What's New in Moonshine?

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Based on joint work with
Anagiannis, Duncan, Harrison, Kachru, Mertens, Volpato, Zimet
and work by **many others**.

European Research Council
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More Umbral Moonshine Modules Constructed!

K3 Elliptic Genus Symmetries Classified

Genus zero mystery of Monstrous Moonshine explained!

Experts agreed: moonshine proliferation is inevitable!

Moonshine for a Pariah!

M5 branes moonshine

Yet Another Case?

Moonshine news on both the physical and the mathematical front, at a conceptual and a technical level!

Outline

I. Old News
II. Umbral Moonshine and K3 String Theory
III. Moonshine Proliferation

Monstrous Moonshine

Member of the Asscoiated Press .
Aenean commodo ligula eget dolor.
Aenean. Aenean commodo ligula eget
dolor. Aenhswse. Cejhciebce fcdcdcd.

Est. 1869 | Late 70's ~Early 90's, last century | Price 6d

J and M. In 1973 a finite simple group of monstrous size was suspected to exist. Its character table was computed in 1978, and it was soon realised that it has a bizarre relation with the coefficients of the canonical modular function, called *J*-invariant.

$$J : \mathbb{H} \longrightarrow \mathbb{C}$$

$$J(\tau) = J(\tau + 1) = J(-1/\tau) \qquad (q := e^{2\pi i \tau})$$

$$= (1)q^{-1} + (1 + 196883)q^2 + (1 + 196883 + 21296876)q^3 + \ldots$$

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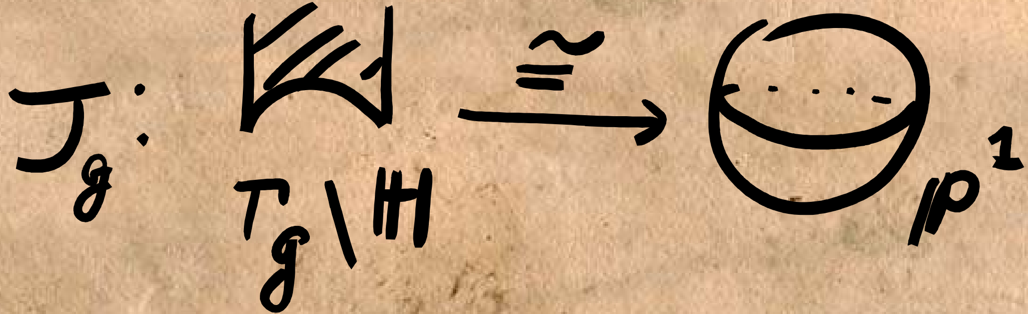
The Monstrous Moonshine Conjecture of Conway-Norton ('79) states the existence of a Monster representation $V^\natural = \bigoplus_n V_n^\natural$ such that its graded character

$$J_g(\tau) := \sum_{n=-1}^{\infty} q^n \; (Tr_{V_n^\natural} g)$$

is the **unique**



and $J_g(\tau) = q^{-1} + 0 + \cdots$ for some $\Gamma_g \subset SL_2(\mathbb{R})$.

Monstrous Moonshine

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The Theory. In 1984, Frenkel, Lepowsky and Meurman constructed $V^\natural$ by $\mathbb{Z}_2$-orbifolding the c=24 chiral CFT constructed by 24 bosons on a 24-dimensional torus defined by the *Leech lattice*.

Niemeier (1973): There are 24 even, self-dual lattices in 24 dimensions with a def. signature.

Λ_L (Leech Lattice)

N^X (labelled by 23 *ADE* root systems *X*)

Monstrous Moonshine

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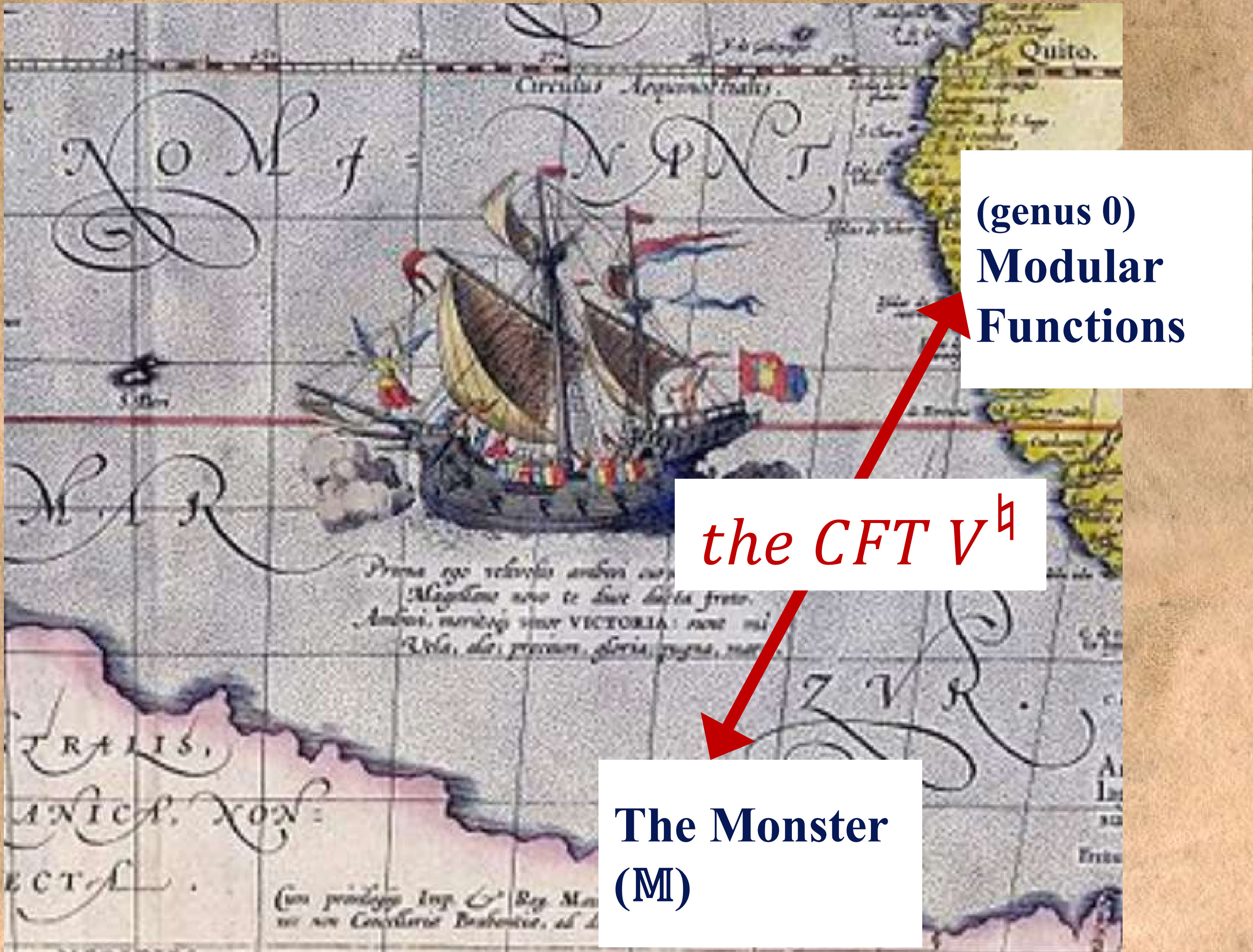
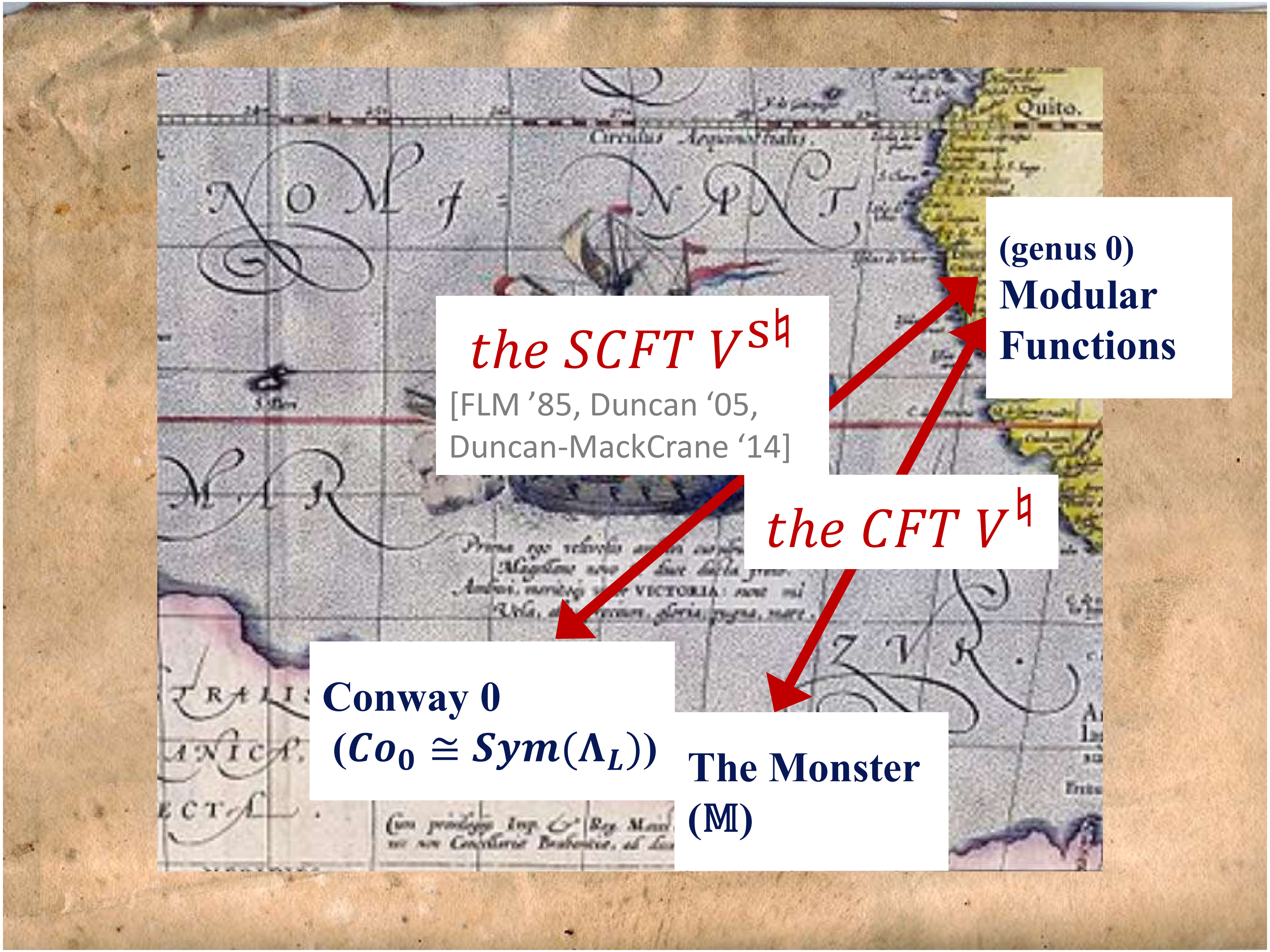
$$\begin{cases} \Lambda_L & \text{(Leech Lattice)} \\ N^X & \text{(labelled by 23 } ADE \text{ root systems } X) \end{cases}$$

The FLM construction was verified by Borcherds in '92 using ideas from string theory.

The physical meaning of Borcherd's proof is further illuminated by the work by Paquette-Persson-Volpato [2016-17], in terms of 1-dim heterotic strings.


(genus 0)
Modular
Functions
the CFT $V^\natural$
The Monster
($\mathbb{M}$)


the SCFT $V^{s\natural}$
[FLM '85, Duncan '05, Duncan-MackCrane '14]
the CFT $V^{\natural}$
(genus 0) Modular Functions
Conway 0 ($Co_0 \cong Sym(\Lambda_L)$)
The Monster ($\mathbb{M}$)
Quito.
Circulus Aequinoctialis
VICTORIA

MATHIEU AND UMBRAL MOONSHINE

***K3* elliptic genus** displays moonshine behavior! Recall that the non-linear sigma model on K3 – the non-trivial CY 2-fold – is a N=(4,4) SCFT. The elliptic genus computes the BPS spectrum and is rigid. When decomposed into N=4 characters, we obtain a nice infinite *q*-series that encodes representations of another sporadic group, M_{24} .

[Eguchi-Ooguri-Tachikawa, '10]

$$EG(K3;\tau,z):=Tr_{H_{RR}}\left((-1)^F q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{c}{24}}e^{2\pi i z J_0}\right)$$

$$=\frac{\theta_1(\tau,z)^2}{\eta(\tau,z)^3}\left(24\mu(\tau,z)+H(\tau)\right)$$

\# of massive rep of N=4 SCA

$$H(\tau)=2\,q^{-\frac{1}{8}}(-1+45\,q+231\,q^2+770q^3+\ \dots.)$$

dim. of irreps of M_{24}

MATHIEU AND UMBRAL MOONSHINE

Mock modular property is another important feature of this *q*-series.

$$\widehat{H}(\tau) := H(\tau) + \frac{1}{\sqrt{4i}} \int_{-\bar{\tau}}^{\infty} d\tau' \; (\tau + \tau')^{-\frac{1}{2}} \left(24\, \eta^3(\tau')\right)$$

Non-holom, transforms well

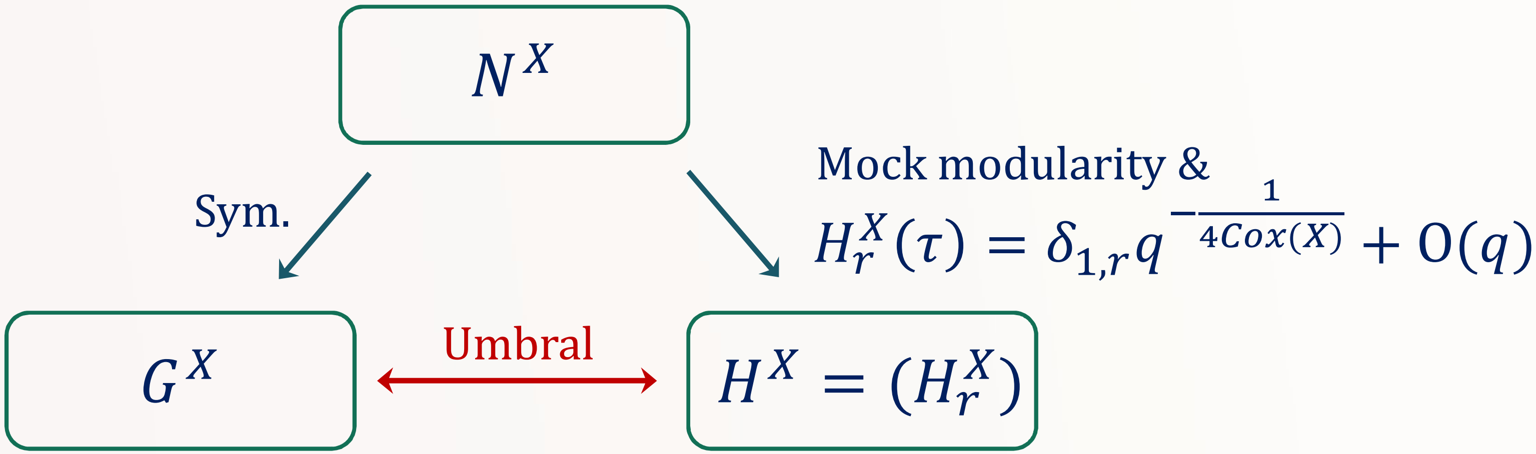
mock modular form

shadow

$$\sim \widehat{H}\left(\frac{a\tau + b}{c\tau + d}\right)$$

MATHIEU AND UMBRAL MOONSHINE

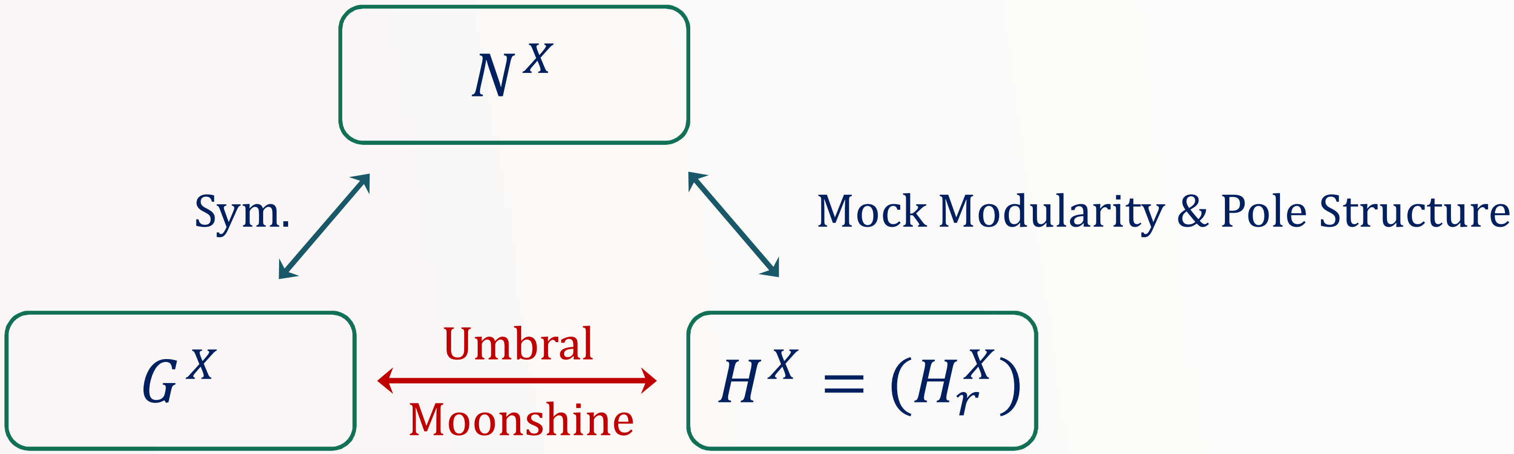
Umbral Moonshine was found to be the natural generalisation of Mathieu moonshine. For each of the 23 Niemeier lattices N^X, the lattice symmetries define a finite group G^X, while the root system X determines a vector of mock modular forms $H^X = (H_r^X)$.
[MC-Duncan-Harvey, ‘13]



MATHIEU AND UMBRAL MOONSHINE

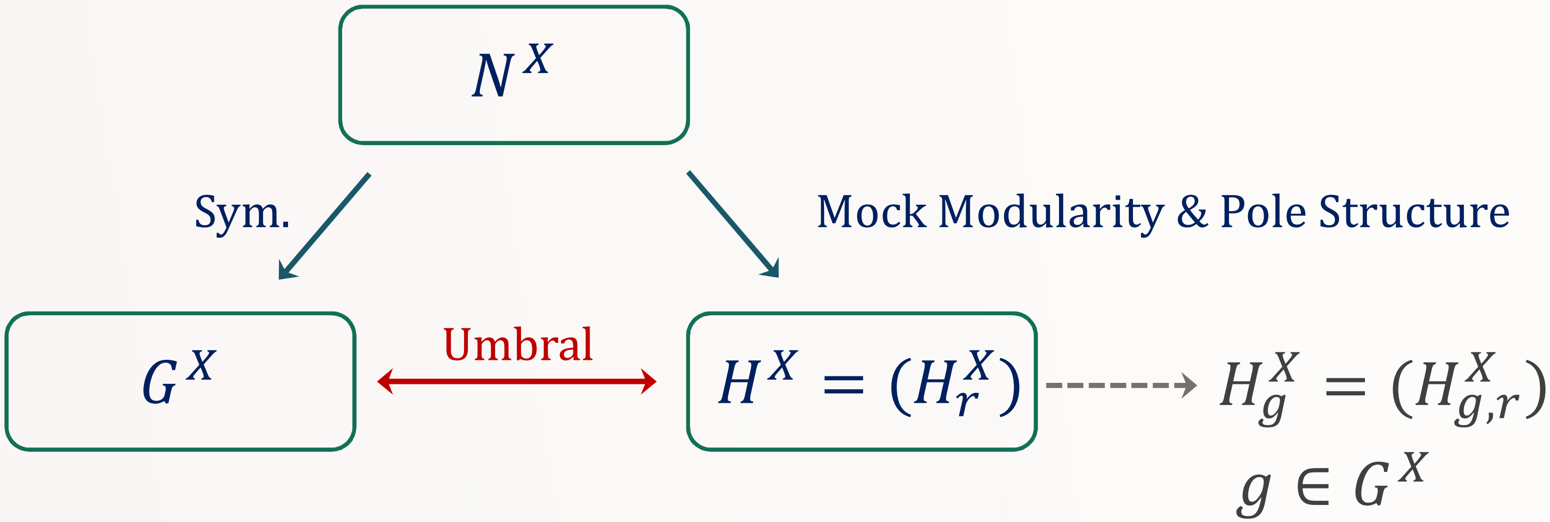
Umbral Moonshine was found to be the natural generalisation of Mathieu moonshine. *e.g.* When taking $X = 24\,A_1$, then $G^X \cong M_{24}$,

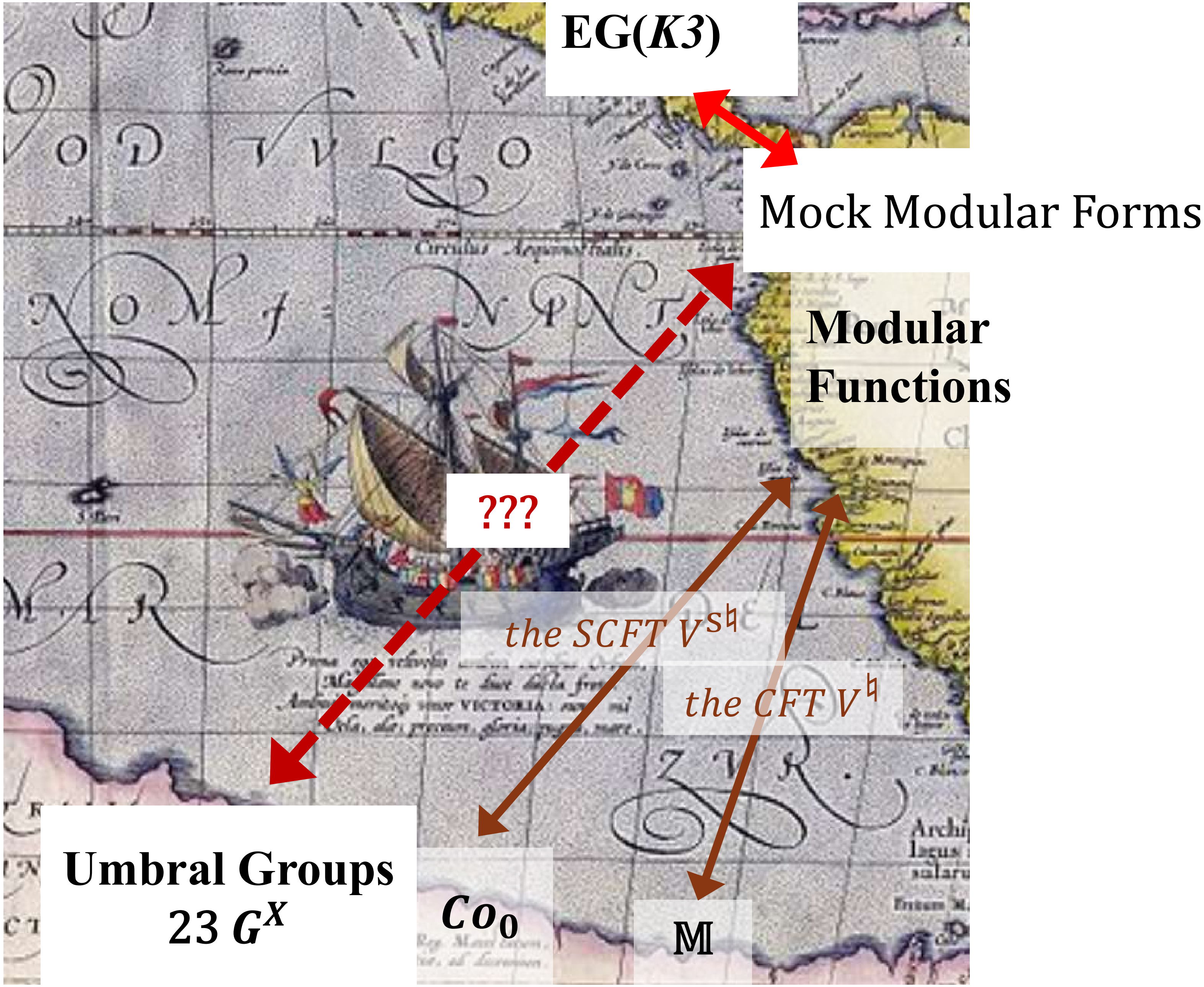
$$H_1^X(\tau) = H(\tau) = 2\,q^{-\frac{1}{8}}\left(-1 + 45\,q + 231\,q^2 + 770q^3 + \,\ldots.\right).$$



MATHIEU AND UMBRAL MOONSHINE

The Umbral Moonshine Conjecture states the existence of the natural G^X-representations that whose graded character coincides with some specific mock modular form $H_g^X = \left(H_{g,r}^X\right)\ \forall\ g \in G^X$.





EG(K3)
Mock Modular Forms
Modular Functions
???
the SCFT V^{s♮}
the CFT V^♮
Umbral Groups
23 G^X
Co_0
M

Part II

Umbral Moonshine (UM)
and K3 String Theory

All cases of UM related to K3 string theory

For all 23 X:

[MC-Harrison '14]

$$EG(K3;\tau,z) = EG(X;\tau,z) + \frac{\theta_1(\tau,z)^2}{\eta(\tau)^6}\left(\frac{1}{2\pi i}\frac{\partial}{\partial w}\Psi^X(\tau,w)\Big|_{w=0}\right)$$

EG of corresponding ADE singularities

Contribution from the mmf

$\Psi^X(\tau,z) = H^X(\tau) * \theta_{Cox(X)}(\tau,z)$

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EG of corresponding ADE singularities

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$$\Psi^X(\tau,z) = H^X(\tau) * \theta_{Cox(X)}(\tau,z)$$

twining

$$EG_g^X(\tau,z) = EG_g(X;\tau,z) + \frac{\theta_1(\tau,z)^2}{\eta(\tau)^6}\left(\frac{1}{2\pi i}\frac{\partial}{\partial w}\Psi_g^X(\tau,w)|_{w=0}\right)$$

$$EG_g := Tr_{H_{RR}}(g \;\ldots)$$

What are the symmetry groups* of SM(K3)?

* : we restrict to sym. preserving (4,4) susy.

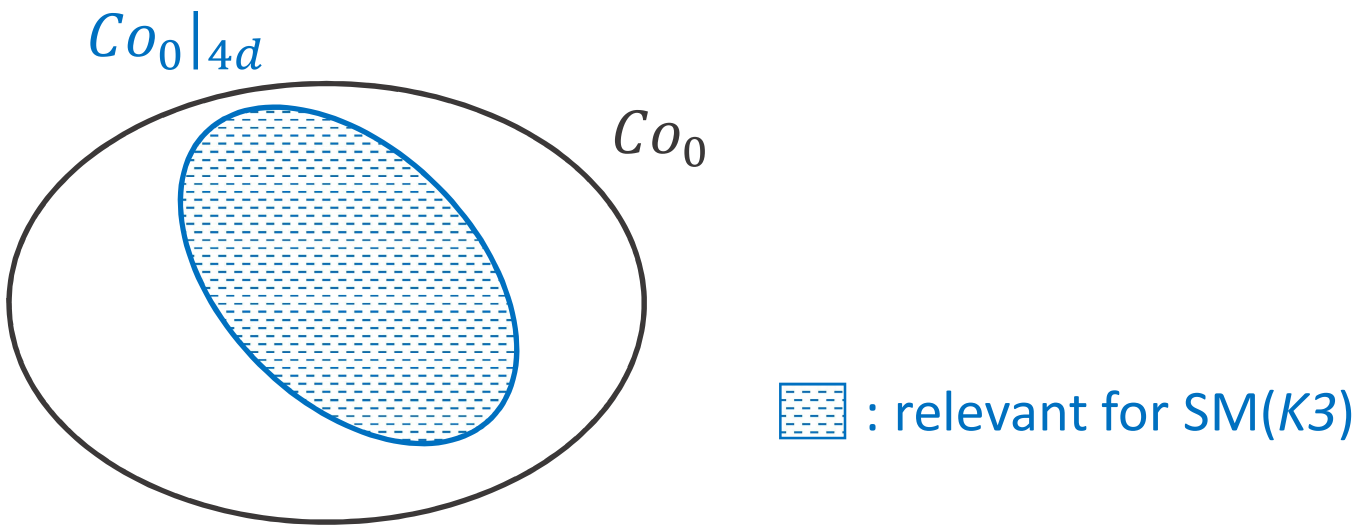
$G \cong Sym(SM_x(K3))$ for some point x in the moduli space

$\Updownarrow$

[Gaberdiel-Hohenegger-Volpato '11]

$G \in Co_0|_{4d}$

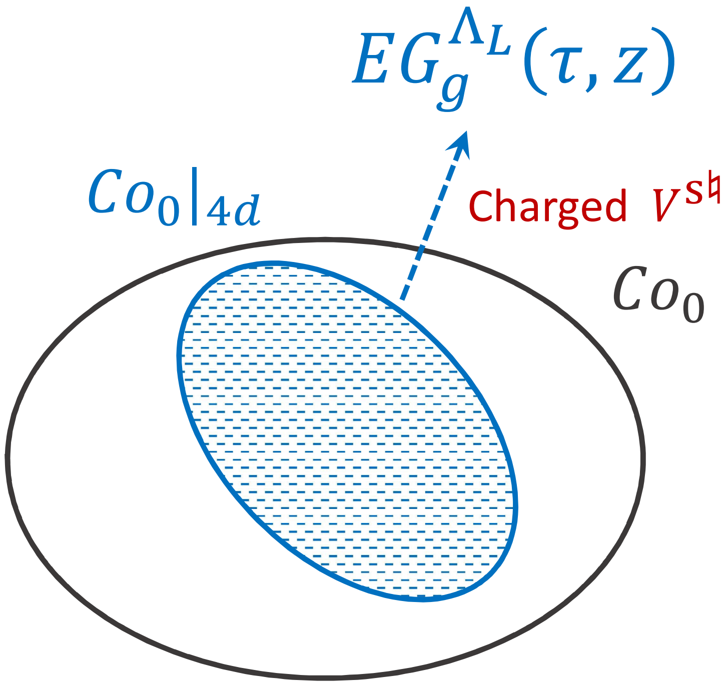
$:= \{Co_0$ subgroups fixing pointwise a 4-dim subspace in $\Lambda_L \otimes \mathbb{R}\}$



Charged Conway Moonshine and EG(K3)

Charging $V^{s\natural}$: build $U(1)$ current from 4 of the 24 fermions, add fugacity.

Charged P.F. = $EG\,(K3;\tau,z)$



[Duncan-MackCrane '15]

How do the symmetry groups act on SM(K3)?

$G \cong G' \;\not\Rightarrow G, G'$ equivalent for SM($K3$) $\Leftarrow G, G'$ conjugate under T-duality group

- Classifying conjugate classes of $O^+(4,20;\mathbb{Z})|_{4d}$ $\Rightarrow$ at most **81** distinct (physical) $EG_g(K3;\tau,z)$ at can be realised at a given point of mod. spc. [MC-Harrison-Volpato-Zimet '16]

- **56** have been explicited realised in actual SM($K3$) computations.

$$\cap$$

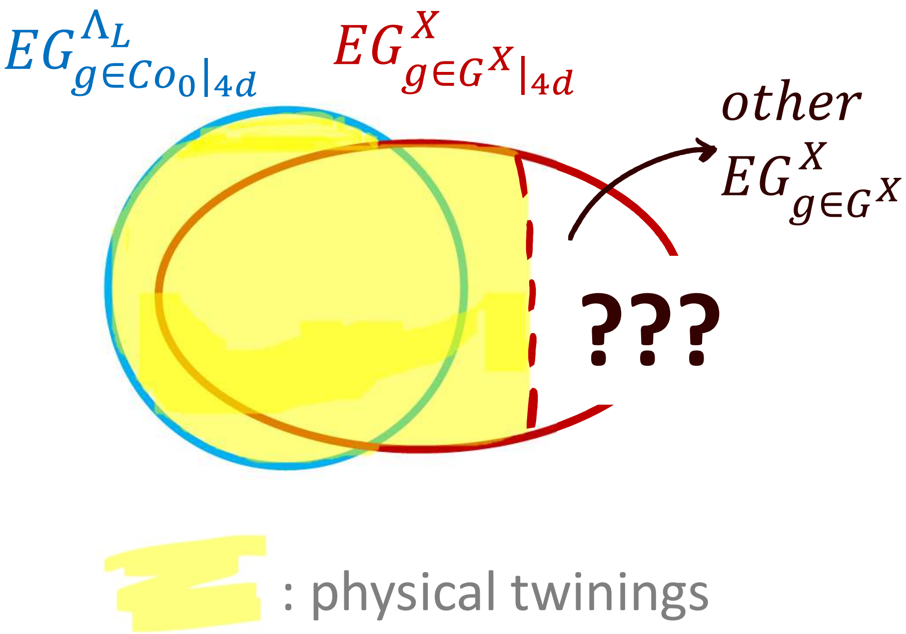
- From UM & Co-moonshine we obtain **69** distinct (fictious) elligible twining functions EG_g^X & $EG_g^{\Lambda_L}$.

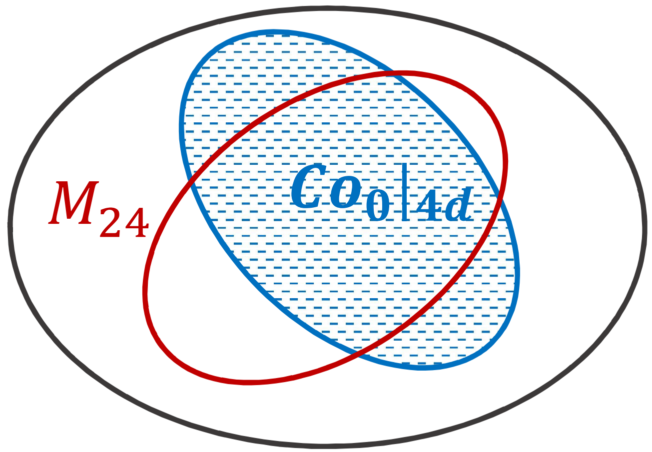
How do the symmetry groups act on SM(K3)?

A : dicated by moonshine!

From UM & Co-moonshine we obtain **69** distinct (fictious) elligible twining functions EG_g^X & $EG_g^{\Lambda_L}$. These are in fact *all there* & *all there is* for all SM(*K3*)s!

[Conjectured in MC-Harrison-Volpato-Zimet '16 proven in Paquette-Volpato-Zimet '17]





UM Modules So Far … 8 out of 23

X	G^X	max. dim. Irrep.
$3E_8$	Sym_3	2
$2A_{12}$	$SL_2(\mathbb{F}_3)$	3
$4A_6$	$\mathbb{Z}/4$	1
$4D_6$	Sym_4	3
$3D_8$	Sym_3	2
$2D_{12}$	$\mathbb{Z}/2$	1
D_{24}	1	1
$6D_4$	$3.Sym_6$	30

UM Modules So Far … 8 out of 23

[Duncan-Harvey '14]

$$H_1^{3E_8}(\tau) = \frac{2\, q^{-\frac{1}{120}}}{\prod_n (1-q^n)^2} \sum_{k,l,m} q^{b(k,l,m)} \ldots$$

$G^{3E_8} \cong Sym_3$

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[Duncan-O'Desky, MC-Duncan '17]

$\Psi^X_{mer.}$ = P.F. of chiral ghost CFT

E.g. $\Psi^{4D_6}_{mer.} \sim$

$$\prod_n \frac{\{(1-y^{-2}q^{n-1})(1-y^2q^n)\}^2}{\{(1-y^{-1}q^{n-1})(1-y^1q^n)\}^3}$$

$$\Psi^X_{mer.}(\tau,z) = \Psi^X_{polar} + \Psi^X,$$
$$\Psi^X(\tau,z) = H^X(\tau) * \theta_{Cox(X)}(\tau,z)$$

[Zwegers '05,
Dabolkar-Murthy-Zagier '12]

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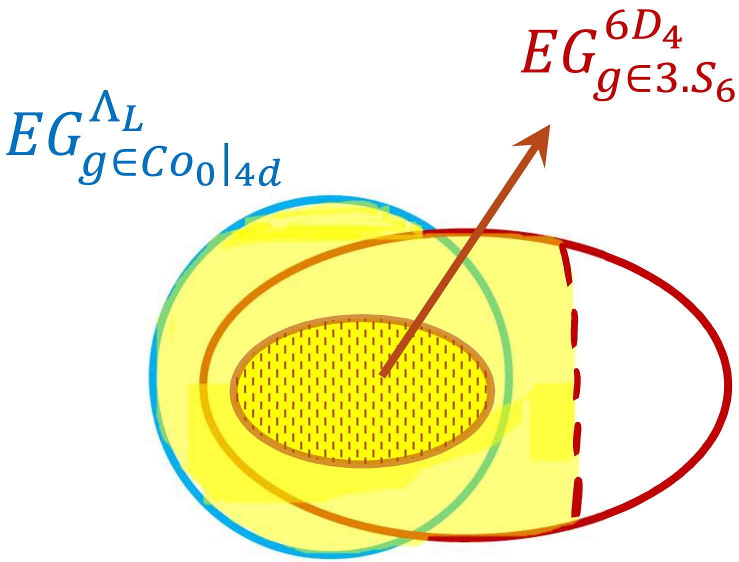
UM Modules So Far … 8 out of 23

[Anagiannis-MC-Harrison '17]

Build a ghosts + fermion orbifold which gives



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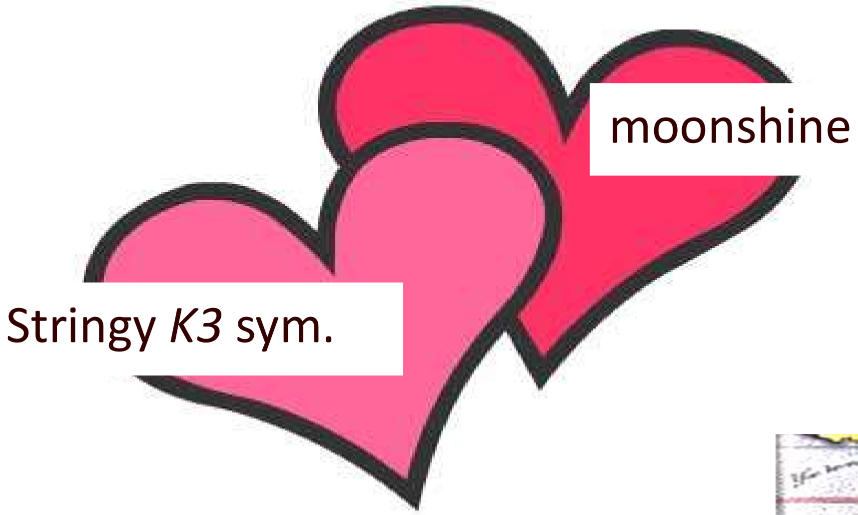
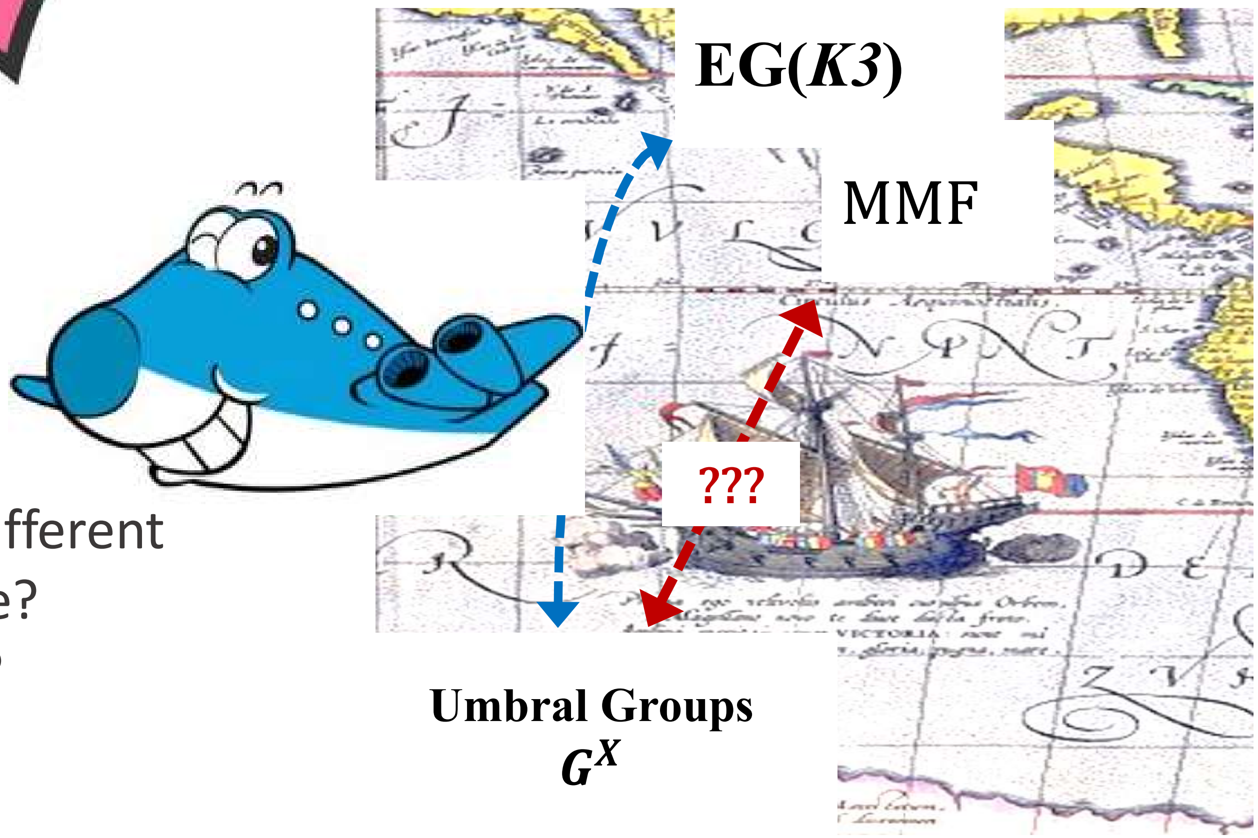




1. Combining sym at different points in mod. space?
2. Down to lower dim?
3. Reduce susy?

[Gaberdiel, Kachru, Keller, Ooguri, Paquette, Paul, Taormina, Volpato, Wendland, … ‘16-’17]

Part III
Moonshine Proliferation

- New examples keep appearing.
- A Paradigm Shift: the New Order?

Some Examples of New Moonshine Cases

Tweets

Moonshine News @realMoonshineNews 2 years

Thompson moonshine found in weight ½ modular forms! @HarveyRayhaun #moremoonshine

617 1.6K 4.9K

Moonshine News @realMoonshineNews 4 months

Moonshine for a Pariah group! O'Nan moonshine found in weight 3/2! @DuncanMertensOno #moremoonshine

179 609 1.8K

Moonshine News @realMoonshineNews 2 days

Moonshine for class numbers too! Seen on M5 branes! @MCDuncanHarrisonKachruetal #moremoonshine

12 63 219

Moonshine News @realMoonshineNews 2 min

New Moonshine on M5 Branes

An M5 branes on divisor $P \subset CY_3$ is described by an effective "MSW String", an N=(0,4) SCFT. Define generalised EG:

$$Z(\tau,\zeta) := Tr'_R(F^2(-1)^F q^{L_0-\frac{c}{24}} \bar{q}^{\widetilde{L_0}-\frac{\tilde{c}}{24}} e^{2\pi i\, \zeta\cdot q})$$

[Maldacena-Strominger-Witten '97]

E.g. wrapping 2 M5 branes on $P = \mathbb{CP}^2$

$$Z_{\mathbb{CP}^2,2}(\tau,\zeta) = \frac{3\,\hat{h}_0(\tau)}{\eta(\tau)^6}\theta_{1,1}(-\bar{\tau},\zeta) - \frac{3\,\hat{h}_1(\tau)}{\eta(\tau)^6}\theta_{1,0}(-\bar{\tau},\zeta)$$

$$h_i(\tau) = \sum_{n=0}^{\infty} H(4n+3i)q^{\frac{4n+3i}{4}}, \qquad i = 0,1.$$

[Vafa-Witten '94]

a mock modular form — Hurwitz class number

$48 \times h_i(\tau)$: M_{23} moonshine function

[MC-Duncan-Mertens ??]

New Moonshine on M5 Branes

[Minahan-Nemeschansky-Vafa-Warner '98]

E.g. wrapping a single M5 branes on $P = del\ Pezzo\ B_9\ (\frac{1}{2}K3\ )$

$$Z_{\frac{1}{2}K3}(\tau,\zeta) = \frac{f_1(\tau)}{\eta(\tau)^6}\theta_{1,1}(-\bar{\tau},\zeta) - \frac{f_0(\tau)}{\eta(\tau)^6}\theta_{1,0}(-\bar{\tau},\zeta)$$

$$\underline{f_0(\tau) + f_1(\tau)} = \frac{E_4(\tau)}{\eta(\tau)^6}\sum_{n\in\mathbb{Z}}(-1)^n q^{\frac{n^2}{4}}$$

a modular form

M_{11} moonshine function

New Paradigm: Is this Moonshine?

Connections between modular objects and finite groups seem more common if we allow for the following new features:

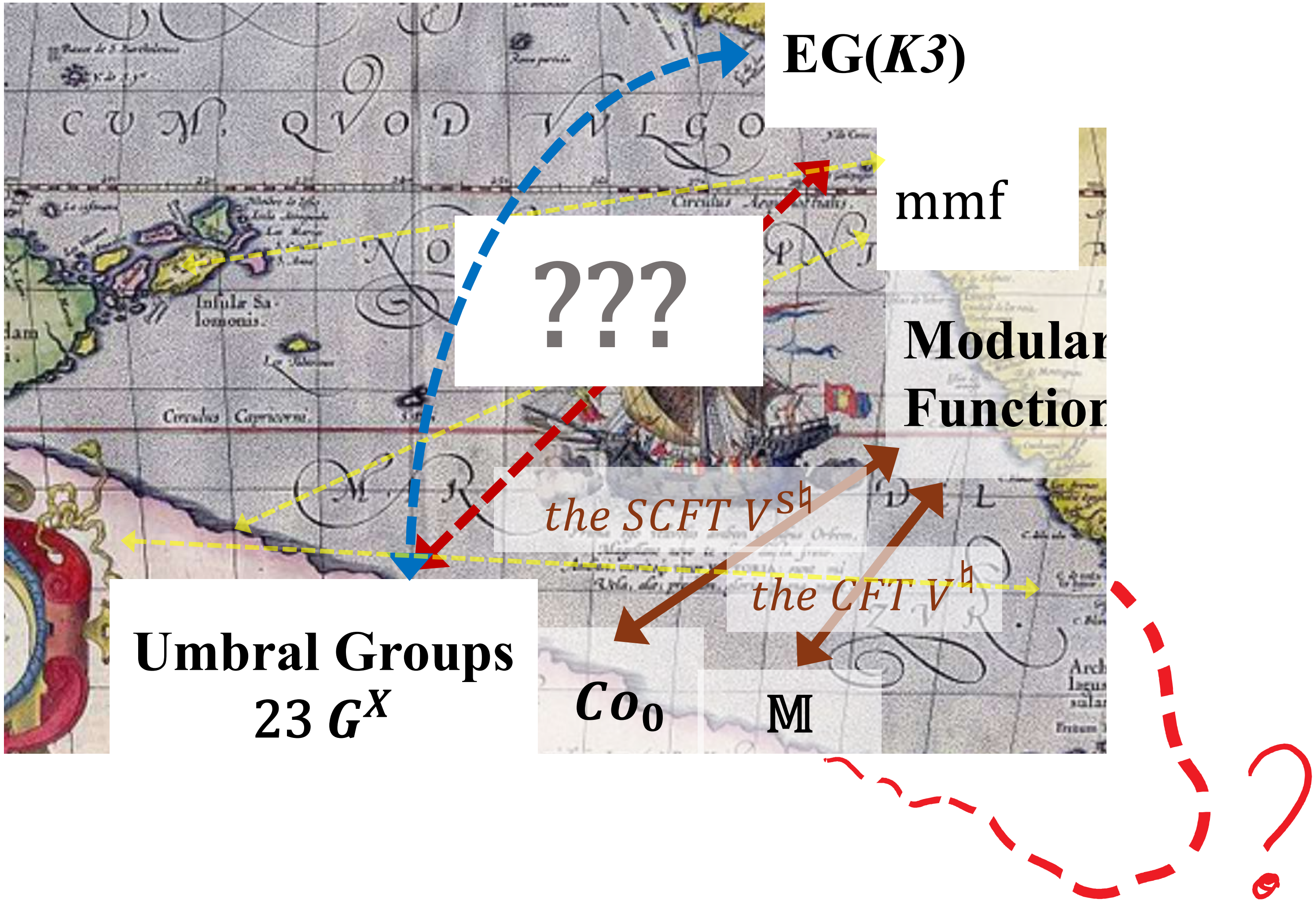
1. Group rep's come with signs:

$V = \oplus_n \left(V_{0,n} \oplus V_{1,n}\right), \qquad f_g(\tau) := \sum_n q^n \sum_{i\in\{0,1\}} (-1)^i \left(Tr_{V_{i,n}} g\right)$

2. The functions are no longer determined uniquely by modularity and pole structure; there are cusp forms to add.

A much more flexible game!

What's the landscape? What's the Physics?



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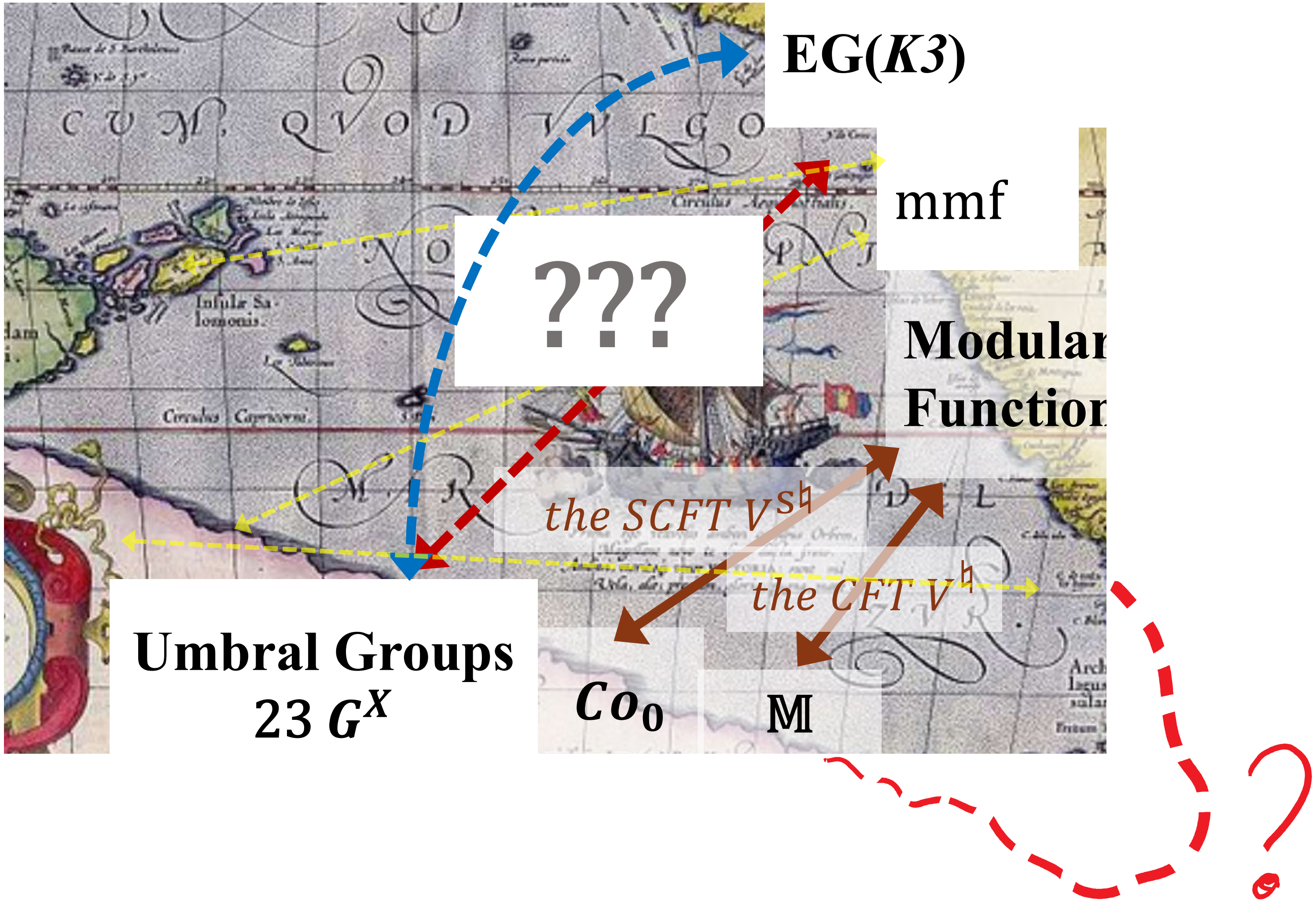

THANK YOU!