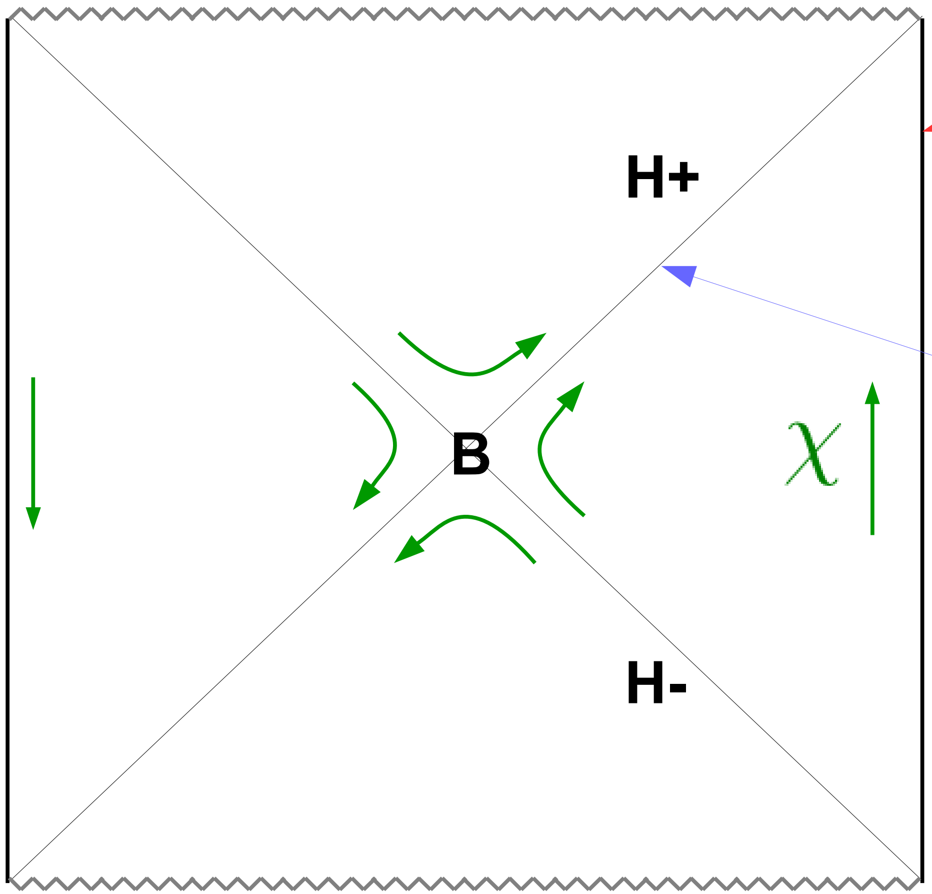
Progress in Horizon Thermodynamics

Aron Wall
DAMTP, Cambridge

Static
Black Hole
thermodynamics

(this part of the talk mostly reviews older material, but there will be some recent results at the end)

Penrose Diagram of an Eternal Static Black Hole



asymptotically AdS boundary conditions, since this is Strings! But most of what I say generalizes to all Killing horizons: dS, Rindler...

area A constant along future and past horizons **H(+/-)** because black hole is stationary

This spacetime has multiple Killing vectors (symmetries) satisfying

$$\nabla_a \chi_b + \nabla_b \chi_a = 0$$

But the most important one is the "horizon generating" χ which is null along **H** (this vector looks like a time translation @ ∞ but like a Lorentz boost near **B**). If we pick χ to be normalized w.r.t. some boundary clock time, then the surface gravity $\kappa = |\nabla_a \chi_b|$ is constant along H (the *Zeroth Law*).

Black Hole Thermodynamics

$$E = \mathrm{ADM\,mass}$$ (already clear in GR)

$$T = \frac{\hbar}{2\pi}\kappa$$ (Hawking radiation)

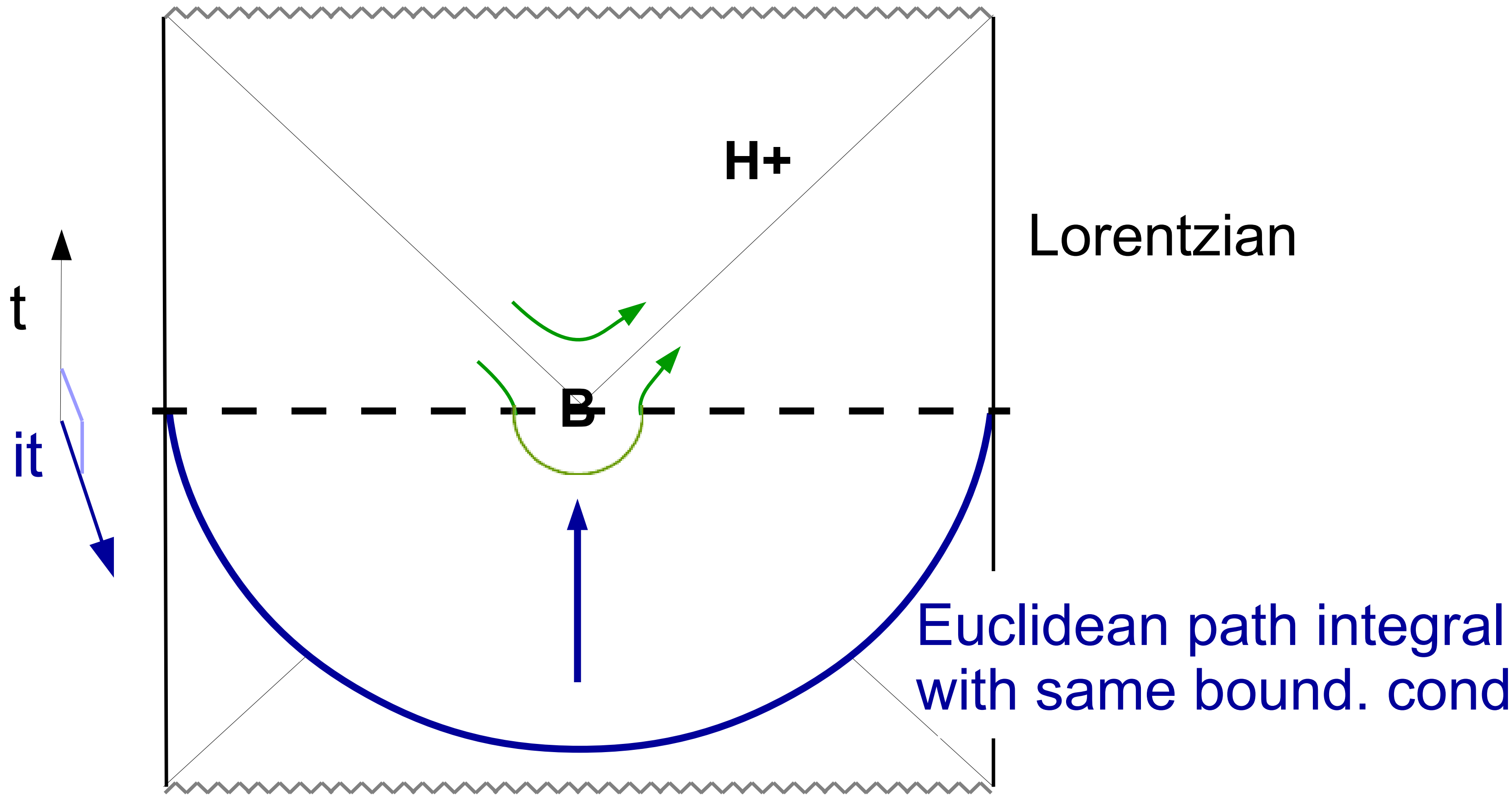
$$S = \frac{A[H]}{4G\hbar}$$ + quantum & string corrections (entropy increases with time)

combination of these quantities satisfy *First Law* (i.e. Clausius relation):

$$dE = T dS$$

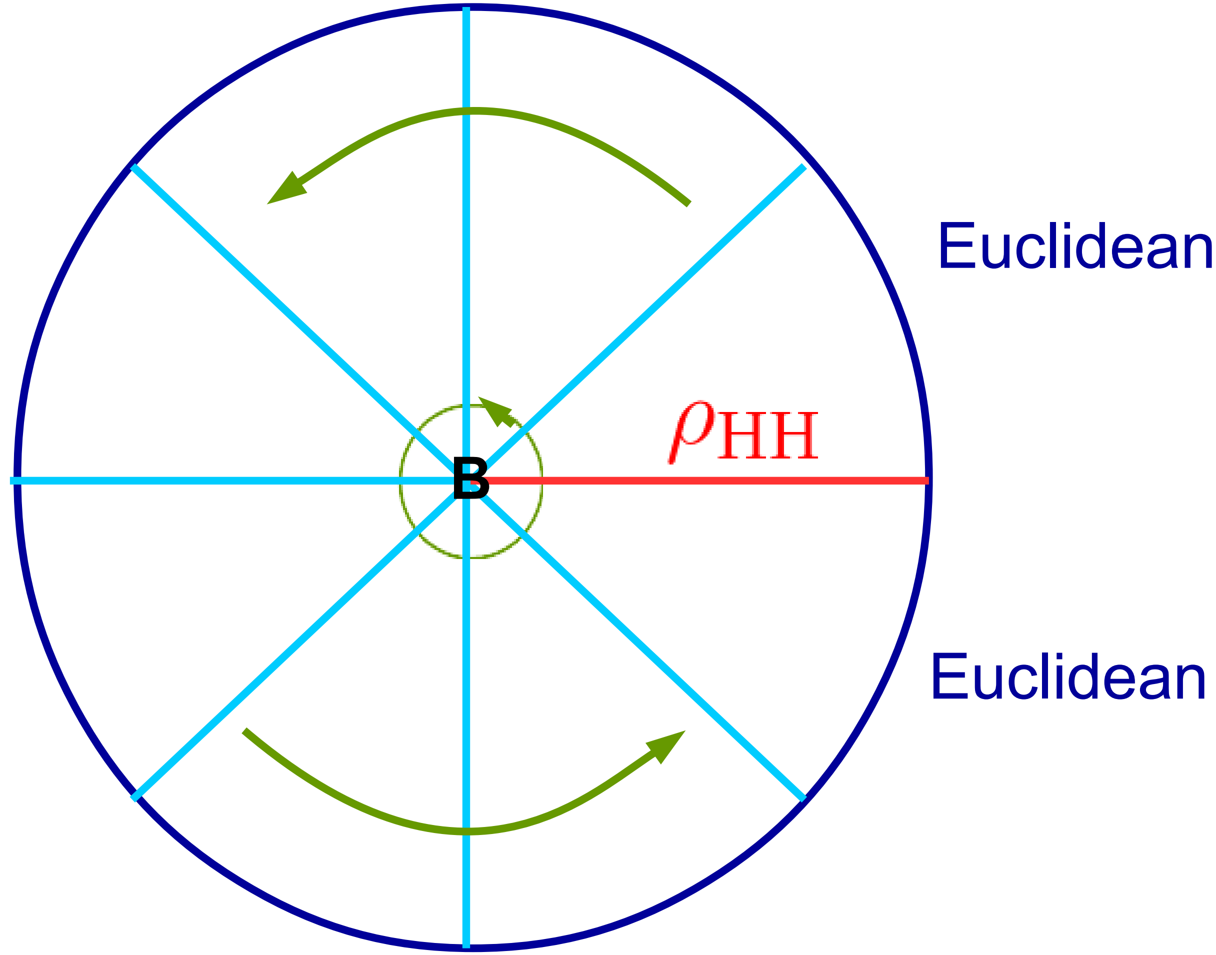
[some extra terms are needed if black hole is rotating or charged]

Hartle-Hawking Path Integral



There is a special "Hartle-Hawking" state of QFT on the BH background obtained by Wick rotating to Euclidean path integral:

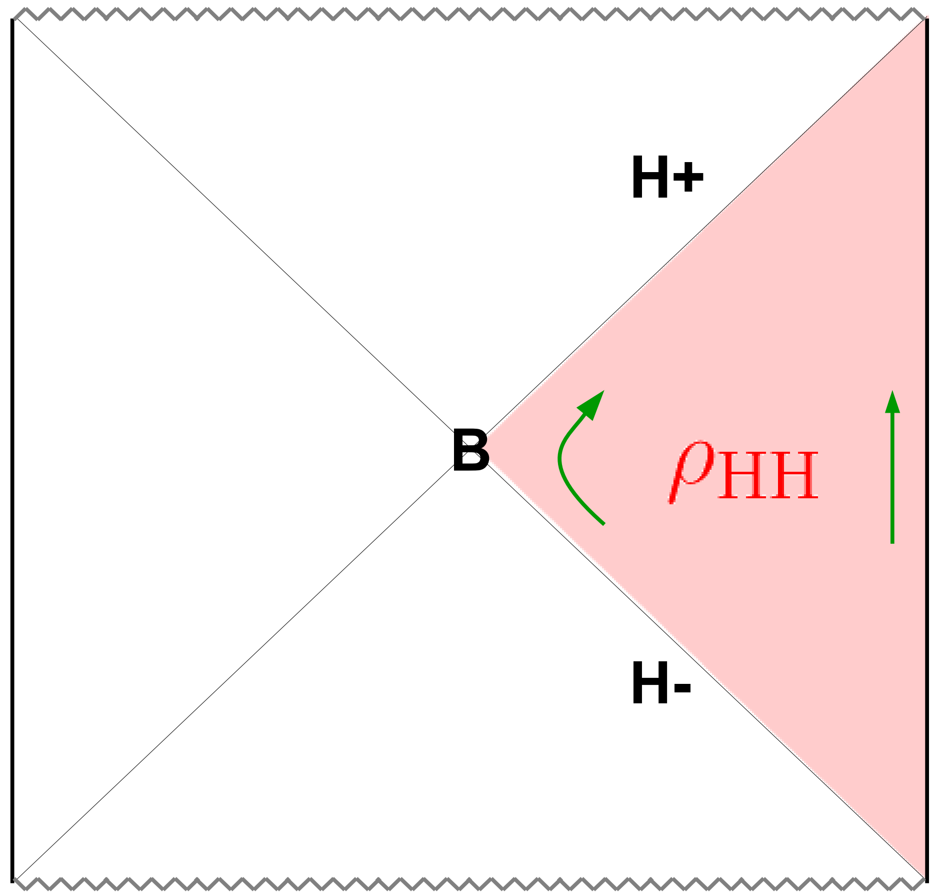
$$\Psi_{\mathrm{HH}}[\Phi(t_0)] = \int_{\Phi(t_0)} D\Phi \, e^{-I[\Phi]/\hbar}$$



Because the double sided Euclidean path has rotational symmetry,

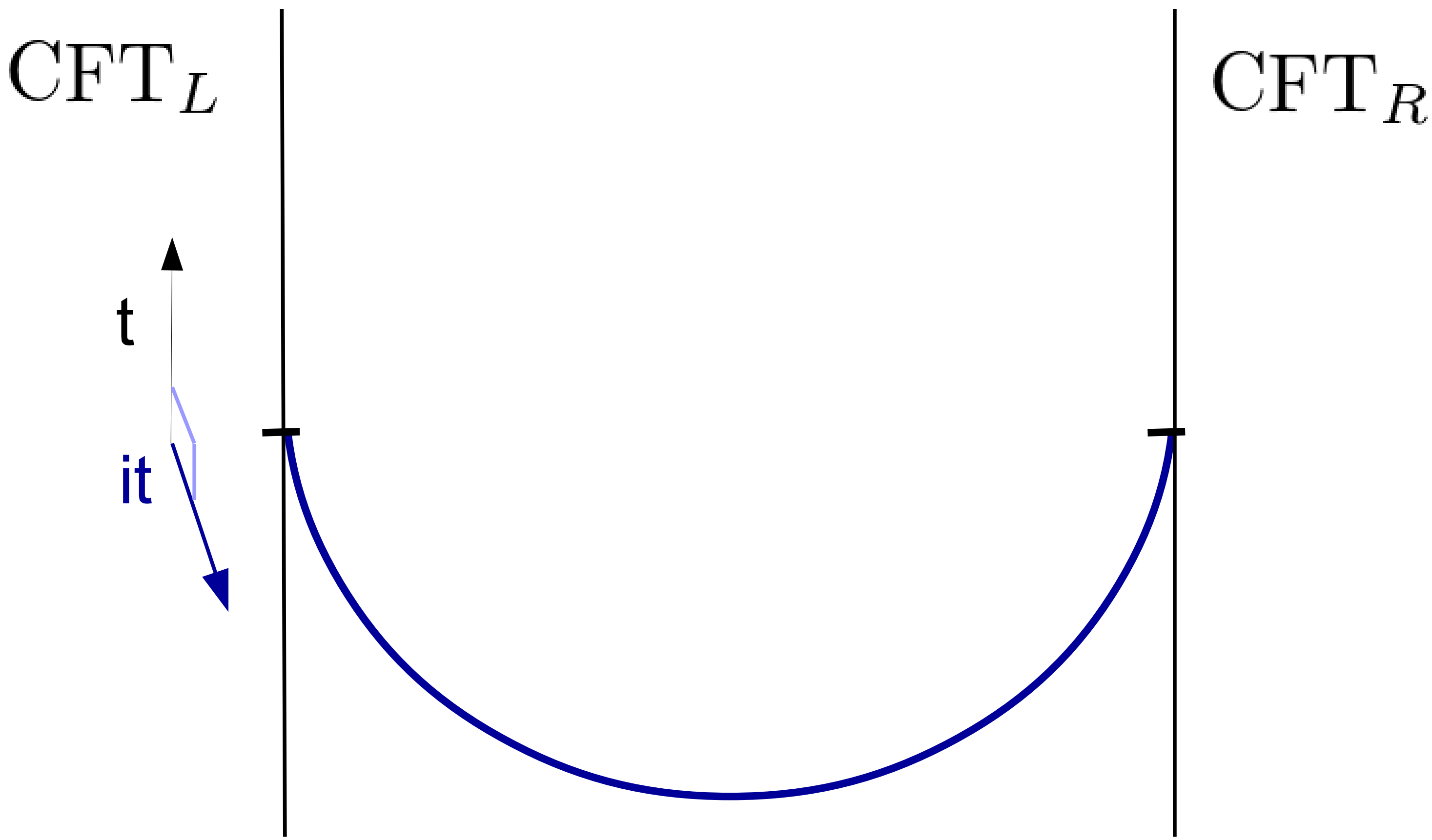
it follows that if Ψ_{HH} is restricted to one side of B, it is thermal:

$$\rho \propto e^{-2\pi K/\hbar}$$



this wedge
is thermal
wrt Killing
energy K

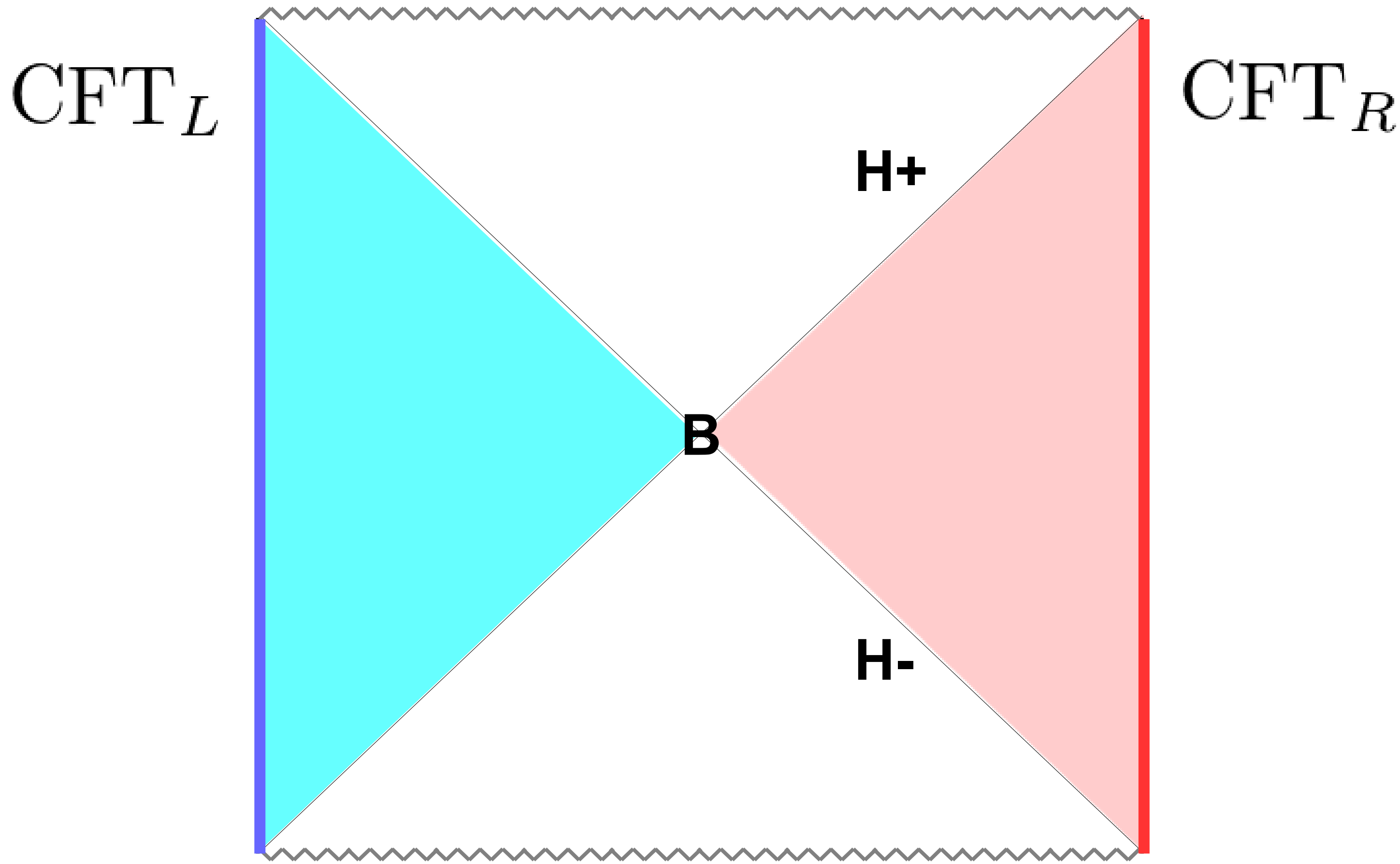
Thermofield Double State



The thermality of HH is related to the fact that in AdS/CFT,
the dual boundary CFT path integral gives the thermofield double state:

$$|\Psi\rangle = e^{-E/2T} |E_L\rangle |E_R\rangle$$

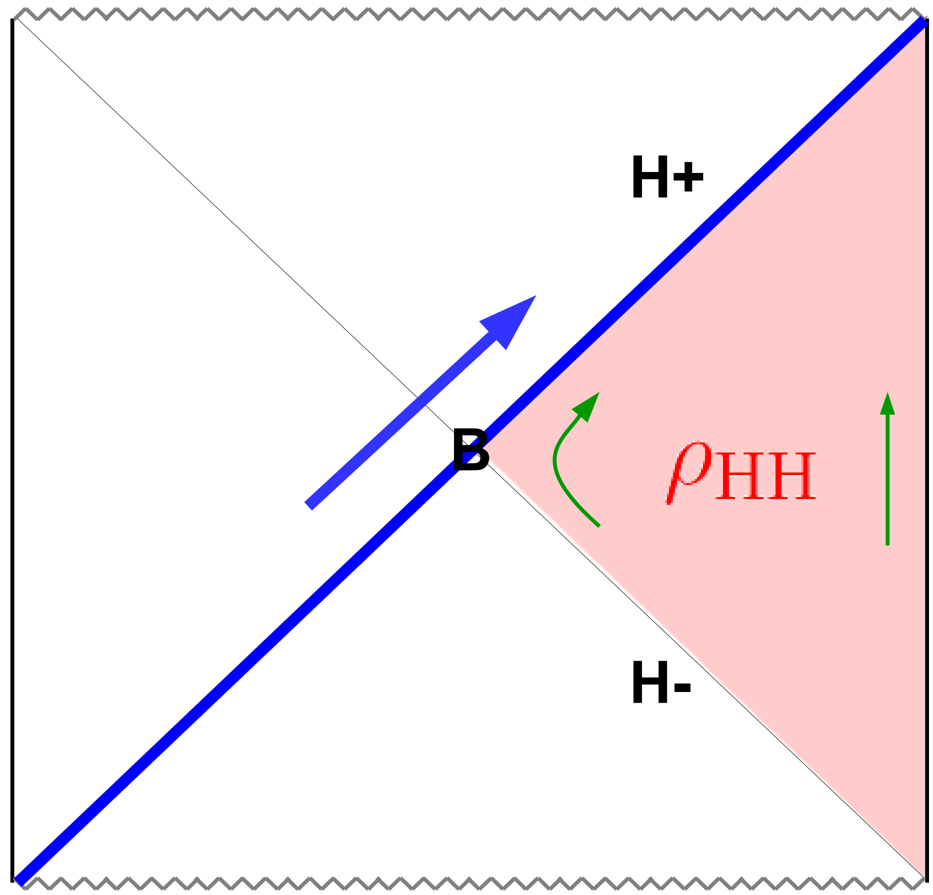
with $S_{\mathrm{CFT}} = S_{\mathrm{BH}}$



this might make you think that the left and right wedges are dual to the left and right CFTs

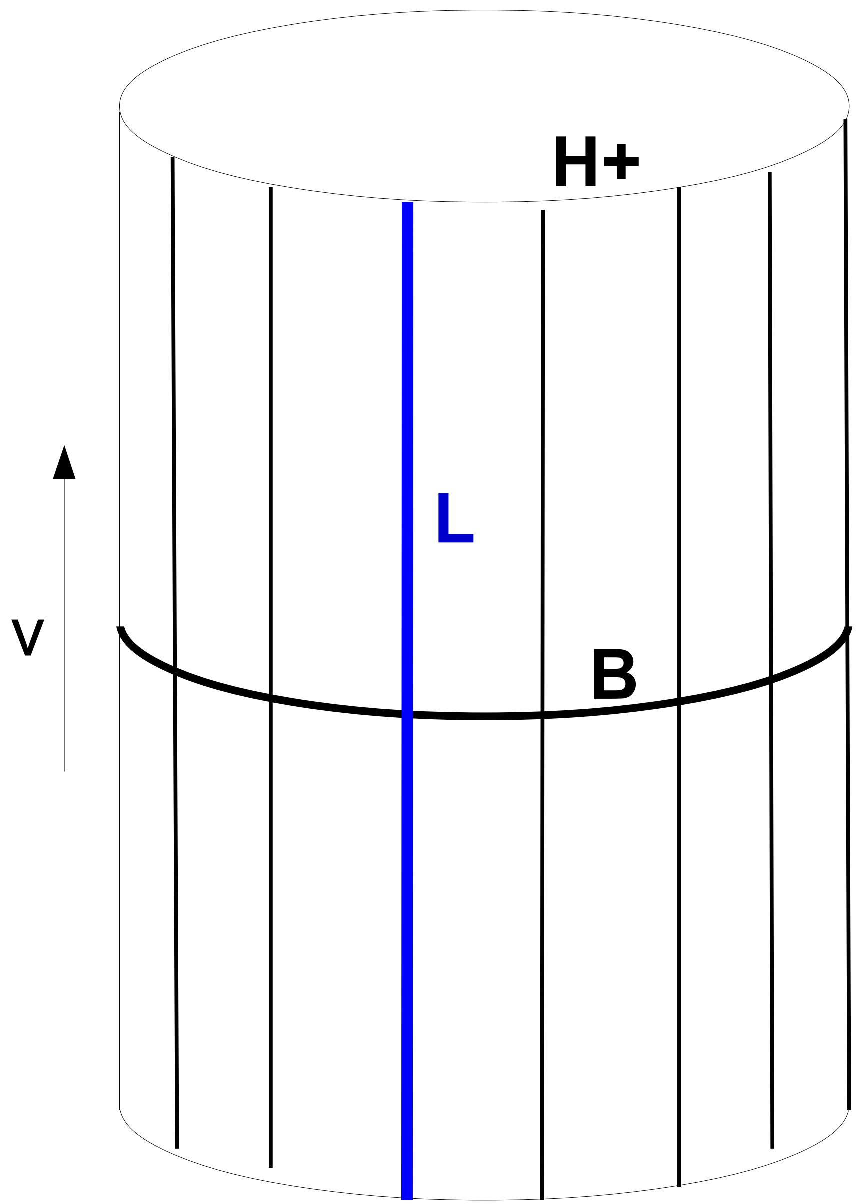
—this turns out to be not only correct, but part of a bigger statement called “entanglement wedge reconstruction” that I will mention later.

back to statements which are true independently of holography...



On the other hand, if we look at all of H+ (including behind H-), then Ψ_{HH} is a pure state and is in fact the *ground state* w.r.t. null translations on the horizon (not a Killing symmetry of the whole spacetime)

Israel, Kay-Wald, Sewell...



Actually a stronger statement is true. Ψ_{HH} is a ground state with respect to the null energy integrated along ANY lightray L on the horizon H.

If *v* is an (affine) null coordinate, the (renormalized) QFT stress-tensor exactly satisfies

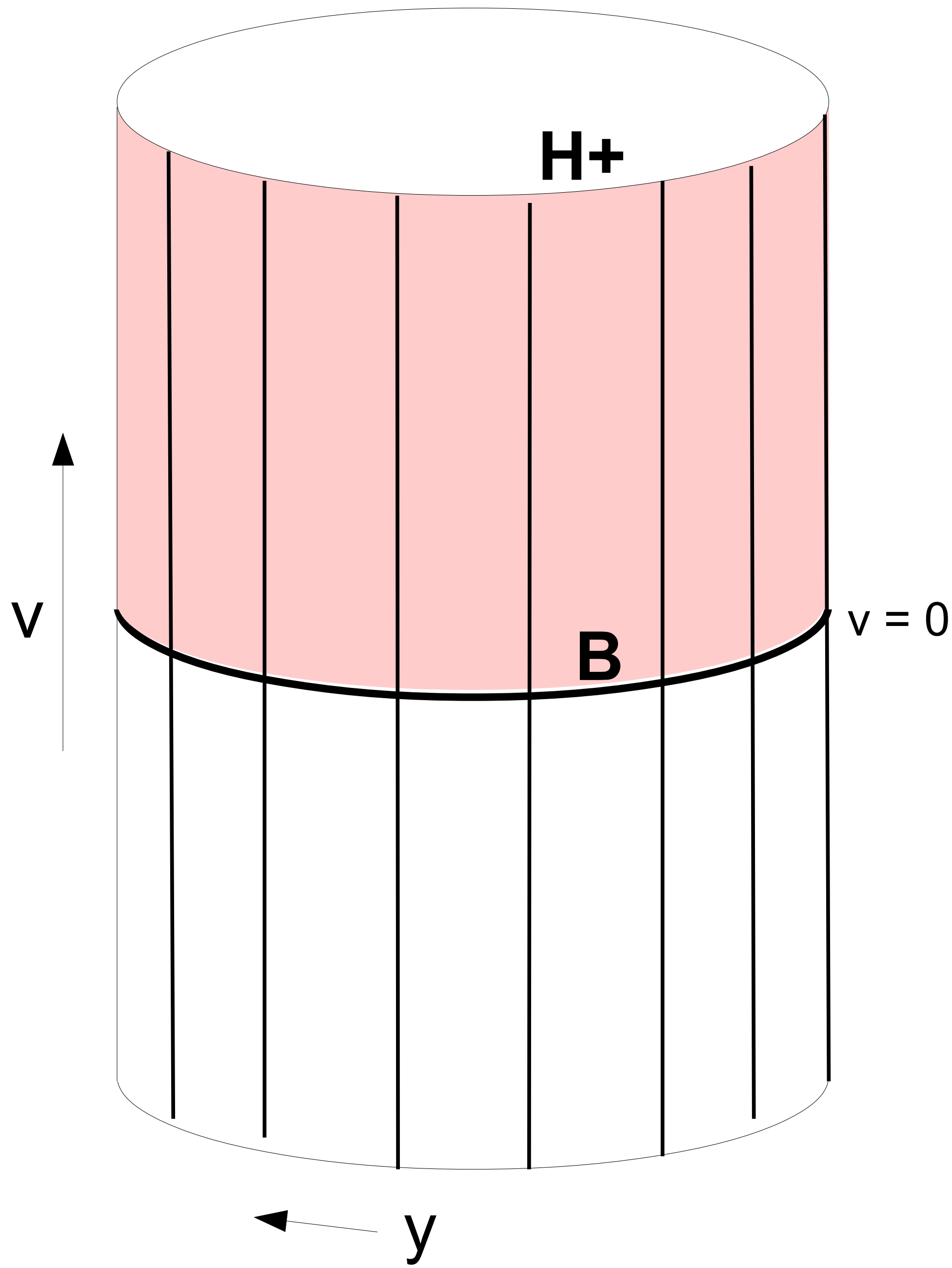
$$\int_L T_{vv}\, dv \,|\Psi_{\mathrm{HH}}\rangle = 0$$

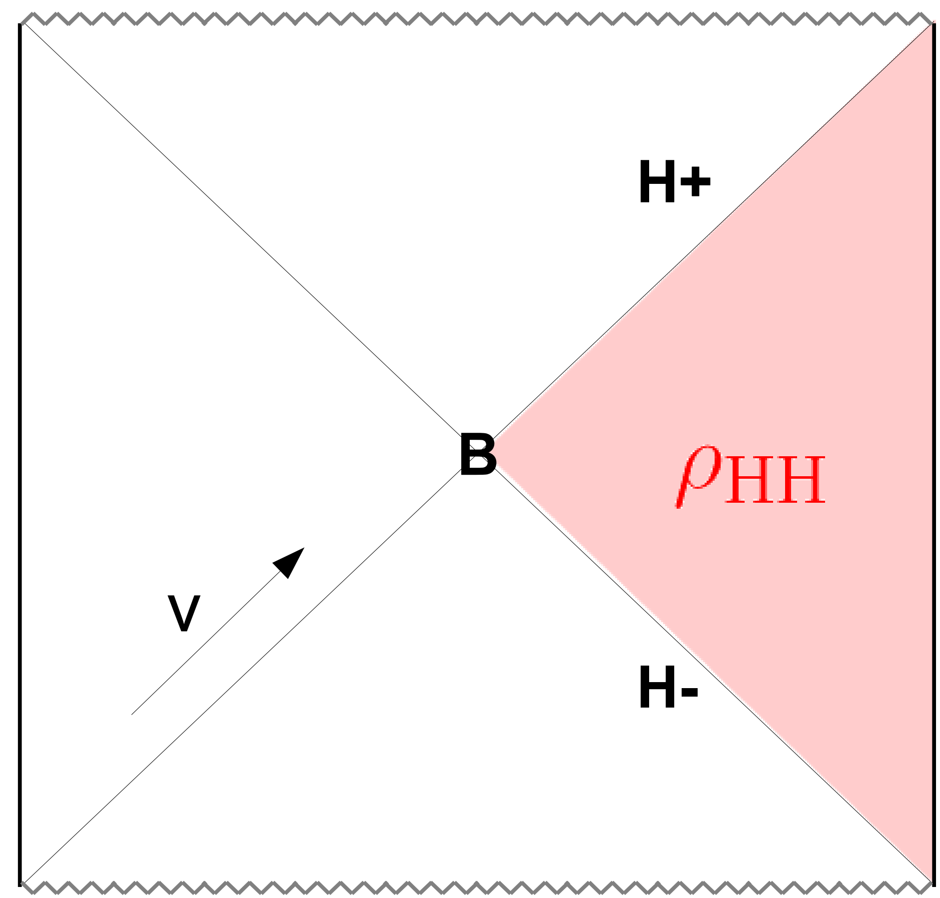
which saturates the lowest bound for all states:

$$\int_L \langle T_{vv}\rangle\, dv \geq 0 \quad \text{(ANEC)}$$

furthermore, because the ANE generates null translations, HH is thermal not just in the wedge outside the bif. surface B...

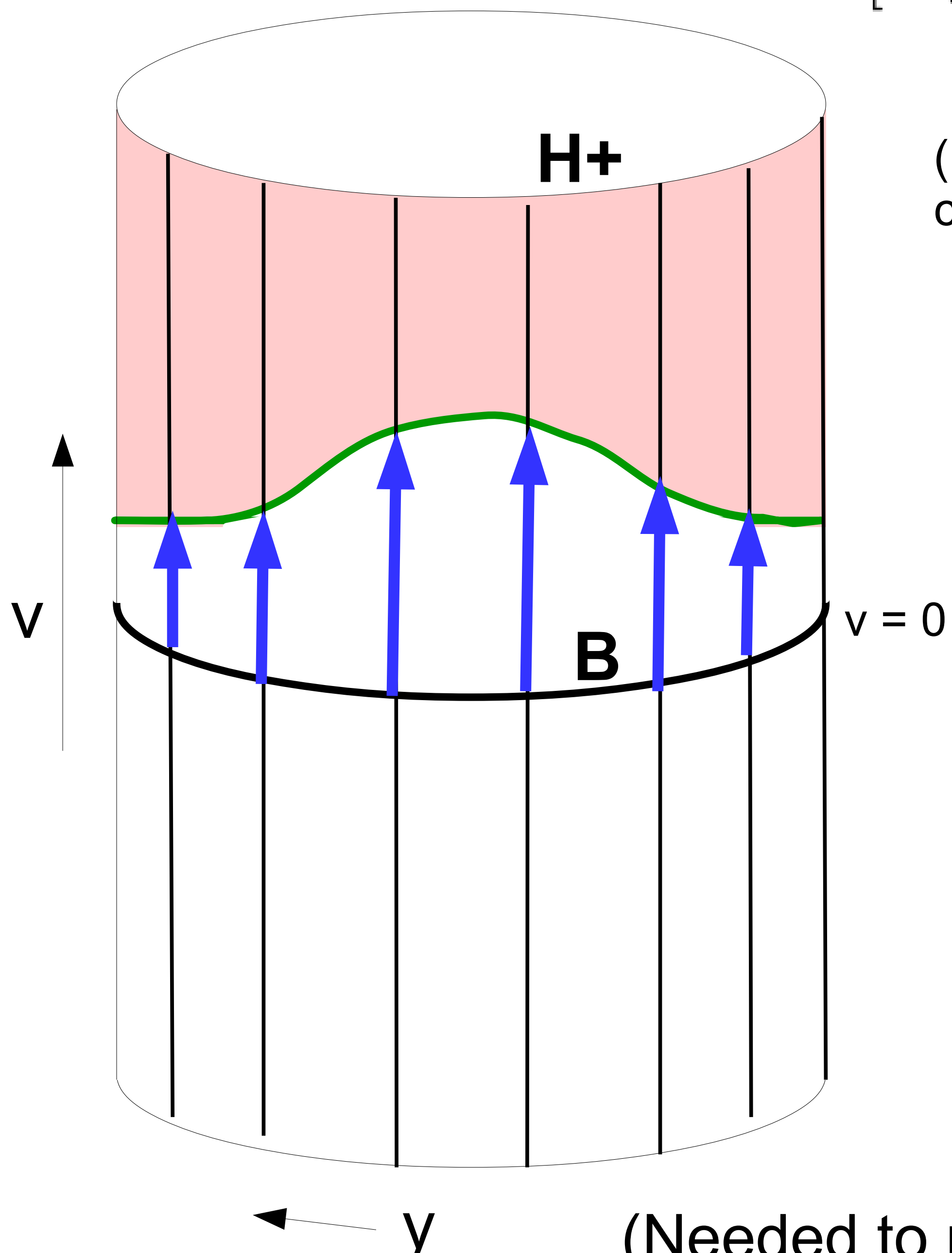
$$K = \int_{v>0} \underbrace{T_{vv}\, v}_{\text{Killing flux}}\, dv\, d^{d-2}y$$



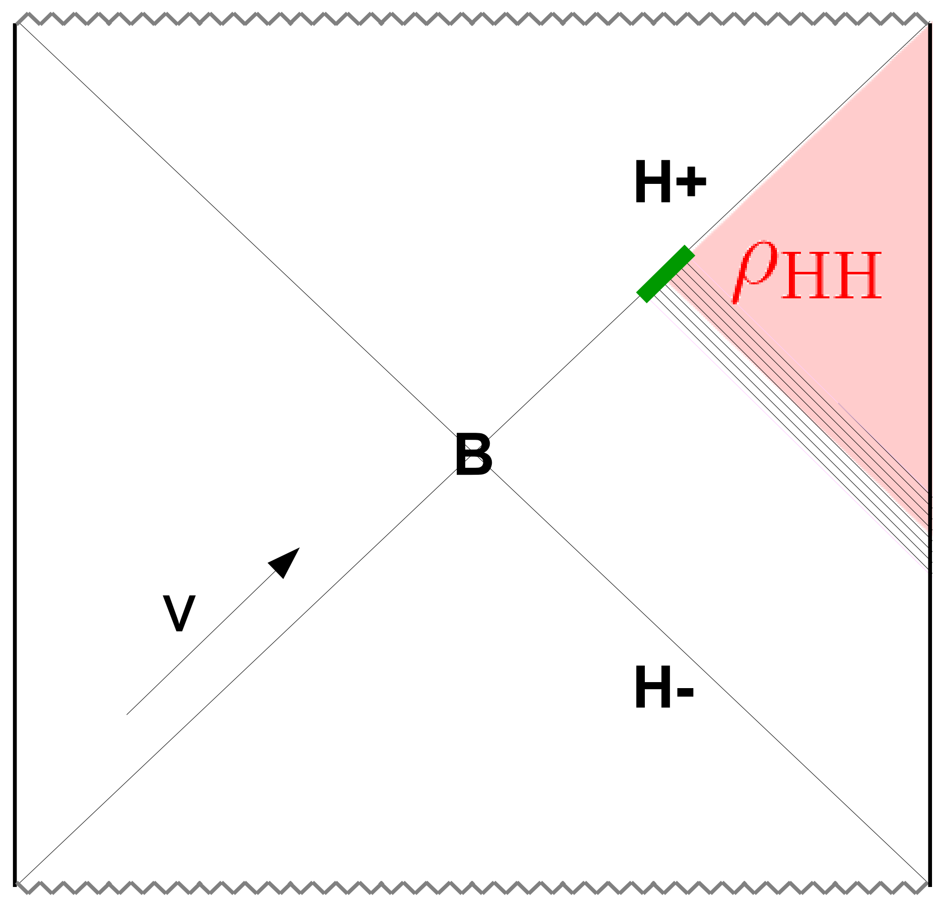


...but also above *any* slice of the horizon.

$$K[V(y)] = \int_{v>0} T_{vv}\,(v - V(y))\, dv\, d^{d-2}y$$



(in asymptotic flat case, this is true up to a constant related to matter that escapes to infinity)

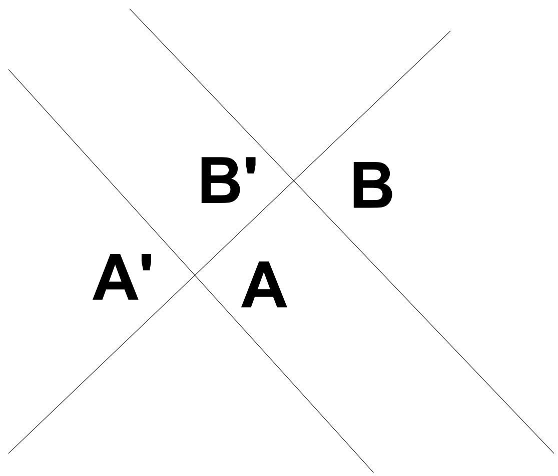


(Needed to prove quantum Second Law, Wall '11)

What's new in the last few years?

Mainly, we now know that these statements should continue to hold for general *interacting* QFTs (with a UV fixed point).

Proofs of the ANEC

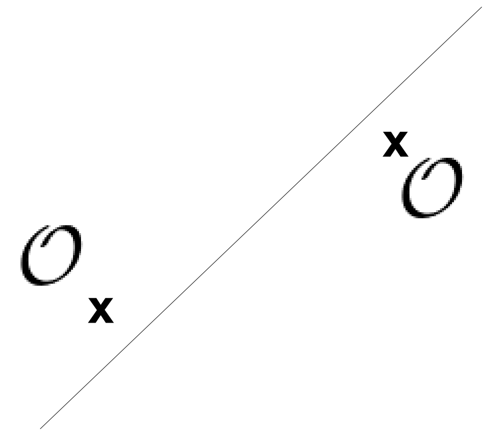


Faulkner, Leigh, Parrikar, Wang '16

relative entropy $\langle \Delta K \rangle_\rho - \Delta S(\rho)$

decreasing from A to B, and B' to A

(K calculated perturbatively using CFT)



Hartman, Kundu, Tajdini '16

causality constraint applied to OPE of operators as they approach null separation

(twist gap, chaos bound, sum rules...)

Modular Hamiltonian on Null Slices

Casini Testé Torroba '17 derived K on null slices, in followup paper proved a-theorem

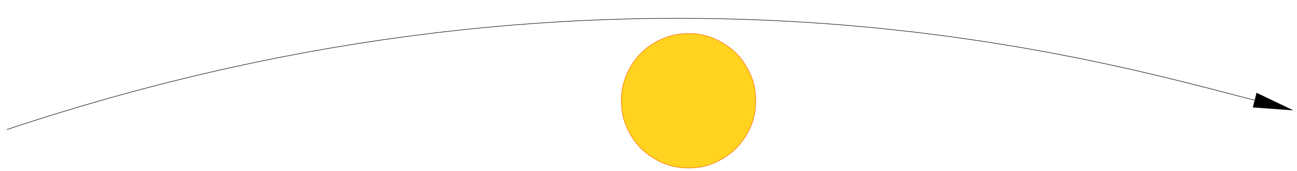
For QFT in curved spacetime ANEC has exceptions

(but not on Killing horizons)

The ANEC only required along “achronal” complete null geodesics meaning that no 2 points are related by timelike curves

In a gravitational field, most lightrays are not “fastest possible”.

Graham & Olum '07
imposing achronal ANEC
generically implies that that NO achronal null geodesics exist,
but this very fact implies most of the GR proofs that require the ANEC!



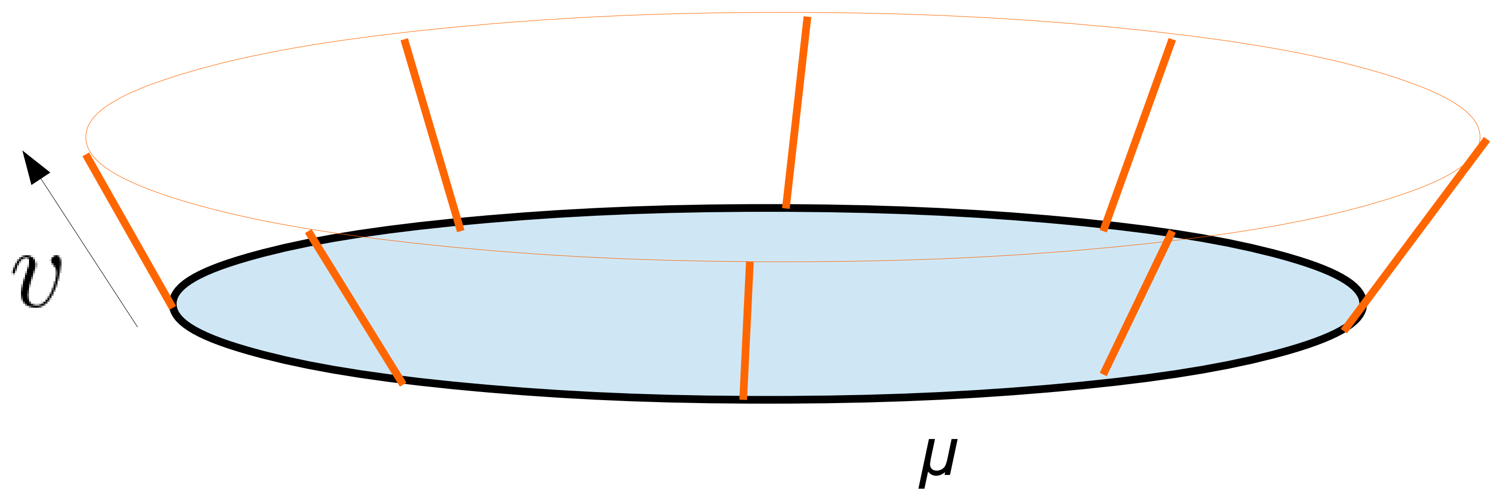
lightrays slowed down by passing through gravity wells are *chronal;*
a timelike observer can catch up to them by going around

Dynamics
(Classical)

Expansion

It is helpful to define the "expansion" of a codim-2 surface as the rate of area increase of lightrays shot outwards from it

$$\theta_v = \frac{1}{\delta A} \frac{d\,\delta A}{dv}$$



In the case where the outgoing lightrays satisfy $\theta_v = 0$
(while the ingoing lightrays satisfy $\theta_u \leq 0$)
we call the surface μ marginally trapped.

(These surfaces play an important role in the Penrose singularity Thm.)

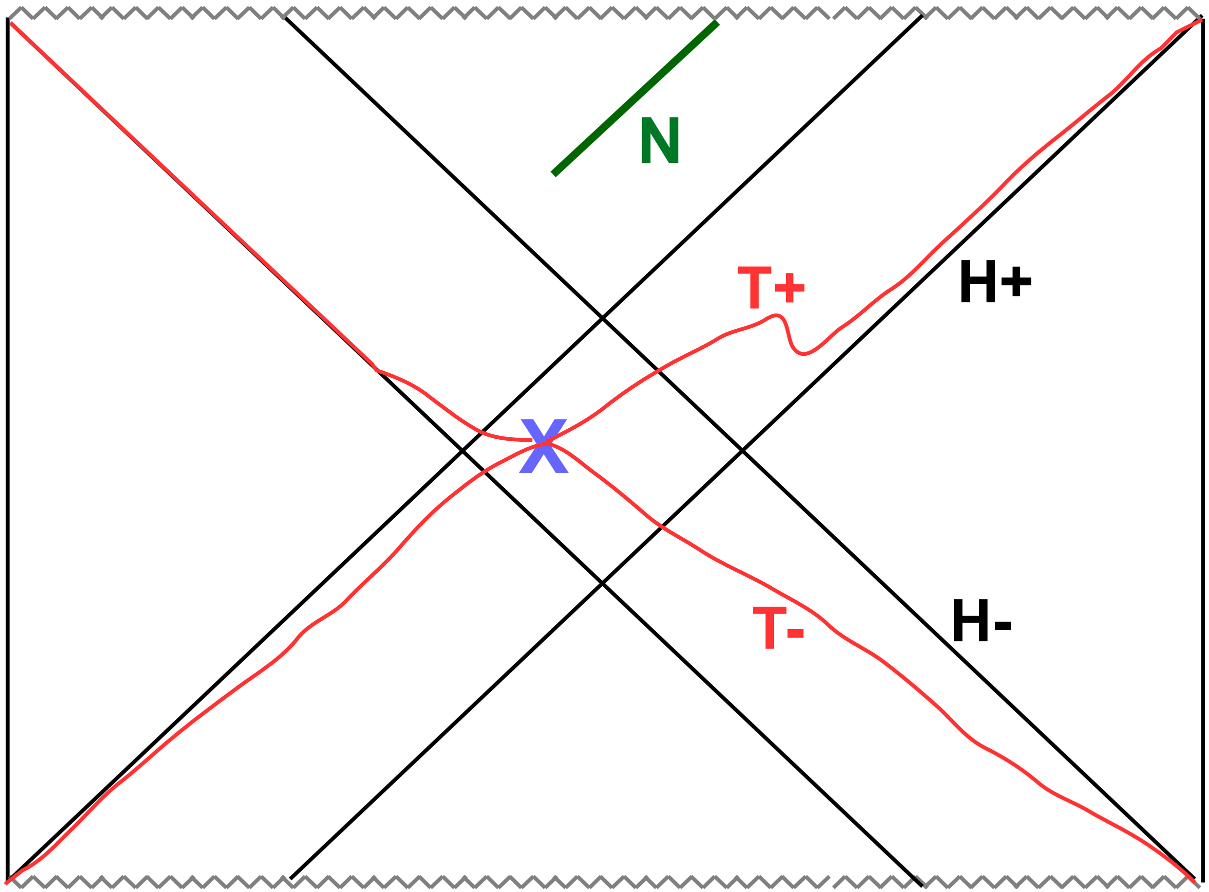
When we add infalling matter (not shown), black hole is not stationary...
different notions of “horizon” separate from one another.

1. Extremal (HRT) Surface (both θ's = 0)

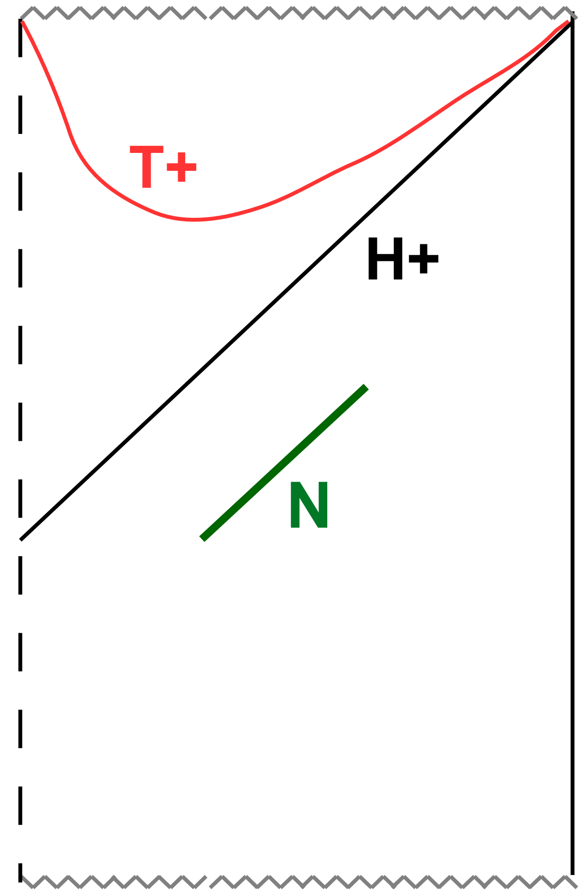
2. Trapping Horizon (one θ = 0)
(a.k.a. dynamical horizon, holographic screen...)

3. The Event Horizon

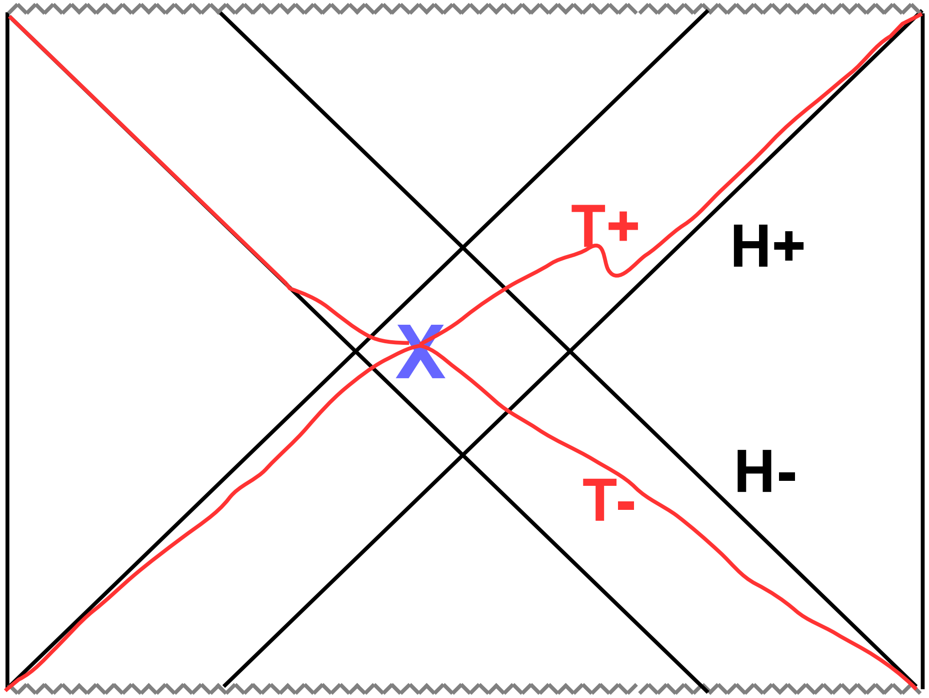
4. General Null Surface



long wormhole



collapse



Classically, if we assume the Null Energy Condition

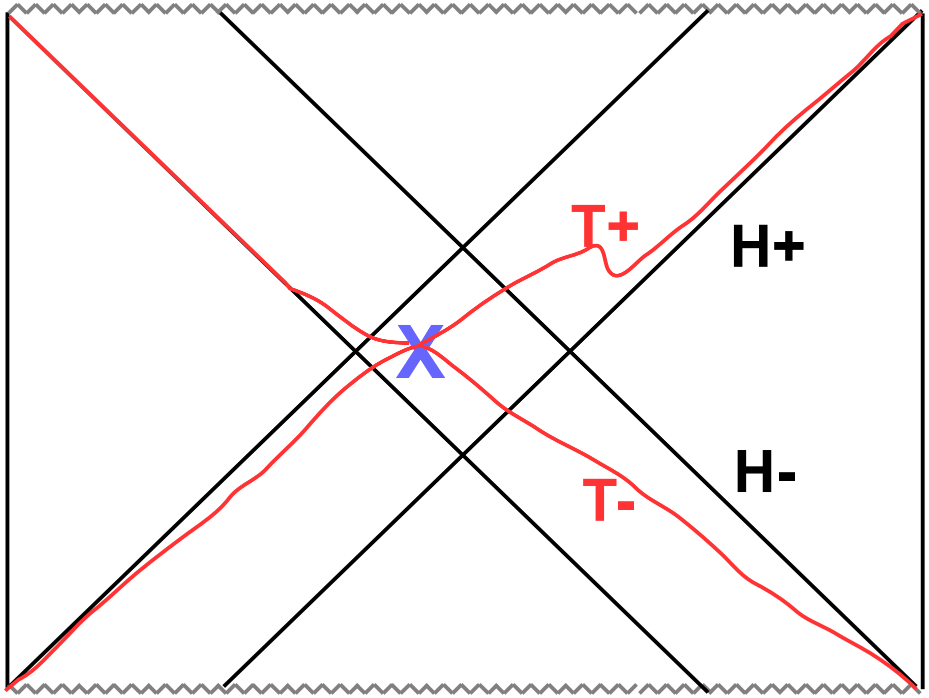
$$T_{vv} \geq 0$$

then the following statements are generically true, as are their time-reversals.

(Nongenerically, can saturate ineq's.)

1. X and T+ always lie inside of H+.

Trapped Surfaces always lie inside of event horizons
(cf. Hawking-Ellis, Wald books)



Classically, if we assume the Null Energy Condition

$$T_{vv} \geq 0$$

then the following statements are generically true, as are their time-reversals.

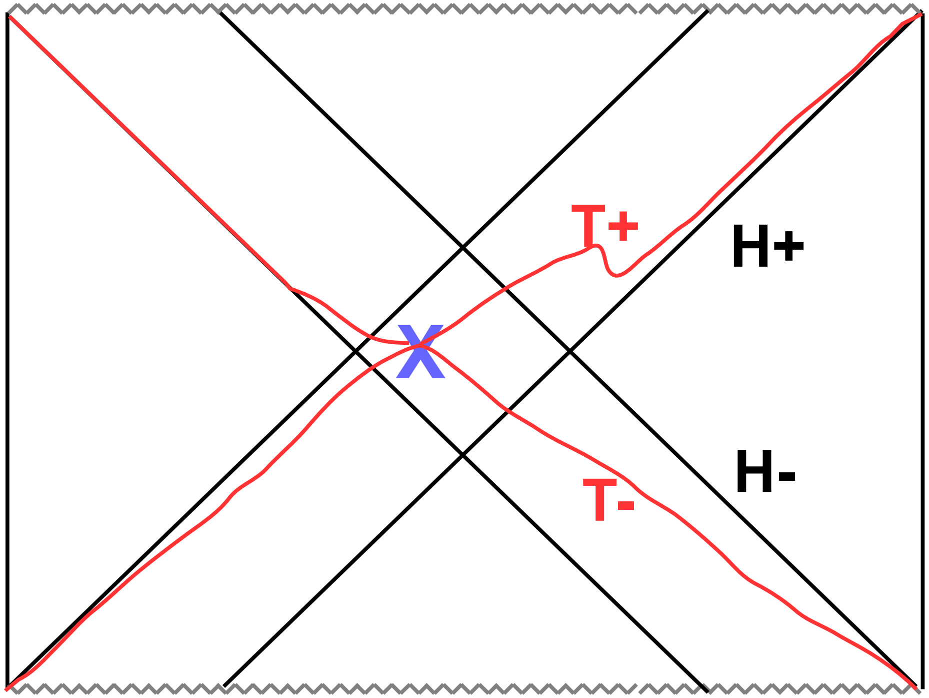
(Nongenerically, can saturate ineq's.)

1. X and T+ always lie inside of H+.

2. H+ has increasing area (**Hawking '71**)

3. T+ has increasing area timelike-pastward and spacelike-outward (**Hayward**) and even for mixed signature, area is monotonic (**Bousso-Engelhardt '15**)

Two versions of the “Second Law”—entropy increases



Classically, if we assume the Null Energy Condition

$$T_{vv} \geq 0$$

then the following statements are generically true, as are their time-reversals.

(Nongenerically, can saturate ineq's.)

1. X and T+ always lie inside of H+.

2. H+ has increasing area (**Hawking '71**)

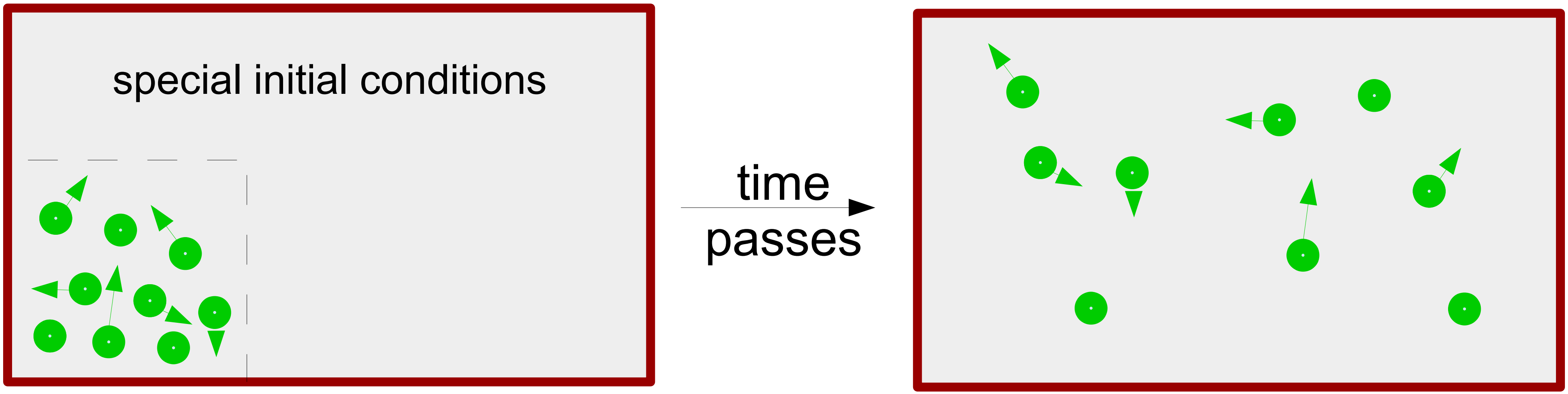
3. T+ has increasing area timelike-pastward and spacelike-outward (**Hayward**) and even for mixed signature, area is monotonic (**Bousso-Engelhardt '15**)

4. X has less area than H+ ∩ H- (**Hubeny-Rangamani '12**, **Wall '12**)

5. Area[X] gives the (leading order in 1/N) entropy of each dual CFT (**HRT, LM**).

If X gives the “fine grained” S, then H or T must involve a “coarse-grained” S!

How the Ordinary Second Law works



$S_{\text{fine}} = -\text{tr}(\rho \ln \rho)$ is conserved under unitary time evolution

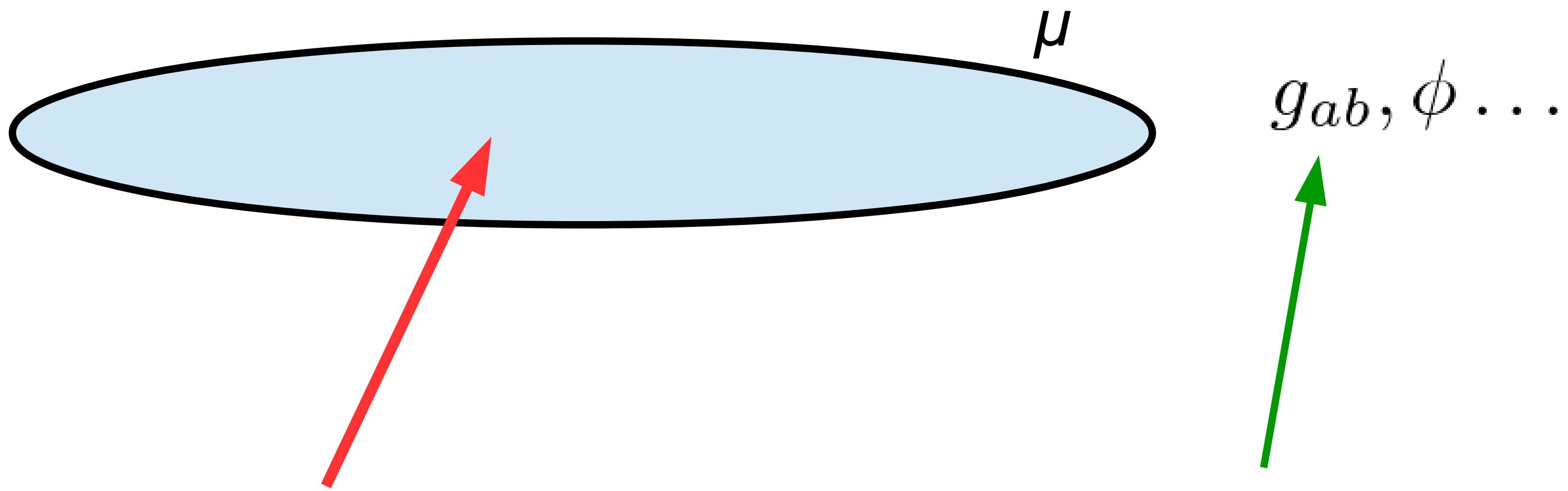
Hence no (nontrivial) second law that allows $\dfrac{dS}{dt} > 0$

Solution is “coarse graining”—must find a way to “forget” detailed correlation of molecules, i.e. find “coarse-grained” procedure for calculating entropy such that

$$S_{\text{coarse}} > S_{\text{fine}}$$

So far there is only a story along these lines for T+, not H+

Outer Entropy:
maximize the area of the stationary surface *X*,
given knowledge of all classical field data outside surface

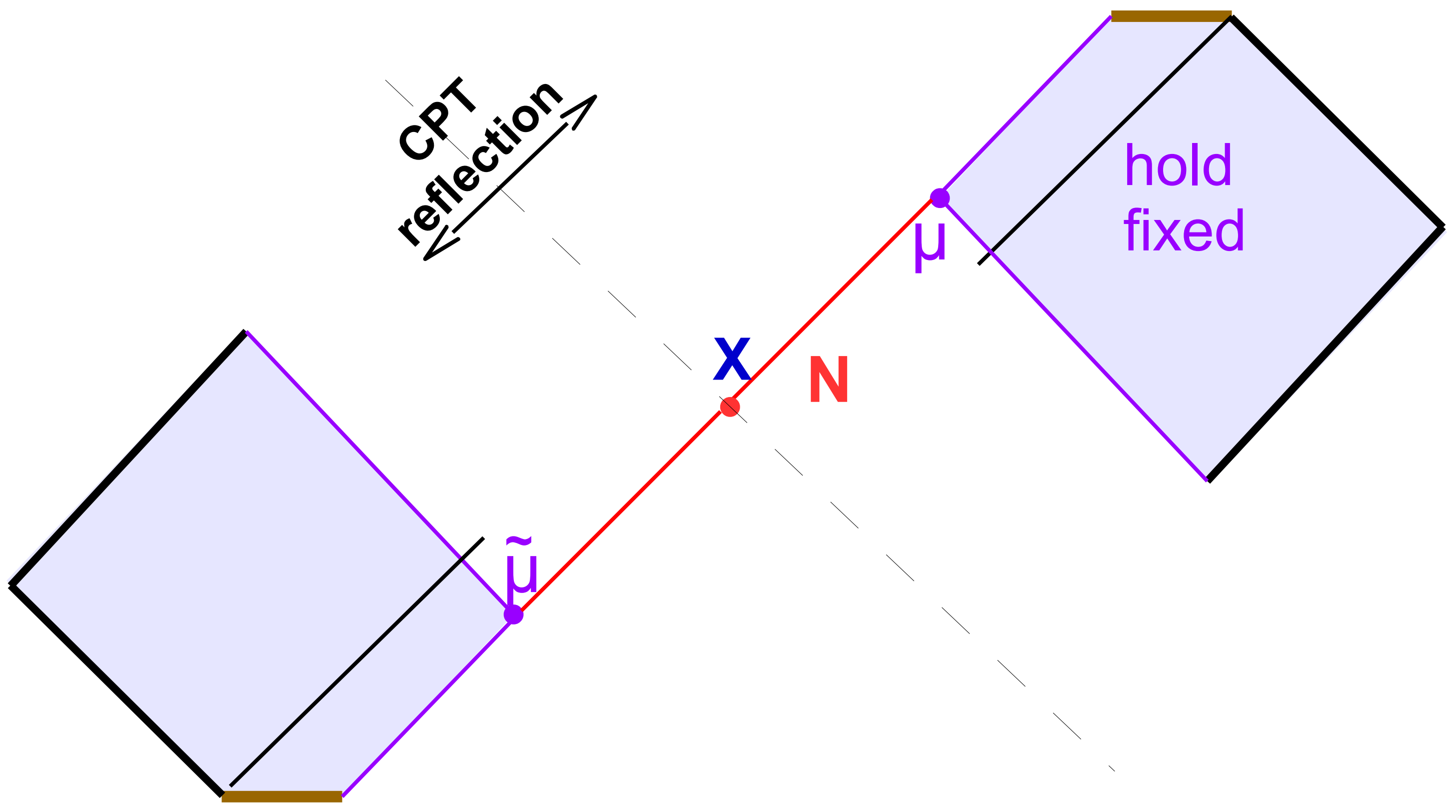


pick ANY interior compatible with the field data outside

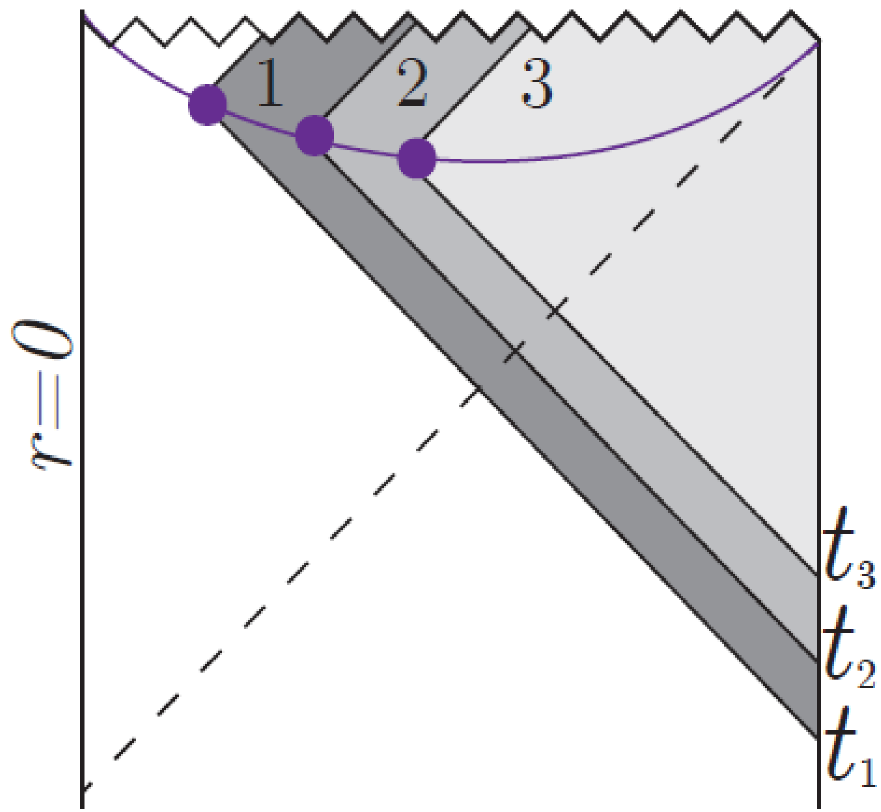
Engelhardt-Wall '17, '18: for a wide class of marg. trapped μ 's,
OuterS[μ] = Area[μ], hence can interpret as coarse-grained entropy.
(This does *not* work for H+)

Explicit solution maximizes X behind μ

(they are connected by a stationary null surface N)



Statistical Explanation for Area Law when T+ spacelike



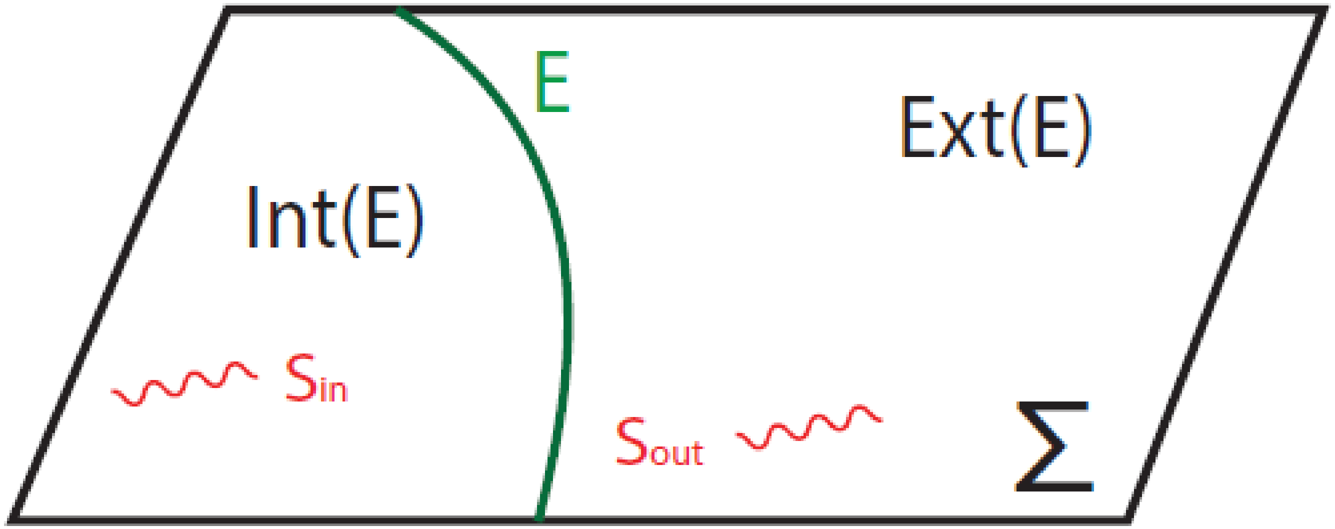
Maximizing entropy subject to fewer constraints → increases

Statistical explanation for Hayward area law

What coarse-grained entropy corresponds to event horizon???

Quantum Corrections

Entanglement Entropy



Given any Cauchy surface Σ, and a surface E which divides it into two regions Int(E) and Ext(E), can define entanglement entropy:

$$S_{\rm ent} = -{\rm tr}(\rho \ln \rho)$$

where ρ is the density matrix restricted to one side or the other. for a pure total state, doesn't matter which side ($\rho_{\rm out}$ or $\rho_{\rm in}$), since $S_{\rm in} = S_{\rm out}$.

but for a mixed state, it does matter ($S_{\rm out} \neq S_{\rm in}$)

$S_{\rm ent}$ is UV divergent, but divergences are local.

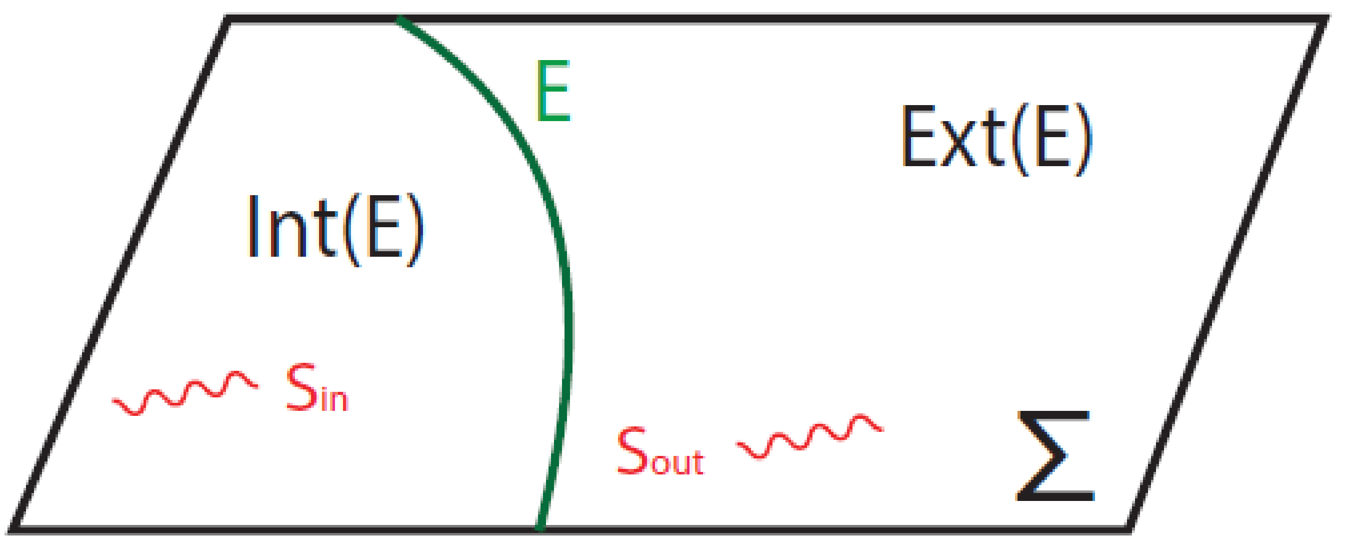
The Generalized Entropy

If the theory is GRAVITATIONAL, then we can also define a finite "generalized entropy" of E:

$$S_{\text{gen}} = \frac{\langle A \rangle}{4G\hbar} + S_{\text{out}} + \text{counterterms}$$

(or we can use S_{in}, which equals S_{out} for a pure state.)

counterterms are *local* geometrical quantities used to absorb EE divergences, (e.g. leading order area law divergence corrects 1/G)



Suggests way to extend classical GR proofs to "semiclassical" situations involving quantum fields...

just replace the area with the generalized entropy!

$$A \to 4G\hbar S_{\mathrm{gen}}$$

Quantum Expansion

classical: area increase (per unit area) of :

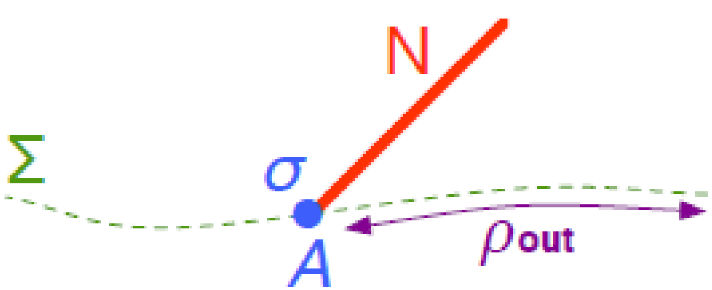
$$\theta = \lim_{A \to 0} \frac{1}{A} \frac{dA}{d\lambda}$$

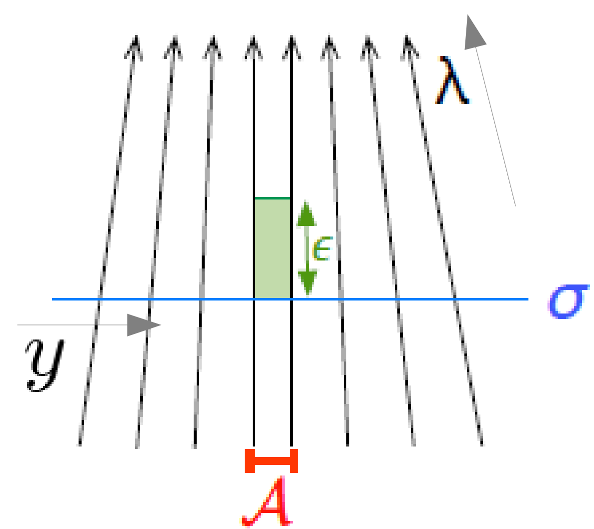
quantum: generalized entropy increase (still per unit area!)

$$\Theta = \lim_{A \to 0} \frac{4G\hbar}{A} \frac{dS_{\text{gen}}}{d\lambda} = \frac{4G\hbar}{a} \frac{\delta S_{\text{gen}}}{\delta\lambda(y)}$$

functional derivative of nonlocal quantity

finite area element





Quantum Focussing

asserts that a *second* functional derivative is negative:

QFC

$$\frac{\delta}{\delta\lambda(y)}\Theta(y')|_\sigma \leq 0$$

for *any* null surface, not just event horizons

$G \to 0$ limit

quantum perturbation to class. stat. null surf.
just look at y → y' contact term

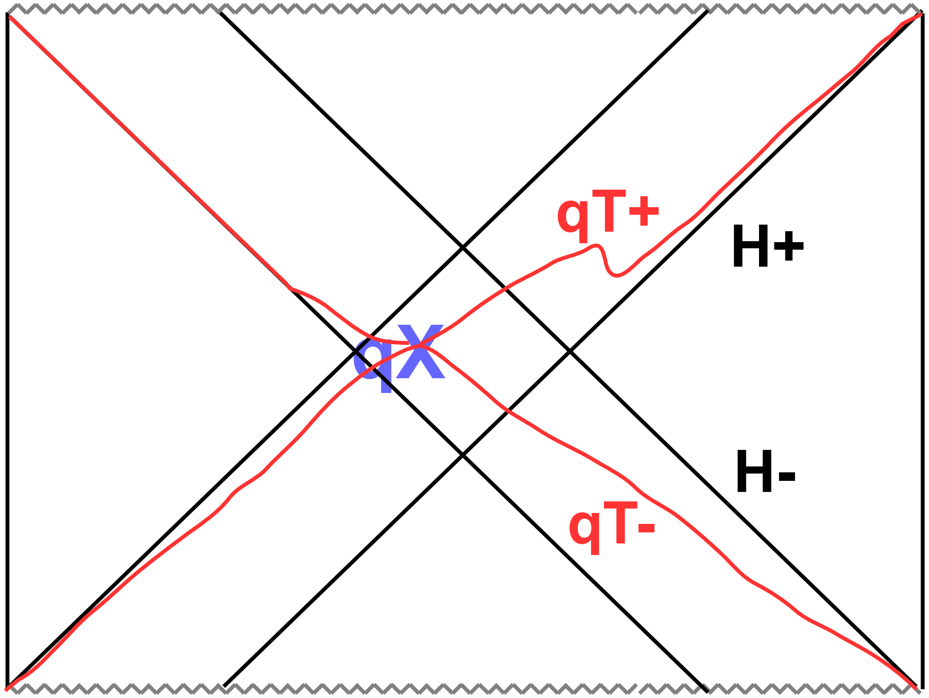
QNEC

$$\langle T_{vv} \rangle \geq \frac{\hbar}{2\pi} S''$$

this is now an assertion about QFT on a fixed background

* QNEC now proven for general QFT's (**Ceyhan & Faulkner '18**,
(see citations therein for many partial proofs)

* Surprisingly, QNEC saturated for interacting d > 2 CFTs!
(**Leichenauer, Levine, Shahbazi-Moghaddam '18**)



In semiclassical regime, ought to redefine X and T+/- using quantum expansion Θ:

1. q Extremal (both Θ's = 0)

2. q Trapping Horizon (one Θ = 0)

Then the “quantum” version of the previous statements hold:

1. qX and qT+ always lie inside of H+ (**Engelhardt-Wall '15,** from #2 below)

Generalized Second Laws:

2. H+ has increasing S_{gen} (**Wall '11,** from monotonicity of relative entropy)

3. qT+ has increasing S_{gen} (**Bousso-Engelhardt '15**, from QFC)

4. qX has less S_{gen} than H+ ∩ H- [not sure if anyone has shown this one yet]

5. S_{gen} gives the entropy of the dual CFT *<u>to all orders in 1/N ~ $\hbar$!</u>*
(**FLM '13**, **Engelhardt-Wall '15**, **Lewkowycz-Dong '17**)

At the *first* subleading (quantum) order in hbar,
for states expanded around a single spacetime background,
the following remarkable relations hold:

FLM: $S_{\mathrm{CFT}} = S_{\mathrm{gen}}[X]$ (derived from path integral)

linearize around any *ρ*

JLMS: $K^{\rho}_{\mathrm{CFT}} = K^{\rho}_{\mathrm{gen}}[X]$

where the modular Hamiltonian is $K^{\rho} \equiv -\ln \rho$ (viewed as an operator)

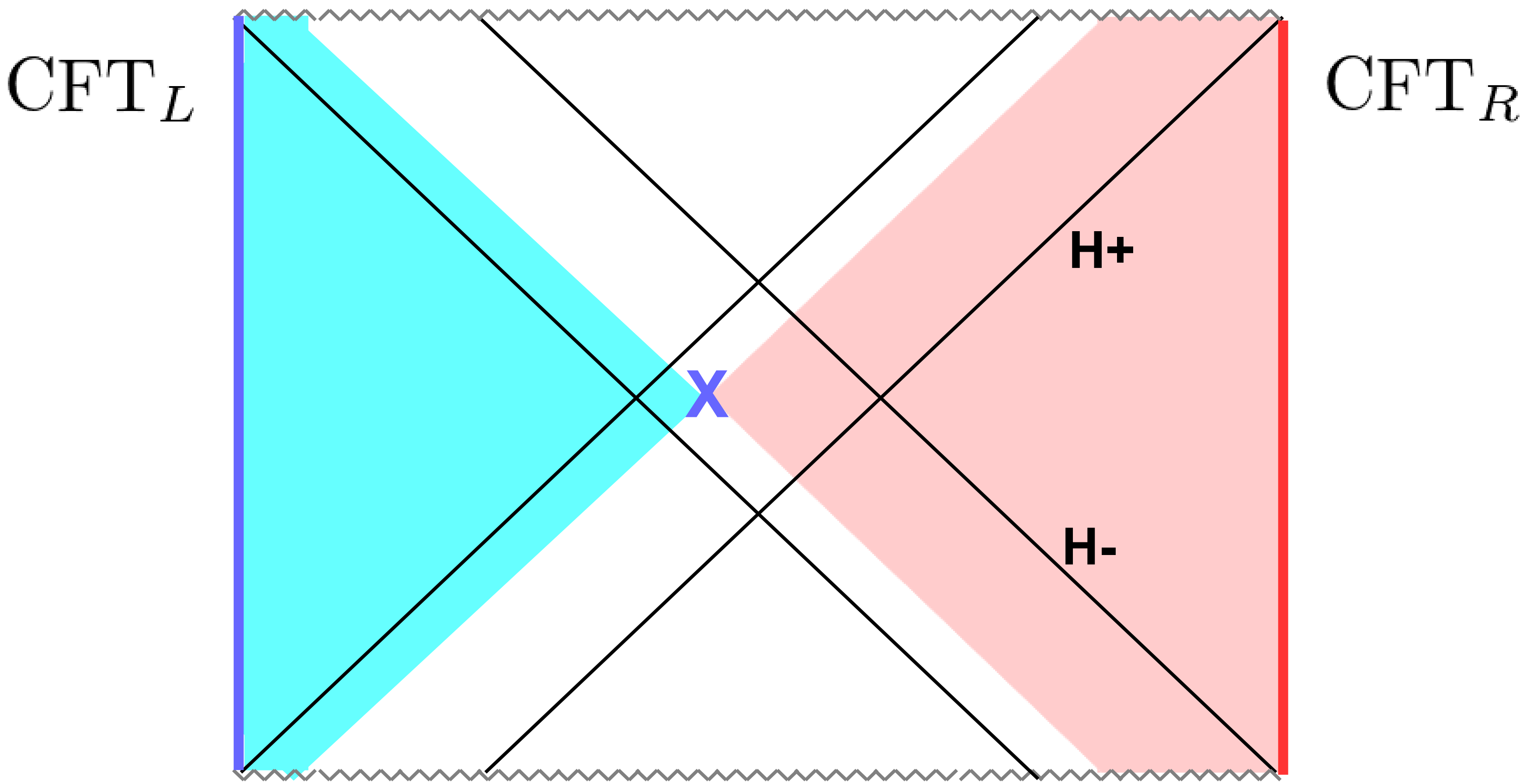
This gives an enormous amount of additional information about AdS/CFT and is useful for reconstructing information behind H(+/-)

Also implies relative entropies $\langle \Delta K^{\sigma} \rangle_{\rho} - \Delta S(\rho)$ agree:

$$S_{\mathrm{rel}}(\rho\,|\,\sigma)_{\mathrm{CFT}} = S_{\mathrm{rel}}(\rho\,|\,\sigma)_{\mathrm{bulk}}[X]$$

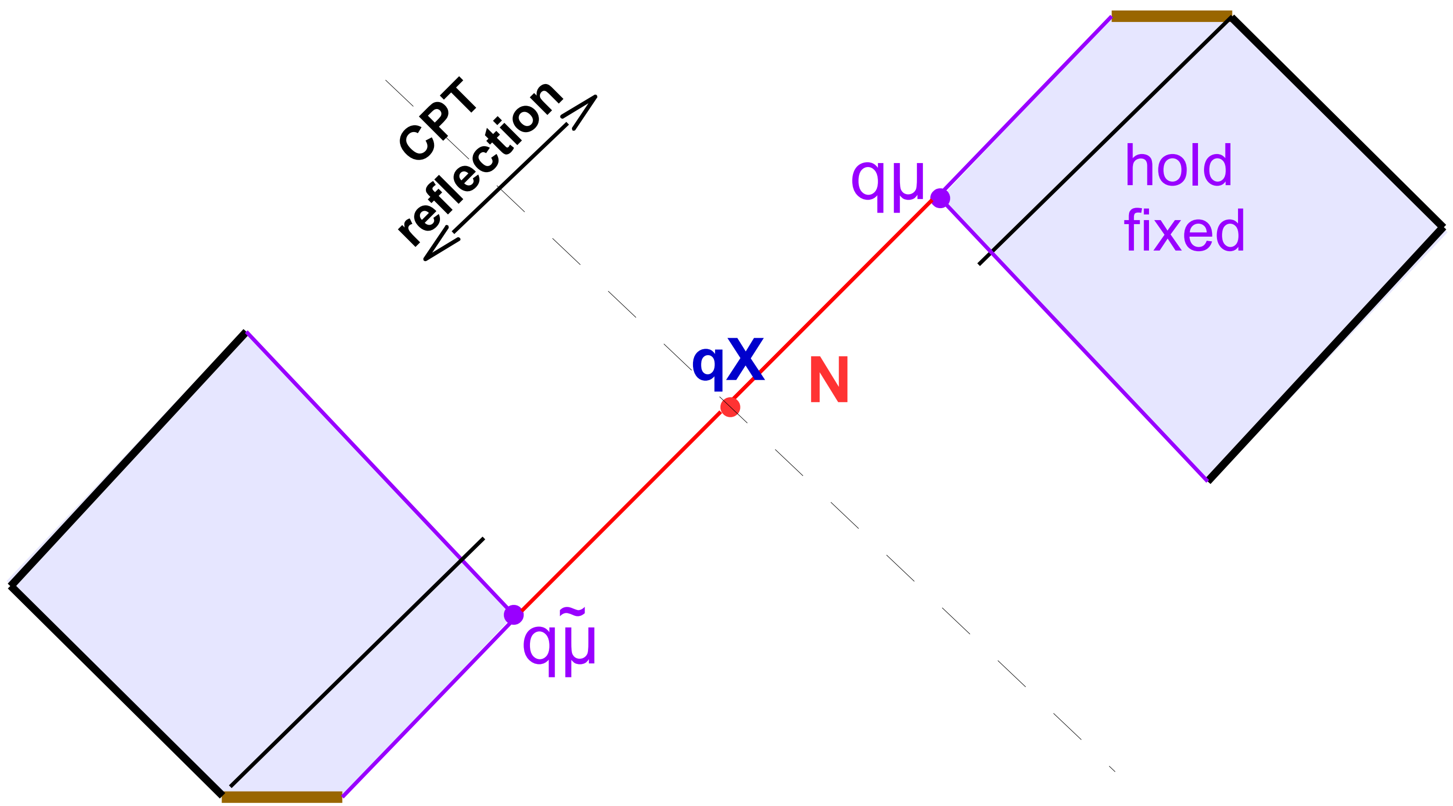
which in turn implies

Entanglement Wedge Reconstruction



Jafferis, Lewkowycz, Maldacena, Suh '15
Dong, Harlow, Wall '16

Also possible to generalize coarse-graining setup to quantum extremal surfaces...



(Bousso, Chandrasekaran, Shahbazi-Moghaddam '19)

Stringy / Higher-Curvature Corrections

Higher Curvature Gravity

$$S_{\text{gen}} = \frac{\langle A \rangle}{4G\hbar} + S_{\text{out}} + \text{counterterms}$$

starting with a local correction to the GR action, e.g:

$$I = \int d^D x \sqrt{g}\, f(R_{abcd})$$

can derive entropy functional $\downarrow$ (in null coordinates v, u)

$$S = -\frac{2\pi}{\hbar} \int d^{D-2} x \sqrt{g} \left[\underbrace{4 \frac{\partial L_g}{\partial R_{uvuv}}}_{\text{Wald}} + \underbrace{16 \frac{\partial^2 L_g}{\partial R_{uiuj} \partial R_{vkvl}} K_{ij(u)} K_{kl(v)}} \right] + \mathcal{O}(K^4)$$

$$= \frac{A}{4G\hbar} \text{ for GR}$$

Solodukhin, FPS, Dong, Miao...
(extrinsic curvature corrections only matter for nonstationary null surfaces)

Higher Curvature Focussing

In any metric-scalar theory of gravitation w/ arbitrarily complex action

$$I = \int d^D x \sqrt{g}\, L(g^{ab}, R_{abcd}, \nabla R \ldots \phi, \nabla\phi \ldots) + I_{\mathrm{matter}}$$

for a linearized perturbation of g_{ab}, ϕ about a Killing horizon, one can always construct an entropy density s that focusses:

$$T_{vv} = H_{vv} = -\frac{2\pi}{\hbar}\frac{d^2 s}{dv^2}$$

obtain s by repeatedly differentiating by parts, at least 2 ∂_v's end up outside:

∂_v

$$\delta H_{vv}{}^{(2)} = \sum_{n \geq 0} X^{(-n)} \cdot \delta Y^{(2+n)}$$

(i) = Killing weight

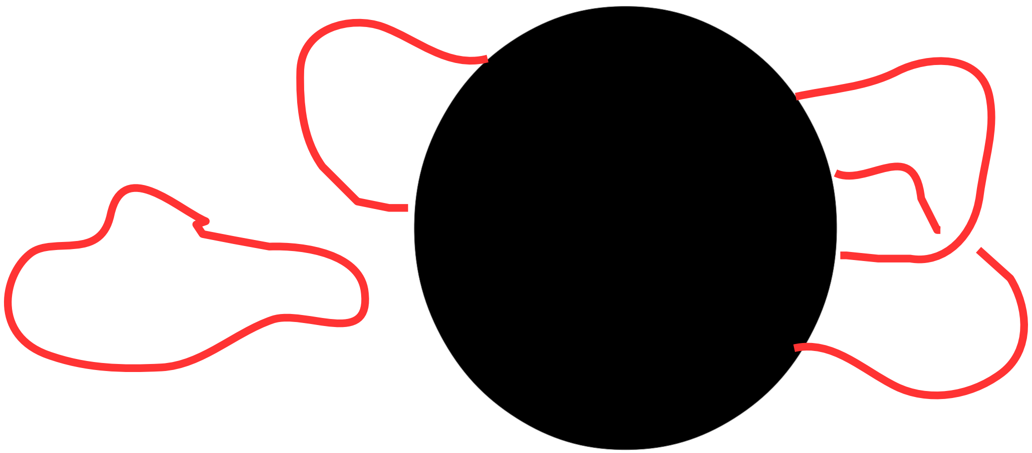
the integral of this s agrees with “Dong entropy” for f(Riemann) actions!

Beyond Semiclassical Quantum Gravity?

True Meaning of Generalized Entropy?

For a static horizon $S_{\text{gen}}[H]$, plausibly counts the total entropy of all degrees of freedom including Planck/string d.o.f. (**Sorkin**, **Jacobson, Susskind & Uglum**), assuming QG cuts off contributions below the Planck scale.

Using known relations between action & entropy, this scenario is equivalent to the “induced gravity” hypothesis of Sakharov that the gravitational action R/G comes entirely from quantum loop corrections i.e. the “bare” 1/G = 0



Susskind & Uglum argued that the Bekenstein-Hawking entropy comes mainly from strings that cross the horizon, but their calculation of A/4 requires off-shell string theory

Tempting to think that more generally, $S_{\text{gen}}[\partial R]$ counts the QG entropy of a general region *R* (**Bianchi-Myers '12**) but because the CFT entropy is fixed this can probably only true of the holographic entropy surface...

Q & A