

Four comments on the relation to AdS_2 and other things

(1) Nearly AdS_2 gravity

A simple theory of 2d gravity described by $g_{\mu\nu}$:

$$I = -\frac{f_0}{G}\left[\int_{bulk}\sqrt{g}R + 2\int_{bdy}K\right]$$

(1) Nearly AdS_2 gravity

A too-simple theory of gravity in AdS_2 described by $g_{\mu\nu}$:

$$I = -\frac{f_0}{G}\overbrace{\left[\int_{bulk}\sqrt{g}R + 2\int_{bdy}K\right]}^{2\pi} = -S_0$$

(1) Nearly AdS_2 gravity

A simple theory of gravity in AdS_2 described by $(g_{\mu\nu}, f)$:

$$I_{JT} = -S_0 - \frac{1}{G}\left[\int_{bulk} \sqrt{g}\,(R+2)\,f + 2\int_{bdy} f\,K\right]$$

[Teitelboim][Jackiw][Almheiri,Polchinski]

(1) Nearly AdS_2 gravity

A simple theory of gravity in AdS_2 described by $(g_{\mu\nu}, f)$:

$$I_{JT} = -S_0 - \frac{1}{G}\left[\int_{bulk} \sqrt{g}\,(R+2)\,f + 2\int_{bdy} f\,K\right]$$

[Teitelboim][Jackiw][Almheiri,Polchinski]

Reduces to the Schwarzian theory!

(1) Nearly AdS_2 gravity

A simple theory of gravity in AdS_2 described by $(g_{\mu\nu}, f)$:

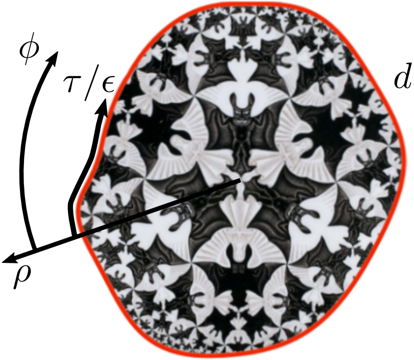
$$I_{JT} = -S_0 - \frac{1}{G}\left[\int_{bulk} \sqrt{g}\,(R+2)\,f + 2\int_{bdy} f\,K\right]$$

[Teitelboim][Jackiw][Almheiri,Polchinski]

Reduces to the Schwarzian theory!

Step 1: integral over f implies $R+2=0$. Step 2: integral over metrics then reduces to cut-outs from hyperbolic disk.

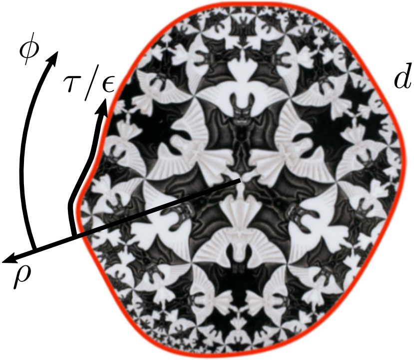
(1) Nearly AdS_2 gravity



$$ds^2 = d\rho^2 + \rho^2 d\phi^2$$

total length = β/ϵ

(1) Nearly AdS_2 gravity



$$ds^2 = d\rho^2 + \rho^2 d\phi^2$$

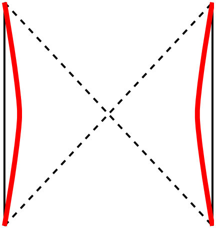
total length = β/ϵ

$$I_{JT} = -S_0 - \frac{2}{G}\int_{bdy} f\, K \quad \longrightarrow \quad -S_0 - \frac{2f_r}{G}\int_0^\beta d\tau\, \mathrm{Sch}(\tan\phi/2, \tau)$$

[Maldacena, DS, Z. Yang] see also [Jensen][Engelsoy, Mertens, Verlinde]

(1) Nearly AdS_2 gravity

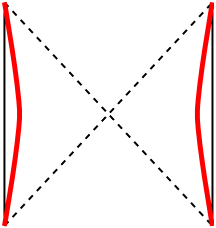
Euclidean



Lorentzian

(1) Nearly AdS_2 gravity

Euclidean



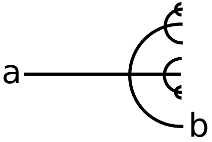
Lorentzian

SYK is a "QM completion" of the JT black hole.

(2) Chaos and the Schwarzian theory

Chaos can be diagnosed using e.g.

$$\langle \{\psi_a(0), \psi_b(t)\}^2 \rangle \propto \frac{1}{N} e^{\lambda_L t}$$

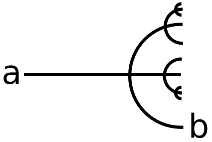


SYK saturates the bound $\lambda_L \leq \frac{2\pi}{\beta}$.

(2) Chaos and the Schwarzian theory

Chaos can be diagnosed using e.g.

$$\langle \{\psi_a(0), \psi_b(t)\}^2 \rangle \propto \frac{1}{N} e^{\lambda_L t}$$



SYK saturates the bound $\lambda_L \leq \frac{2\pi}{\beta}$.

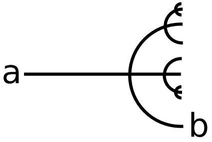
Easy to see in variables of Schwarzian theory. Expand $\phi(\tau) = \tau + \epsilon(\tau)$. Then have solutions

$$\epsilon(\tau) = 1, \tau, e^{\frac{2\pi}{\beta} i\tau}, e^{-\frac{2\pi}{\beta} i\tau} \qquad \Longrightarrow \qquad \epsilon(t) \propto e^{\frac{2\pi}{\beta} t}.$$

(2) Chaos and the Schwarzian theory

Chaos can be diagnosed using e.g.

$$\langle \{\psi_a(0), \psi_b(t)\}^2 \rangle \propto \frac{1}{N} e^{\lambda_L t}$$



SYK saturates the bound $\lambda_L \leq \frac{2\pi}{\beta}$.

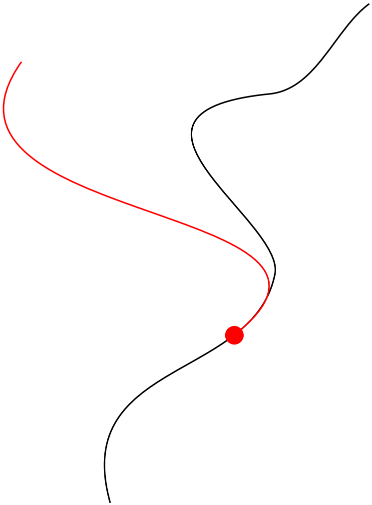
Easy to see in variables of Schwarzian theory. Expand $\phi(\tau) = \tau + \epsilon(\tau)$. Then have solutions

$$\epsilon(\tau) = 1, \tau, e^{\frac{2\pi}{\beta} i\tau}, e^{-\frac{2\pi}{\beta} i\tau} \qquad \Longrightarrow \qquad \epsilon(t) \propto e^{\frac{2\pi}{\beta} t}.$$

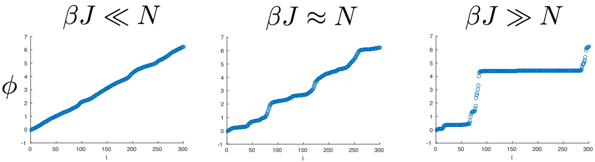
Linearized $SL(2,R)$ gauge transformations are

$$\delta\epsilon(\tau) = 1, e^{\frac{2\pi}{\beta} i\tau}, e^{-\frac{2\pi}{\beta} i\tau}.$$

(2) Chaos and the Schwarzian theory

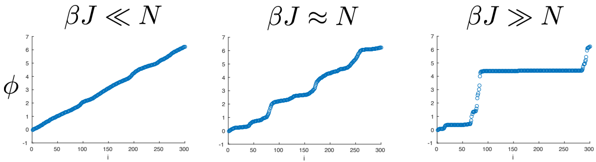


(3) Very low temperatures



For $\beta J \gg N$, large fluctuations in ϕ.

(3) Very low temperatures



For $\beta J \gg N$, large fluctuations in ϕ. However, Z_{Sch} turns out to be one-loop exact,

$$Z_{Sch}(\beta) = \int \frac{d\mu[\phi]}{SL(2,R)} e^{\frac{N\alpha}{J}\int_0^\beta d\tau \mathrm{Sch}(\tan\phi/2,\tau)} = \frac{\#}{(\beta J)^{3/2}} e^{\frac{2\pi^2 N\alpha}{\beta J}}.$$

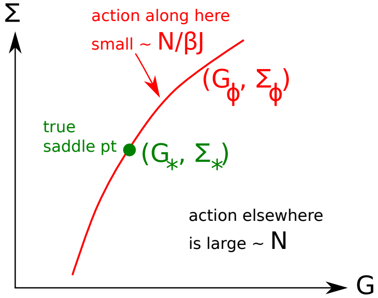
[CGHPSSSST][DS,Witten][Bagrets,Altland,Kamenev][Z. Yang][Mertens,Turiaci,Verlinde]

Gives us control of low-energy density of states:

$$Z_{Sch}(\beta) = \int_{E_0}^{\infty} dE \rho(E) e^{-\beta E}, \qquad \rho(E) \propto \sinh\sqrt{C(E-E_0)}.$$

(4) Massive modes

Other directions in G, Σ space correspond to roughly integer-spaced spectrum of massive modes propagating in AdS_2.



Interactions between modes will be important for sorting out bulk theory! Requires higher point functions of fermions [Gross,Rosenhaus].

Generalizations of SYK

Generalizations of SYK

- global symmetry (e.g. complex fermions) [Sachdev][Davison,Fu,Georges,Gu,Jensen,Sachdev]
- more flavors [Gross,Rosenhaus]
- additional quadratic fermions [Banerjee,Altman] [Chen,Fan,Chen,Zhai,Zhang]
- lattices of SYK [Gu,Qi,DS][Song,C.M.Jian,Balents][S.K.Jian,Yao]
- supersymmetry [Fu, Gaiotto, Maldacena, Sachdev]
- models without disorder [Witten][Klebanov,Tarnopolsky][Gurau][Peng, Spradlin,Volovich] [Ferrari][Peng]
- higher d field theory models [Turiaci, Verlinde][Berkooz,Narayan,Rozali,Simon][Murugan,DS,Witten]

Supersymmetry

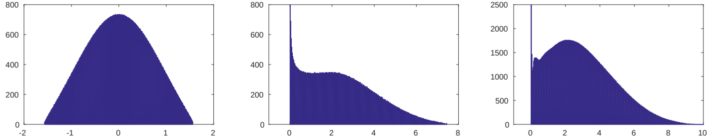
[Fu, Gaiotto, Maldacena, Sachdev]

$\mathcal{N} = 1$ version, using Majorana fermions

$$H = Q^2, \qquad Q = iC_{abc}\psi_a\psi_b\psi_c$$

$\mathcal{N} = 2$ version, using complex fermions

$$H = \{Q, \overline{Q}\}, \qquad Q = iC_{abc}\psi_a\psi_b\psi_c, \qquad \overline{Q} = iC^*_{abc}\overline{\psi}_a\overline{\psi}_b\overline{\psi}_c$$



Low-energy effective theory is $\mathcal{N} = 1$ or $\mathcal{N} = 2$ super-Schwarzian.

Disorder j_{abcd} is unfamiliar for holography

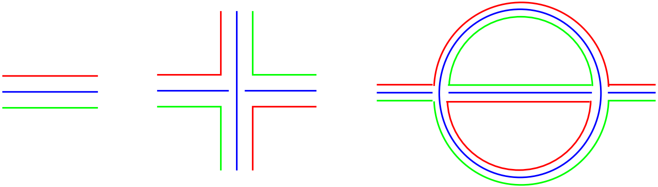
- No global symmetry so no singlet condition to impose.
- What is the bulk interpretation of the different j_{abcd}?

Models without disorder!

[Witten][Gurau][Klebanov,Tarnopolsky]

One version: organize N fermions into a tensor χ_{abc} where $a = 1, ..., n$ and $N = n^3$. The Hamiltonian is

$$H = g\, \chi_{a_1 b_1 c_1} \chi_{a_1 b_2 c_2} \chi_{a_2 b_1 c_2} \chi_{a_2 b_2 c_1}$$



- Same as SYK at order one and order $1/N$.
- $O(n)^3$ symmetry. Can gauge, consider only singlet operators. (Their number grows very rapidly with energy.)

Higher dimensions

How to generalize to continuum models in higher d?

1. Try with fermions: [Turiaci, Verlinde][Berkooz,Narayan,Rozali,Simon]

$$I = \int d^2x \left[\psi_a \bar{\partial} \psi_a + \bar{\psi}_a \partial \bar{\psi}_a + J_{ab;cd} \psi_a \psi_b \bar{\psi}_c \bar{\psi}_d\right]$$

2. Try with bosons: [Klebanov,Tarnopolsky][Murugan,DS,Witten]

$$I = \int d^2x \left[\partial \phi_a \bar{\partial} \phi_a + J_{abcd} \phi_a \phi_b \phi_c \phi_d\right]$$

3. Try with superfields, $(1,1)$ supersymmetry [Murugan,DS,Witten]

$$I = \int d^2x d^2\theta \left[D_\theta \Phi_a D_{\bar{\theta}} \Phi_a + C_{abc} \Phi_a \Phi_b \Phi_c\right]$$

$$\supset \int d^2x \; C_{abc} \phi_a \psi_b \bar{\psi}_c + C_{abc} C_{ab'c'} \phi_b \phi_c \phi_{b'} \phi_{c'}$$

Twist of spin 4: $E - J \approx 0.29$. Chaos exp.: $\lambda_L \approx 0.58 \times \frac{2\pi}{\beta}$.

A puzzle!

What are the corrections to a large N theory that tell us the spectrum is discrete at finite N?

- For large $|\beta|$, good approximation to Z just from Schwarzian:

$$Z_{Sch}(\beta) = \frac{\#}{\beta^{3/2}} e^{C/\beta}.$$

- In QM $Z(\beta_0 + it)$ should not vanish for large t. What fixes this in the full G, Σ theory?[1]

[1] See talk by Shenker for more precise statement with two replicas.

Summary

- SYK is a solvable but strongly interacting model.
- Low energy theory is Schwarzian $=$ JT gravity in AdS_2.
- Many interesting generalizations, puzzles remain!

Higher dimensions

These flow to a CFT at large N. Sketch of four point function:

$$\langle 4pt\rangle(\chi, \bar{\chi}) = \frac{1}{N}\sum_J \int_{1+i\mathbb{R}} dE\, C(E, J) G_{E,J}(\chi, \bar{\chi}).$$

$C(E, J) =$ bunch of gamma functions, $G_{E,J} =$ conformal block.

- Can deform E contour to get OPE expansion, defined by poles in $C(E, J)$. Twist of lightest spin 4 op. is $E - J \approx 0.29$.
- Can represent J sum as integral and deform J contour to get Regge/Chaos limit, exponent is $\lambda_L \approx 0.58 \times \frac{2\pi}{\beta}$.

Theory has $O(1)$ interaction strength at large N. Not enough for a local gravity dual.