# Green-Schwarz superstring on a lattice

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### **Junior Research Group "Gauge fields from strings"**

**1601.04670, 1605.01726, with M. Bianchi, L. Bianchi, Leder, Vescovi 1702.02164, 1707.xxxxx, with L. Bianchi, Leder, Töpfer, Vescovi** 

# **Strings 2017, Tel Aviv**

## **Motivation**

*g<sup>S</sup>* = g witl Within *Nao*/Of Troband Chaol in the coupling **R**  $\overline{\phantom{a}}$  (2)  $\overline{\phantom{a}}$  (2)  $\overline{\phantom{a}}$  (2)  $\overline{\phantom{a}}$  (2)  $\overline{\phantom{a}}$  (3)  $\overline{\phantom{a}}$ Beautiful progress in obtaining within AdS/CFT results exact in the coupling



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- *<sup>N</sup>* (5)  $\bm{m}$  integrability and  $\bm{m}$  and  $\bm{m}$  $\mathcal{L}(\mathcal{L})$  from supersymmetric localization (BPS observable) (BPS observable) (BPS observable) (BPS observable)  $\blacktriangleright$  from integrability
- p  $\lambda$ <sup>*m*</sup> supersymr <sup>4</sup>⇡↵<sup>0</sup> (7)  $\blacktriangleright$  from supersymmetric localization

## **Motivation**

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- *<sup>N</sup>* (5) **Firm integrability (assumed)**  $\mathcal{I} = \mathcal{I} \cup \mathcal{I}$
- **•** from supersymmetric localization (BPS observable) 4 (7)

In the world-sheet string theory integrability only classically, localization not formulated.

Call for genuine 2d QFT to cover the finite-coupling region.

## Lattice QFT methods in AdS/CFT

Consolidated program on 4d CFT side, subtleties with supersymmetry, control on the perturbative region. Lattice 4d production<br>production<br>production

 $\frac{1}{4}$ [Catterall, Damgaard, DeGrand, Giedt, Schaich…]



## Lattice QFT methods in AdS/CFT



**[previous study: Roiban McKeown 2013]**

### Features:

- $\triangleright$  2d: computationally cheap
- $\triangleright$  no supersymmetry (only in target space!)
- $\blacktriangleright$  all gauge symmetries are fixed, only scalar fields

Non-trivial 2d qft with strong coupling analytically known, finite-coupling (numerical) prediction.

## Green-Schwarz string action in  $AdS_5 \times S^5$  + RR flux

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Symmetries:

- $\triangleright$  global  $PSU(2, 2|4)$ , local bosonic (diffeomorphism) and fermionic ( $\kappa$ -symmetry)
- $\blacktriangleright$  classical integrability

manifest for  $\sigma$ -model on  $G/H = \frac{PSU(2,2|4)}{SO(1,4) \times SO(5)}$ .

**Explicitly** 

$$
S = g \int d\tau d\sigma \left[ \partial_a X^\mu \partial^a X^\nu G_{\mu\nu} + \bar{\theta} \Gamma (D + F_5) \theta \partial X + \bar{\theta} \partial \theta \bar{\theta} \partial \theta + \dots \right]
$$

Quantized semiclassically

$$
X = X_{\text{cl}} + \tilde{X} \qquad \longrightarrow \qquad \Gamma = g \left[ \Gamma_0 + \frac{\Gamma_1}{g} + \frac{\Gamma_2}{g^2} + \dots \right]
$$

Formally power-counting non-renormalizable, judicious choice of regularization is needed to verify UV finiteness.

# The cusp anomaly of *N* = 4 SYM from string theory The cusp anomaly of *N* = 4 SYM from string theory

Completely solved via integrability. [Beisert Eden Staudacher 2006] Completely solved via integrability.

Expectation value of a light-like cusped Wilson loop Expectation value of a light-like cusped Wilson loop Previous study

$$
\langle W[C_{\rm cusp}] \rangle \sim e^{-f(g)\phi \ln \frac{L_{\rm IR}}{\epsilon_{\rm UV}}}
$$
\nAdSCFT

\n
$$
Z_{\rm cusp} = \int [D\delta X][D\delta\theta] e^{-S_{\rm IIB}(X_{\rm cusp} + \delta X, \delta\theta)}
$$



String partition function with "cusp" boundary conditions.

 $A a 35$  approach preferred values of  $z^2$ h<sub>S</sub>cus</sub>pin =  $\frac{1}{2}$  $x^{\top} = \tau$   $x^{-} = -\frac{\tau}{2\sigma}$ [*DX*][*D* ] *eS*cusp *dg*  $\frac{u}{2}$   $-\frac{1}{2}$  $\sqrt{\sigma}$  2 $\sigma$  9 2  $x^+ = \tau$   $x^- = -\frac{1}{2\pi}$  $\alpha$   $=$   $-\frac{1}{2}$ *X*cusp is the minimal surface  $ds^2_{AdS_5} =$  $dz^{2} + dx^{+}dx^{-} + dx^{*}dx$  $\frac{dx}{z^2} + dx \frac{dx}{z^2}$   $x^{\pm} = x^3 \pm x^0$   $x = x^1 + ix^2$  $z =$  $\sqrt{\tau}$  $\sigma$  $x^+ = \tau \qquad x^- = -\frac{1}{26}$  $2\sigma$  $x^+x^- = -\frac{1}{2}$ 2 *z*2 ending on a null cusp, since  $x^+x^- = 0$  at the boundary  $z = 0$ . [Giombi Ricci Roiban Tseytlin 2009]

### The cusp anomaly of  $\mathcal{N}=4$  SYM from string theory *xmx<sup>m</sup>* = *x*+*x* + *x*⇤*x, x<sup>±</sup>* <sup>=</sup> *<sup>x</sup>*<sup>3</sup> *<sup>±</sup> <sup>x</sup>*<sup>0</sup> *, x* <sup>=</sup> *<sup>x</sup>*<sup>1</sup> <sup>+</sup> *ix*<sup>2</sup>

*<u>Completely solved via integrability. [Beisert Eden Staudacher 2006]*</u> **Completely solved via integrability** [Beisert Eden Staudacher 2006]

Expectation value of a light-like cusped Wilson loop

AlsoCFT

\n
$$
\langle W[C_{\text{cusp}}] \rangle \sim e^{-f(g)\phi \ln \frac{L_{\text{IR}}}{\epsilon_{\text{UV}}}}
$$
\n
$$
Z_{\text{cusp}} = \int [D\delta X][D\delta \theta] e^{-S_{\text{IIB}}(X_{\text{cusp}} + \delta X, \delta \theta)} = e^{-\Gamma_{\text{eff}}} \equiv e^{-f(g\sqrt{V_2})}
$$

String partition function with "cusp" boundary conditions, String partition function with "cusp" boundary conditions.

### Perturbatively analytic calculations are available for the scaling functions are available functions a

$$
f(g)|_{g\to 0} = 8g^2 \left[ 1 - \frac{\pi^2}{3} g^2 + \frac{11\pi^4}{45} g^4 - \left( \frac{73}{315} + 8\zeta_3 \right) g^6 + \ldots \right]
$$
 [Bern et al. 2006]  

$$
f(g)|_{g\to \infty} = 4g \left[ 1 - \frac{3\ln 2}{4\pi} \frac{1}{g} - \frac{K}{16\pi^2} \frac{1}{g^2} + \ldots \right]
$$
 [Gubser Klebanov Polyakov 02]  
[Frolov Tseytlin 02][Giombi et al. 2009]

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\n
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Z_{\text{cusp}} = \int [D\delta X][D\delta \theta] e^{-S_{\text{IIB}}(X_{\text{cusp}} + \delta X, \delta \theta)} = e^{-\Gamma_{\text{eff}}} \equiv e^{-f(g)} V_2
$$

String partition function with "cusp" boundary conditions.

### A lattice approach prefers expectation values

$$
\langle S_{\text{cusp}} \rangle = \frac{\int [D\delta X][D\delta \Psi] S_{\text{cusp}} e^{-S_{\text{cusp}}}}{\int [D\delta X][D\delta \Psi] e^{-S_{\text{cusp}}} } = -g \frac{d \ln Z_{\text{cusp}}}{dg} \equiv g \frac{V_2}{8} f'(g)
$$
  

$$
S_{\text{cusp}} = g \int \mathcal{L}_{\text{cusp}}
$$

#### Cohwarz otripa in the pull euen hookareund  $\frac{d}{dt}$  dispersion relation for the classical string surface. For example, the classical s Green-Schwarz string in the null cusp background

AdS lightcone) gauge-fixed a The (AdS lightcone) gauge-fixed action for fluctuations above the null cusp is In the continuum, the *AdS*<sup>5</sup> ⇥ *<sup>S</sup>*<sup>5</sup> superstring "cusp" action, which describes quantum fluctuations above the null cusp is an action of the null cusp is a background can be written action for the multipl The (AdS lightcone) gauge-fixed action for fluctuations above the null cusp is

$$
S_{\text{cusp}} = g \int dt ds \mathcal{L}_{\text{cusp}}
$$
\n[Giombi Ricci Roiban Tseytlin 2009]\n
$$
\mathcal{L}_{\text{cusp}} = |\partial_t x + \frac{1}{2}x|^2 + \frac{1}{z^4} |\partial_s x - \frac{1}{2}x|^2 + \left( \partial_t z^M + \frac{1}{2} z^M + \frac{i}{z^2} z_N \eta_i \left( \rho^{MN} \right)^i{}_j \eta^j \right)^2 + \frac{1}{z^4} \left( \partial_s z^M - \frac{1}{2} z^M \right)^2
$$
\n
$$
+ i \left( \theta^i \partial_t \theta_i + \eta^i \partial_t \eta_i + \theta_i \partial_t \theta^i + \eta_i \partial_t \eta^i \right) - \frac{1}{z^2} \left( \eta^i \eta_i \right)^2
$$
\n
$$
+ 2i \left[ \frac{1}{z^3} z^M \eta^i \left( \rho^M \right)_{ij} \left( \partial_s \theta^j - \frac{1}{2} \theta^j - \frac{i}{z} \eta^j \left( \partial_s x - \frac{1}{2} x \right) \right) + \frac{1}{z^3} z^M \eta_i (\rho^{\dagger}_M)^{ij} \left( \partial_s \theta_j - \frac{1}{2} \theta_j + \frac{i}{z} \eta_j \left( \partial_s x - \frac{1}{2} x \right)^* \right)
$$

- 8 bosons:  $x, x^*, z^M$   $(M=1,\cdots, 6),$   $z=\sqrt{z_Mz^M};$  $\mathbf{R} \text{ becomes } \mathbf{R} \text{ becomes } \mathbf{R} \text{ is }$ **b** 8 bosons:  $x, x^2, z^{11}$  ( $M = 1, \dots, 6$ ),  $z = \sqrt{2M}z^{11}$ ; ▶ 8 bosons:  $x, x^*, z^M$  ( $M = 1, \cdots, 6$ ),  $z = \sqrt{z_M z^M}$ ;
- 8 fermions:  $\theta^i=(\theta_i)^\dagger\,$   $n^i=(n_i)^\dagger\,$   $i=1,2,3,4\,$  complex Graßmann: ▶ 8 fermions:  $\theta^i = (\theta_i)^\dagger$ ,  $\eta^i = (\eta_i)^\dagger$ ,  $i = 1, 2, 3, 4$ , complex Graßmann;

symmetry and do not carry (Lorentz) spinor indices. The matrices ⇢*<sup>M</sup>*

 $M$  are off diedened bleeke of  $C_{\mathcal{O}}(c)$ . Directmentices introduces 7 auxiliary fields, one scalar and a *SO*(6) vector field *M*, with the following  $\rho^M$  are off-diagonal blocks of  $SO(6)$  Dirac matrices

 $\frac{M}{2}$ 

 $\blacktriangleright$   $(\rho^{MN})^i_j$  are the  $SO(6)$  generators

*decorranced*<br>*decisionmetr i* is  $SO(6) \times SO(2)$ ⇠ interactions at most  $\overline{\phantom{a}}$ **Fermionic interactions at most quartic.** Remnant global symmetry is  $SO(6) \times SO(2)$ . **0 <u>∤</u> ∤** 10 **∤ 10 ∤** 10 **∤** Fermionic interactions at most quartic.

### Lattice QFT basics

Discretize Euclidean worldsheet in a grid of lattice spacing *a*, size *L* = *N a*. Fields  $\phi \equiv \phi_n$  defined at  $\xi = (a n_1, a n_2) \equiv a n$ . a) natural cutoff  $-\frac{\pi}{a} < p_\mu \leq \frac{\pi}{a}$ b) path integral measure  $[D\phi] = \prod_n d\phi_n$ .

Then  $\int \prod_n d\phi_n \, e^{-S_{\rm discr}}$  via Monte Carlo: generate an ensamble  $\{\Phi_1,\ldots,\Phi_K\}$ of field configurations, each weighted by  $P[\Phi_i] = \frac{e^{-S_E[\Phi_i]}}{Z}$ .

Ensemble average  $\langle A \rangle = \int [D\Phi] P[\Phi] A[\Phi] = \frac{1}{K}$  $\sum_{i=1}^K A[\Phi_i] + \mathcal{O}(\frac{1}{\sqrt{\epsilon}})$ 1 *K*  $\overline{)}$ 

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Graßmann-odd fields are formally integrated out:  $P[\Phi_i] = \frac{e^{-S_E[\Phi_i]}\det \mathcal{O}_F}{Z}$ *Z* it:  $P[\Phi_i] = \frac{e-\mu}{\sigma}$ *Z* Graßma

 $\blacktriangleright$  action must be quadratic in fermions

$$
\mathbf{X} = \mathbf{X} - \mathbf{X}
$$
 Introduce auxiliary fields (complex bosons)

**• determinant must be definite positive** 

$$
\det O_F \ \longrightarrow \ \sqrt{\det(O_F^{\dagger} \, O_F)} \equiv \int \stackrel{\cdot }{D\zeta} \, D\bar{\zeta} \, \, e^{- \int d^2 \xi \, \bar{\zeta} \left(O_F^{\dagger} \, O_F \right)^{- \frac{1}{2}} \zeta}
$$

### Lattice QFT basics Linearization and discretization: first setting  $T_{\rm eff}$  and  $T_{\rm eff}$  action has  $q_{\rm eff}$  and  $q_{\rm eff}$  action has  $q_{\rm eff}$  interactions interactions interactions in

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$$
\mathsf{M} = \mathsf{M} \cdot \
$$

 $\blacktriangleright$  determinant must be definite positive *<sup>F</sup> O<sup>F</sup>* ) ⌘

$$
\text{Pf } O_F \ \longrightarrow \ (\det O_F^{\dagger} \ O_F)^{\frac{1}{4}} \equiv \int D\zeta \ D\bar{\zeta} \ e^{-\int d^2\xi \ \bar{\zeta} \left(O_F^{\dagger} O_F\right)^{-\frac{1}{4}} \zeta}
$$

# Linearization

## Four-fermion interactions<br>Four-fermion interactions

- Linearization via Hubbard-Stratonovich transformation Linearization via Hubbard-Stratonovich transformation *<sup>z</sup>*<sup>2</sup> (⌘2) *z*2

$$
\exp\left\{-\,g\int dt\,ds\,\mathcal{L}_4\right\}\sim\int d\phi\,d\phi^M\,\exp\Big\{-\,g\int dt\,ds\,\mathcal{L}_{\rm aux}\,\Big\}
$$

$$
\exp\left\{-g\int dtds\Big[-\frac{1}{z^2}\left(\eta^i\eta_i\right)^2+\left(\frac{i}{z^2}z_N\eta_i\rho^{MN^i}{}_j\eta^j\right)^2\Big]\right\}
$$
  
\$\sim \int D\phi D\phi^M \exp\Big\{-g\int dtds\Big[\frac{1}{2}\phi^2+\frac{\sqrt{2}}{z}\phi\,\eta^2+\frac{1}{2}(\phi\_M)^2-i\frac{\sqrt{2}}{z^2}\phi^M\left(\frac{i}{z^2}z\_N\eta\_i\rho^{MN^i}{}\_j\eta^j\right)\Big]\Big\}\$.

*.*

 $\blacktriangleright$  +7 bosonic auxiliary fields  $\phi,$   $\phi^M$  ( $M=1,\cdots,6)$ 

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$$

$$
\exp\left\{\bigodot g \int dt ds \Big[-\frac{1}{z^2} \left(\eta^i \eta_i\right)^2 + \left(\frac{i}{z^2} z_N \eta_i \rho^{MN}{}^i{}_j \eta^j\right)^2\Big]\right\}
$$
hermitian  

$$
\sim \int D\phi D\phi^M \exp\Big\{-g \int dt ds \Big[\frac{1}{2}\phi^2 + \frac{\sqrt{2}}{z}\phi \eta^2 + \frac{1}{2}(\phi_M)^2 - \left(\frac{\sqrt{2}}{z^2}\phi^M \left(\frac{i}{z^2} z_N \eta_i \rho^{MN}{}^i{}_j \eta^j\right)\right)\Big\}.
$$

*.*

- $\blacktriangleright$  +7 bosonic auxiliary fields  $\phi,$   $\phi^M$  ( $M=1,\cdots,6)$ 
	- $\lambda_{\text{u} \text{x}}$  is not hern ⌘*†*  $\blacktriangleright$   $\mathcal{L}_{\text{aux}}$  is not hermitian,  $e^{-\frac{b^2}{4a}} = \int dx e^{-a x^2 + i b x}, b \in \mathbb{R}$ .

### Green-Schwarz string in the null cusp background question being whether the latter is treatable via standard reweighting. Below we will see

After linearization the Lagrangian reads  $(m\sim P_+)$  $\Delta$ ttar linoarization the Lagrangian reade  $(m_{\alpha}, D_{\alpha})$ linearization) should be provided to explore the full nonperturbative region. inieanzation the Lagrangian reads  $\left\langle m\sim P_+\right\rangle$ After linearization the Lack After inicalization the Lagrangian reads  $(n\ell \sim 1 + p)$ 

$$
\mathcal{L}_{\text{cusp}} = |\partial_t x + \frac{m}{2} x|^2 + \frac{1}{z^4} |\partial_s x - \frac{m}{2} x|^2 + (\partial_t z^M + \frac{m}{2} z^M)^2 + \frac{1}{z^4} (\partial_s z^M - \frac{m}{2} z^M)^2
$$
  
+  $\frac{1}{2} \phi^2 + \frac{1}{2} (\phi_M)^2 + \psi^T O_F \psi$ ,

 $\lim_{n \to \infty} \frac{1}{n!} = \lim_{n \to \infty} \frac{1}{n} \int_{0}^{n} \frac{1}{n} \, d^{n} \$  $\psi = (\sigma, \sigma_i, \eta, \eta)$  and *,* ✓*i,* ⌘*<sup>i</sup>*  $\frac{1}{2}$  $\overline{a}$  $\equiv (\theta^i,$  $\overline{a}$  $\overline{a}$  $\theta^{\pmb{i}}, \theta_{\pmb{i}}, \eta^{\pmb{i}}, \eta_{\pmb{i}}$  $v_i$ ) and  $\overline{c}$ where  $\psi \equiv (\theta^i, \theta_i, \eta^i, \eta_i)$  and



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$$
  
+  $\frac{1}{2} \phi^2 + \frac{1}{2} (\phi_M)^2 + \psi^T O_F \psi$ ,

 $\lim_{n \to \infty} \frac{1}{n!} = \lim_{n \to \infty} \frac{1}{n} \int_{0}^{n} \frac{1}{n} \, d^{n} \$  $\psi = (\sigma, \sigma_i, \eta, \eta)$  and *,* ✓*i,* ⌘*<sup>i</sup>*  $\frac{1}{2}$  $\overline{a}$  $\equiv (\theta^i,$  $\overline{a}$  $\overline{a}$  $\theta^{\pmb{i}}, \theta_{\pmb{i}}, \eta^{\pmb{i}}, \eta_{\pmb{i}}$  $v_i$ ) and  $\overline{c}$ where  $\psi \equiv (\theta^i, \theta_i, \eta^i, \eta_i)$  and



conjugate <sup>11</sup>, thus formally integrating out generates a Pfaan Pf *O<sup>F</sup>* rather than a determi-As  $A^{\dagger} \neq A$ , Pfaffian is complex.  $Pf(\mathcal{O}_F) = e^{i\theta} (O_F O_F^{\dagger})^{\frac{1}{4}}$ . As  $A^\dagger \neq A$ , Pfaffian is complex:  ${\rm Pf}({\cal O}_F) = e^{i\theta} \, (O_F O_F^{-\dagger})^{\frac{1}{4}}$  .  $\frac{1}{4}$ .

# **Phase problem bilinear** *b b b b b b b b b b b b b b b b b b b b b b b b b b b b b b b b b*

 $\mathsf{Even}$  with  $\mathrm{Pf}(\mathcal{O}_F) = e^{i\theta} \, (O_F O_F^{-\dagger})^{\frac{1}{4}}$  $\bar{4}$  , vev's can be still obtained via reweighting:

$$
\langle A \rangle = \frac{\int D\Phi \, A \, \text{Pf}(O_F) \, e^{-S[\Phi]}}{\int D\Phi \, \text{Pf}(O_F) \, e^{-S[\Phi]}} \\
= \frac{\int D\Phi \, D\zeta \, D\bar{\zeta} \, A \, e^{i\theta} \, e^{-S[\Phi] - \int d^2\xi \, \bar{\zeta} \left(\mathcal{O}_F \mathcal{O}_F^\dagger\right)^{-\frac{1}{4}} \zeta}}{\int D\Phi \, D\zeta \, D\bar{\zeta} \, e^{i\theta} \, e^{-S[\Phi] - \int d^2\xi \, \bar{\zeta} \left(\mathcal{O}_F \mathcal{O}_F^\dagger\right)^{-\frac{1}{4}} \zeta}} = \frac{\langle A \, e^{i\theta} \rangle_{\theta=0}}{\langle e^{i\theta} \rangle_{\theta=0}}
$$

It gives meaningful results as long as the phase does not averages to zero.

# **Phase problem bilinear** *b b b b b b b b b b b b b b b b b b b b b b b b b b b b b b b b b*

 $\mathsf{Even}$  with  $\mathrm{Pf}(\mathcal{O}_F) = e^{i\theta} \, (O_F O_F^{-\dagger})^{\frac{1}{4}}$  $\bar{4}$  , vev's can be still obtained via reweighting: r ə yalı k

Step Problem

\nven with 
$$
Pf(\mathcal{O}_F) = e^{i\theta} \left( O_F O_F^{-\dagger} \right) \frac{1}{4}
$$
, vev's can be still obtained via reweighting:

\n
$$
\langle \mathcal{A} \rangle = \frac{\int D\Phi \, A \, Pf(O_F) \, e^{-S[\Phi]}}{\int D\Phi \, Pf(O_F) \, e^{-S[\Phi]}} = \frac{\int D\Phi \, D\zeta \, D\bar{\zeta} \, A \, e^{i\theta} \, e^{-S[\Phi] - \int d^2\xi \, \bar{\zeta} (\mathcal{O}_F \mathcal{O}_F^{\dagger})^{-\frac{1}{4}} \zeta}}{\int D\Phi \, D\zeta \, D\zeta \, e^{i\theta} \, e^{-S[\Phi] - \int d^2\xi \, \bar{\zeta} (\mathcal{O}_F \mathcal{O}_F^{\dagger})^{-\frac{1}{4}} \zeta}} = \frac{\langle \mathcal{A} \, e^{i\theta} \rangle_{\theta=0}}{\langle e^{i\theta} \rangle_{\theta=0}}
$$
\ngives meaningful results as long as the phase does not averages to zero.

It gives meaningful results as long as the phase does not averages to zero. ha<sub>i a</sub> **e** phase do



Dedicated algorithms: active field of study, no general proof of convergence.

The phase is implicit in the linearization, like  $e^{-\frac{b^2}{4\,a}} = \int dx\, e^{-a\,x^2 + i\,b\,x}$ 

Consider a simple SO(4) invariant four-fermion interaction

**[Catterall 2015]**

$$
\mathcal{L}_{4F} = \frac{1}{2} \epsilon_{abcd} \psi^a(x) \psi^b(x) \psi^c(x) \psi^d(x) \equiv \Sigma^{ab} \, \widetilde{\Sigma}^{ab}
$$

where  $\Sigma^{ab}=\psi^a\psi^b\,,\ \widetilde{\Sigma}^{ab}=\frac{1}{2}\epsilon_{abcd}\,\psi^c\,\psi^d.$  Introducing  $\Sigma^{ab}_{\pm}=\frac{1}{2}$ 2  $( \Sigma^{ab} \pm \widetilde{\Sigma}^{cd} )$ , rewrite the non-positive-definite part of the Boltzmann weight into the observable into t <sup>R</sup> *<sup>D</sup> <sup>A</sup>* Pf(*O<sup>F</sup>* ) *<sup>e</sup>S*[]  $\frac{2}{\sqrt{a+b}}$   $\frac{2}{\sqrt{a+b$ 

$$
\mathcal{L}_{4F}=\pm\,2\left(\Sigma^{ab}_{\pm}\right)^2
$$

just exploiting the Graßmann character of the underlying fermions. **Graßmann character of the underlying fer**  $\overline{1}$  $\mathbf{S}_{\bullet}$ 

It gives meaningful results as long as the phase does not averages to zero. ...I will come back to this later.

In our case,  $(\rho^M)^{im}(\rho^M)^{kn} = 2\epsilon^{imkn}$ , we analogously rewrite

$$
\mathcal{L}_{F4} = -\frac{1}{z^2} (\eta^2)^2 \mp \frac{2}{z^2} (\eta^2)^2 \mp \frac{1}{z^2} \Sigma_{\pm} i \Sigma_{\pm} j
$$

$$
\Sigma_i{}^j = \eta_i \eta^j \,, \qquad \widetilde{\Sigma}_j{}^i = (\rho^N)^{ik} n_N (\rho^L)_{jl} n_L \eta_k \eta^l \,, \qquad \Sigma_{\pm i}{}^j = \Sigma_i^j \pm \widetilde{\Sigma}_i^j
$$

 $\frac{1}{2}$ Choosing the good sign (-), new set of  $1 + 16$  real auxiliary fields

$$
\mathcal{L}_{\text{aux}} = \frac{12}{z} \eta^2 \phi + 6\phi^2 + \frac{2}{z} \Sigma_{\pm} i_j \phi_i^j + \phi_j^i \phi_i^j \qquad \mathcal{L}_{\text{aux}}^{\dagger} = \mathcal{L}_{\text{aux}}
$$

Antisymmetry and 5-hermiticity (*†* <sup>5</sup><sup>5</sup> = **1**, *†* <sup>5</sup> = 5) Antisymmetry and  $\Gamma_5$ -hermiticity ( $\Gamma_5^{\text{T}}\Gamma_5=\mathbb{1}, \Gamma_5^{\text{T}}=-\Gamma_5$ )

$$
O_F^{\dagger} = \Gamma_5 O_F \Gamma_5, \qquad O_F^T = -O_F
$$

.<br>.<br>. ensure positive-definite determinant  $({\rm Pf}O_F)^2=\det O_F\geq 0,$  and a real Pfaffian.

In our case,  $(\rho^M)^{im}(\rho^M)^{kn} = 2\epsilon^{imkn}$ , we analogously rewrite

$$
\mathcal{L}_{F4} = -\frac{1}{z^2} (\eta^2)^2 \mp \frac{2}{z^2} (\eta^2)^2 \mp \frac{1}{z^2} \Sigma \pm i \Sigma \pm j
$$

$$
\Sigma_i{}^j = \eta_i \eta^j \,, \qquad \widetilde{\Sigma}_j{}^i = (\rho^N)^{ik} n_N (\rho^L)_{jl} n_L \eta_k \eta^l \,, \qquad \Sigma_{\pm i}{}^j = \Sigma_i^j \pm \widetilde{\Sigma}_i^j
$$

 $\frac{1}{2}$ Choosing the good sign (-), new set of  $1 + 16$  real auxiliary fields

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$$

Antisymmetry and 5-hermiticity (*†* <sup>5</sup><sup>5</sup> = **1**, *†* <sup>5</sup> = 5) Antisymmetry and  $\Gamma_5$ -hermiticity ( $\Gamma_5^{\text{T}}\Gamma_5=\mathbb{1}, \Gamma_5^{\text{T}}=-\Gamma_5$ ) <u>.</u><br>Latmitici  $\overline{\text{city}}$   $(\Gamma_5^{\dagger} \Gamma_5)$  $\equiv$   $\frac{1}{2}$  $\vec{M}$   $\vec{M}$ *jn<sup>N</sup>* ⌘*<sup>j</sup>* <sup>2</sup>

$$
O_F^{\dagger} = \Gamma_5 O_F \Gamma_5, \qquad O_F^T = -O_F
$$

ensure positive-definite determinant  $({\rm Pf}O_F)^2=\det O_F\geq 0,$  and a real Pfaffian.

Gain in computational costs, but  $\text{Pf}O_F = \pm \sqrt{\det O_F}$ .

#### $\mathbf{r}$ *g* Where are we sign-problem free?



In contare well concreted from zero. International mixing of the Lagrangian mixing of the Lagrangian with lower dimension of the Lagrangian with lo Eigenvalue distribution of fermionic operators well separated from zero, no sign problem for  $g\geq 10,$  where nonperturbative physics is captured.

# **Discretization**

## Guiding lines for discretization

- **Example 2** Lattice perturbation theory  $\frac{a \rightarrow 0}{b}$  continuum perturbation theory
- $\blacktriangleright$  Preserve the symmetries of the model
- $\blacktriangleright$  No complex phases

### Guiding lines for discretization: continuum perturbation of the original perturbation of the original perturbation 16 *real* anti-commuting degrees of freedom. In order to be interpreted as a probability, the Pfaan should be definite positive. For this positive. For this positive,  $\mathcal{L}$

 $\blacktriangleright$  Lattice perturbation theory  $\stackrel{a \to 0}{\longrightarrow}$  continuum perturbation theory  $\overline{a}$ m p z<br>Z *D definition theory* **a** added to the one already in the one alternation in the Hubbard-Stratonovich procedure adopted. We will be will  $f(x) = \frac{1}{2} \int_0^x \int_0^$ 

In the continuum, the free kinetic part of the fermionic operator Looking at the free model in Fourier part of the fermionic operator (2.6) in Fourier transformation (2.6) in Fourier transformation (2.6) in Fourier transformation (2.6) in Fourier transformation (2.6) in Fourier transform In the continuum, the free kinetic part of the fermionic operator

$$
K_F = \begin{pmatrix} 0 & -p_0 \mathbb{1} & (p_1 - i\frac{m}{2})\rho^M u_M & 0 \\ -p_0 \mathbb{1} & 0 & 0 & (p_1 - i\frac{m}{2})\rho_M^{\dagger} u^M \\ -(p_1 + i\frac{m}{2})\rho^M u_M & 0 & 0 & -p_0 \mathbb{1} \\ 0 & -(p_1 + i\frac{m}{2})\rho_M^{\dagger} u^M & -p_0 \mathbb{1} & 0 \end{pmatrix}
$$
  
gives the contribution  $\det K_F = \left(p_0^2 + p_1^2 + \frac{m^2}{4}\right)^8$  to the one-loop partition function  

$$
\Gamma^{(1)} = -\ln Z^{(1)} = \frac{V_2}{a^2} \frac{1}{2} \int_{-\pi}^{\pi} \frac{dp_0 dp_1}{(2\pi)^2} \ln \left[ \frac{(p_0^2 + p_1^2 + m^2)(p_0^2 + p_1^2 + \frac{m^2}{2})^2 (p_0^2 + p_1^2)^5}{(p_0^2 + p_1^2 + \frac{m^2}{4})^8} \right]
$$

(3.1)

$$
=-\frac{3\ln 2}{8\pi}m^2\,V_2
$$

### Guiding lines for discretization

Wilson-like lattice operator

A naive discretization *p<sup>µ</sup>* !

▶ Lattice perturbation theory  $\frac{a\rightarrow 0}{\rightarrow}$  continuum perturbation theory **2** )

A naive discretization  $p_\mu \rightarrow \stackrel{\circ}{p}_\mu \equiv \frac{1}{a} \sin(a \, p_\mu)$  leads to fermion doublers,

$$
K_F=\begin{pmatrix} 0 & -\mathring{p_0}\mathbb{1} & (\mathring{p_1}-i\frac{m}{2})\rho^Mu_M & 0 \\ -(\mathring{p_1}+i\frac{m}{2})\rho^Mu_M & 0 & 0 & (\mathring{p_1}-i\frac{m}{2})\rho_M^{\dagger}u^M \\ 0 & -(\mathring{p_1}+i\frac{m}{2})\rho_M^{\dagger}u^M & -\mathring{p_0}\mathbb{1} & 0 \end{pmatrix}
$$

*<sup>a</sup>* sin(*a pµ*) leads to fermion doublers,

 $\frac{1}{100}$  invariance is maintained, in the invariance is maintained, in the invariance is maintained, in the invariance is maintained in the interval of  $\frac{1}{100}$ Unity of introduced following La dapologramme. spoiling UV finiteness (effective 2d supersymmetry).

### Discretization and lattice perturbation theory A Wilson-like fermion discretization

*a*

2

- **Example 2** Lattice perturbation theory  $\stackrel{a \rightarrow 0}{\longrightarrow}$  continuum perturbation theory  $\alpha \to 0$  and lattice perceptuals of in a theory  $a \to 0$  constitution of perturbation **▶ Lattice perturbation theory**  $\stackrel{a\rightarrow 0}{\longrightarrow}$  **continuum perturbation theory**  $\alpha \rightarrow 0$ **Example 2** Lattice perturbation theory  $\stackrel{a\rightarrow 0}{\longrightarrow}$  continuum perturbation theory
	- **Preserve**  $SO(6)$ , breaks  $U(1) \sim SO(2)$ Preserve  $SO(6)$ , breaks  $U(1) \sim SO(2)$
	- No complex phases:  $(O_F^W)^\dagger = \Gamma_5 O_F^W \Gamma_5$ ,  $(O_F^W)^T = -O_F^W$ A No complex phases:  $(QW)^{\dagger} = \Gamma_{\epsilon} QW \Gamma_{\epsilon}$   $(QW)^{T} = -QW$ i.e. identical propagator at 2*d* points:  $P$  points:  $P$ *<sup>a</sup> ,* 0),(0*,* ⇡ ⇡ *<sup>a</sup>* )( ⇡  $\blacktriangleright$  No complex phases:  $(O_F^W)^{\dagger} = \Gamma_5 O_F^W \Gamma_5$ ,  $(O_F^W)^T = -O_F^W$

Add to the action a "Wilson term",  $K_F + W \equiv K_F^W$ **Wilson te** ⇣sin2(*p*<sup>1</sup> *a*)  $\zeta_F + W$  $=$   $\frac{1}{2}$  $\frac{d^2W}{d^2}$  $\Delta$ dd to the action a "Wilson term"  $K_{\pi}$   $\pm$   $W$   $\pm$   $\sqrt{W}$ Add to the action a "Wilson term"  $K_{B} + W = K^{1}$  $\ddot{\phantom{a}}$ *p*<br>**p**<sup>*n*</sup>/*a* 2010

$$
K_F{}^W = \left(\begin{matrix} W_+ & -\mathring{p_0} \mathbb{1} & (\mathring{p_1} - i\frac{m}{2})\rho^M u_M & 0 \\ -\mathring{p_0} \mathbb{1} & -W_+^\dagger & 0 & (\mathring{p_1} - i\frac{m}{2})\rho_M^\dagger u^M \\ -(\mathring{p_1} + i\frac{m}{2})\rho^M u_M & 0 & W_- & -\mathring{p_0} \mathbb{1} \\ 0 & -(\mathring{p_1} + i\frac{m}{2})\rho_M^\dagger u^M & -\mathring{p_0} \mathbb{1} & -W_-^\dagger \end{matrix}\right)
$$

Į.

(1)

where  $W_{\pm} = \frac{r}{2} \left( \hat{p}_0^2 \pm i \, \hat{p}_1^2 \right) \rho^M u_M$ ,  $|r| = 1$ , and  $\hat{p}_\mu \equiv \frac{2}{a} \sin \frac{p_\mu a}{2}$ , leads t  $\Gamma^{(1)}_{\rm LAT}=$ *V*2  $2\,a^2$ Z  $+\pi$  $-\pi$  $\frac{d^2p}{(2\pi)^2}$  ln  $\left[\frac{4^8(\sin^2\frac{p_0}{2}+\sin^2\frac{p_1}{2})^5(\sin^2\frac{p_0}{2}+\sin^2\frac{p_1}{2}+\frac{M^2}{8})^2(\sin^2\frac{p_0}{2}+\sin^2\frac{p_1}{2}+\frac{M^2}{4})}{(\sin^2p_0+\sin^2p_1+\frac{M^2}{4}+4\sin^4\frac{p_0}{2}+4\sin^4\frac{p_1}{2})^8}\right]$  $\left(\sin^2 p_0 + \sin^2 p_1 + \frac{M^2}{4} + 4\sin^4 \frac{p_0}{2} + 4\sin^4 \frac{p_1}{2}\right)$  $\sqrt{8}$  $\overline{1}$  $\overline{V}$  $\frac{2}{3}$  $p_0$   $\left[ 4^8 (\sin^2 \frac{p_0}{2} + \sin^2 \frac{p_1}{2})^5 (\sin^2 \frac{p_0}{2} + \sin^2 \frac{p_1}{2} + \frac{M^2}{8})^2 (\sin^2 \frac{p_0}{2} + \sin^2 \frac{p_0}{2})^2 \right]$  $\frac{1}{(\sin^2 p_0 + \sin^2 p_1 + \frac{M^2}{4} + 4\sin^4 \frac{p_0}{2} + 4\sin^4 \frac{p_1}{2})^8}$  $-\pi$  $\overline{A}^2$ CCCA *.* (3.7) 2  $(\hat{p}_0^2 \pm i \, \hat{p}_1^2)$  $\left(\int \rho^M u_M, \, |r| = 1, \, \text{and} \, \hat{p}_\mu \equiv \frac{2}{a} \sin \frac{p_\mu a}{2}$  , leads to  $\ln \left[ \frac{4^8 (\sin^2 \frac{p_0}{2} + \sin^2 \frac{p_1}{2})^4}{\sin^2 n_0} \right]$  $(si)$  $\mathbf{s}$ i  $\frac{2}{2} + \sin^2$ <br>2 n<sub>1</sub> +  $\frac{M^2}{2}$  $\frac{1}{2}$  $-\frac{M^2}{8}$  $\frac{4}{3}$  $)^{2}(%$  $\frac{4}{ }$ 

 $\frac{a\rightarrow 0}{\longrightarrow}$   $\frac{3\ln 2}{\longrightarrow}$  $\frac{1}{\sqrt{2}}$  $\frac{a \rightarrow 0}{a}$   $-\frac{3 \ln 2}{8 \pi} V_2 \, m^2$ , cusp anomaly at strong coupling  $\,$  (  $|r| = 1, M = m \, a.$  )  $8\pi$  $V_2 m^2$ , cusp anomaly at strong coupling  $(|r| = 1, M = m a.)$  $a_0$  3 ln 2  $\frac{1}{2}$  $V_2$   $m^2$  even anomaly at strong coupling  $(|n| = 1, M - m)$ 

Simulations, continuum limit: measurements

#### Parameter space, continuum limit  $(a \rightarrow 0)$ Parameter space, continuum limit  $(a \rightarrow 0)$ Parameter space, continuum limit ( $a \rightarrow 0$ ) *L*cusp = *|*@*tx*+  $\overline{2}$ . *x|* +  $\overline{\mathsf{u}}$ @*sx <sup>m</sup>*  $\frac{1}{2}$ + (@*tz<sup>M</sup>* + space, <sup>2</sup> + <sup>2</sup> + *<sup>T</sup> O<sup>F</sup> ,*  $D_{\text{SUSY}}$  string in the number of  $\mathbb{R}^n$  background backgroun

*m*<sup>2</sup>

Parameter space, continuum limit (*a* ! 0) Two bare parameters,  $g =$  $\frac{\sqrt{\lambda}}{4\pi}$  $\ddot{\phantom{1}}$ and  $P + \sim m$ , assume the only additional scale is  $a$ Two bare parameters,  $g =$  $\sqrt{\lambda}$  $\frac{\sqrt{\lambda}}{4\pi}$  and  $P^+ \sim m$ , assume the only additional scale is  $a$ Two bare parameters, *g* = and *<sup>P</sup>* <sup>+</sup> ⇠ *<sup>m</sup>*, assume the only additional scale is *<sup>a</sup>* **I** Two bare parameters,  $g = \frac{\sqrt{\lambda}}{4\pi}$  and  $M \times P^+ \sim m$ , assume the only additional scale is *L*cusp = *|*@*tx*+ ete *x|*  $\mathcal{L}$  $\frac{7}{11}$  $\overline{a}$ and  $P^+ \sim m$ ,  $\epsilon$ + (@*tz<sup>M</sup>* + *m zM*) <sup>2</sup> + **1** Two bare parameters,  $g = \frac{\sqrt{\lambda}}{4\pi}$  and  $P^+ \sim m$ , assume the only additional scale is  $a$ 

$$
F_{\rm LAT} = F_{\rm LAT} (g, M, N) \qquad \qquad M = m a \, , \qquad N = \frac{L}{a}
$$

*L*

*F*LAT  $\frac{1}{2}$  in  $\frac{1}{2}$  and  $\frac{1}{2}$  *FRAT*  $\frac{1}{2}$  *FRAT* The continuum limit must be taken along a line of constant physics: curve in  $\{g, M, N\}$  $\alpha$  and  $\alpha$  physical quantities are neptrixed as  $a \rightarrow 0$ . where physical quantities are kept fixed as  $a \rightarrow 0$ . The continuum limit must be taken along a line of constant physics: curve in *{g, M, N}* where physical quantities are kept fixed as *a* ! 0. **Fig. 2** The continuum limit must be taken along a line of constant physics: curve in  $\{g, M, N\}$ where physical quantities are kept fixed as  $a \rightarrow 0$ . along a line of constant physics: curve in  $\{g,$ Green-Schwarz string in the null cusp background *F*LAT  $\alpha$  and  $\alpha$  are kept fixed as  $a \to 0$ . *L*

E.g. 
$$
m_x^2 = \frac{m^2}{2} \left( 1 - \frac{1}{8g} + \mathcal{O}(g^{-2}) \right)
$$
  
\n $L^2 m_x^2 = \text{const} \longrightarrow (L m)^2 \equiv (NM)^2 = \text{const.}$ 

For a generic observable  
\n
$$
F_{\text{LAT}} = F_{\text{LAT}}(g, M, N) = F(g) + \mathcal{O}\left(\frac{1}{N}\right) + \mathcal{O}\left(e^{-MN}\right)
$$
\nfinite values (~a) effects

Recipe: fix g, fix  $MN$  large enough, evaluate  $F_{\rm LAT}$  for  $N=6,8,10,12,16,\ldots;$ Obtain  $F(g)$  extrapolating to  $N \to \infty$ .

# Measurement I:  $\langle x, x^*\rangle$  correlator

From the correlator of the *x* fields

$$
C_x(t;0) = \sum_{s_1, s_2} \langle x(t, s_1) x^*(0, s_2) \rangle
$$
  

$$
t \ge 1 \qquad e^{-t m_x \text{LAT}}
$$

 $\sim 0$  0.5 1 1.5 2 2 Consistent with large *g* prediction, no clear signal of bending down.  $\frac{1}{0}$   $\frac{1}{0}$   $\frac{1}{0}$   $\frac{1}{1}$   $\frac{1}{1}$   $\frac{1}{0}$   $\frac{1}{2}$  $0\frac{1}{0}$  0.5 1 1.5 2 2.5 3 1 2 3 4 5  $6 \frac{10^{-3}}{2}$  $\begin{bmatrix} 2 & 1 \end{bmatrix}$ •  $Lm = 4$ <br> *www.* PT,  $g_c = 0.04g$ corresponding e↵ective mass *m*e↵ ln *<sup>C</sup>x*(*t*)  $\frac{1}{2}$ as functions of the time *t* in units of *mx*LAT for di↵erent *g* and lattice sizes. The flatness  $\begin{array}{ccc} 0.6 \leftarrow & \tau \\ \tau & \tau \end{array}$ allows for a reliable extraction of the mass of the *x*-excitation. Data points are masked by large errorbars for time scales greater than unity because the signal of the correlator degrades exponentially compared with the statistical noise. The statistical noise  $\frac{1}{\sqrt{2}}$  $\frac{1}{2}$  is  $\frac{1}{2}$  is  $\frac{1}{2}$  is  $\frac{1}{2}$  is  $\frac{1}{2}$  is  $\frac{1}{2}$  is  $\frac{1}{2}$  if  $\frac{1}{2}$  $\frac{1}{\sqrt{1}}$ <br>  $\frac{1}{\sqrt{1}}$ From the correlator of the *x* fields  $\frac{1}{10}$   $\frac{1}{10}$  h*x*(*t, s*1)*x*⇤(0*, s*2)i = X  $\overline{2.5}$ *|cn|* <sup>2</sup>*etEx*(0; *<sup>n</sup>*)  $t~m_{x\, \rm LAT}$ extract the *x*-mass  $\frac{1}{\sqrt{2}}$  **m**  $\frac{1}{\sqrt{2}}$  $\frac{1}{2}$ 2.5 3 *a* log *<sup>C</sup>x*(*t*; 0)  $\frac{1}{4}$   $\frac{1}{4.5}$ <br>*G*  $C_x(t;0)$ h*x*(*t, s*1)*x*⇤(0*, s*2)i  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 &$ <sup>2</sup>*etEx*(0; *<sup>n</sup>*)  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$  $\frac{m x}{\sqrt{2}}$ *t*!1 *m*e↵*<sup>x</sup>*  $0.8$ log *<sup>C</sup>x*(*t*; 0) No infinite renormalization occurring, no need of tuning *m* to adjust for it. This correction is corrected as a corrected our choice of line of line of line of line of line of constant physics. *<sup>x</sup>* )<sup>2</sup>  $m_x^2$  $m^2$ *m*<sup>2</sup> *m*<sup>2</sup> *m*<sup>2</sup>  $\overset{\circ}{g}_c$ 

No infinite renormalization of the infinite model of the infinite model of the interest of tuning and it.

extract the  $x$ -mass

$$
m_{x \text{LAT}} = \lim_{t \to \infty} m_x^{\text{eff}}
$$
  

$$
\equiv \lim_{t \to \infty} \frac{1}{a} \log \frac{C_x(t;0)}{C_x(t+a;0)}
$$

## Measurement I:  $\langle x, x^* \rangle$  correlator

From the correlator of the *x* fields From the correlator of the *x* fields



 $C_x(t; 0)$   $\frac{10^{-3}}{2}$ 

h*x*(*t, s*1)*x*⇤(0*, s*2)i

 $\frac{1}{2}$ 

 $\overset{\circ}{g}_c$ 

 $\int_{0}^{\infty}$  the physical mass  $\int_{0}^{\infty}$  *m*<sup>2</sup> is usefully obtained as a limit of an  $\int_{0}^{\infty}$ *m*e↵ *x* , defining about the discretized timeslice pair (*t, t*+*a*) by the discretized timeslice pair (*t*, t<sub>+</sub>a) by the discretized timeslice pair (*t*, t<sub>+</sub>a) by the discretized timeslice pair (*t*, t<sub>+</sub>a) by the discretiz Consistent with large *g* prediction, no clear signal of bending down.

No infinite renormalization occurring.

We measure 
$$
\langle S_{\text{cusp}} \rangle \equiv g \frac{V_2 m^2}{8} f'(g)
$$
. At large g,

$$
\langle S_{\rm LAT} \rangle \equiv g \, \frac{N^2 \, M^2}{4} \, 4 + \frac{c}{2} (2N^2)
$$

 $q$ uadratic divergences appear, with  $c = n$ ,  $q = 8 \pm 17 = 95$ . quadratic divergences appear, with  $c = n_{\textit{bos}} = 8 + 17 = 25.$ 6.3. Observables



 $\frac{O}{\sigma}$  is a factor  $\sim a^{-\frac{(2R)}{2}}$ ↵:[*O*↵]*D* there is a factor  $\sim g^{-\dfrac{(2N^2)}{2}}$ . In lattice codes, coupling omitted from fermionic part. Indeed,  $\langle S \rangle = -\frac{\partial \ln Z}{\partial \ln g}$  and  $Z \sim \Pi_{\sf n_{\rm bos}} (\det g \, {\cal O})^{-\frac{1}{2}}$ , so for each bosonic species  $\phi=-\frac{\partial\ln Z}{\partial\ln g}$  and  $Z\sim \Pi_{\mathsf{n}_{\text{bos}}}(\det g\,\mathcal{O})^{-\frac{1}{2}}$  , so for each bosonic species  $(2N^2)$ actor  $\sim q^{-\frac{m}{2}}$  . In lattice codes, coupling omitted from fermi *c/*2 of the divergent (<sup>≥</sup> *<sup>L</sup>*2) contribution in (6.3.23).

We measure 
$$
\langle S_{\text{cusp}} \rangle \equiv g \, \frac{V_2 \, m^2}{8} \, f'(g)
$$
. At finite  $g$ ,

 $\mathcal{S}$  is the cusp action:  $\mathcal{S}$  $\sqrt{2}$ *MAI*  $\frac{1}{2}$  **b**  $\frac{1}{4}$  *J*  $\frac{1}{4}$  *J*  $\frac{1}{4}$  *J*  $\frac{1}{4}$  $\langle S_{\rm LAT} \rangle \equiv g \frac{N^2 M^2}{4} f'_{\rm LAT}(g) + \frac{c(g)}{2} (2N^2)$ 



 $\frac{1}{2}$ *g*, the continuum, choing power aivergences are set t Here, expected mixing of the Lagrangian with lower dimension operator  $\overline{\mathsf{I}}$   $\overline{\$ ↵:[*O*↵]*D <sup>Z</sup>*↵ *<sup>O</sup>*↵((*x*)) *, <sup>Z</sup>*↵ ⇠ ⇤(*D*[*O*↵])⇠ *<sup>a</sup>*(*D*[*O*↵]) In continuum, existing power divergences are set to zero (dim. reg.)

$$
\mathcal{O}(\phi(s))_r = \sum_{\alpha:[O_\alpha]\leq D} Z_\alpha \mathcal{O}_\alpha(\phi(x)), \qquad Z_\alpha \sim \Lambda^{(D-[\mathcal{O}_\alpha])} \sim a^{-(D-[\mathcal{O}_\alpha])}
$$

To compare, assume  $g = \alpha \, g_c$ : then from  $f'(g) = f'(g_c)_c$  is  $g_c = 0.04g$ .



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### naresse In progress

### **with L. Bianchi, B. Leder, R. Roiban, E. Vescovi**

as the coupling, and e.g. mass measurements plotted against it. The relation among *gc* and *g* may be non-trivial. Then the cusp may be ``declared''



 $\cos(m^2 - 1)$ related to the *U*(1)-breaking of the discretization.  $T<sub>0</sub>$  condition a single tuning is expected. We are observing an unexpected splitting in the fermionic masses  $(m_F^2=\frac{1}{2})$ related to the *U*(1)-breaking of the discretization. The corresponding Ward identity may be used as renormalization condition, a single tuning is expected.

We are extending our simulations to  $g \leq 5$ .

# On the CFT side On the CFT side

Strong sign problem at strong coupling ( $\lambda \gg 1$ ), one tuning.

The control is in the perturbative region (matching with NNLO). The control is in the perturbative region (matching with NNLO).

### **Courtesy of David Schaich**



 $U(2)$  and  $U(3)$  results less stable — working on further improvements

### **Conclusions**

- I presented a study of lattice field theory methods for gauge-fixed string  $\sigma$ -models  $T$ using is a model is a model is a model in  $T$  and  $T$  methods  $T$  and  $T$  methods  $T$  and  $T$ relevant in AdS/CFT: address ab initio, non-perturbative calculations within them.
	- $\blacktriangleright$  The model  $\blacktriangle$  GS string on GKP vacuum  $\blacktriangle$  is amenable to study. Ongoing work on the property on several open the property of the property include the property of the property ▶ The model – GS string on GKP vacuum – is amenable to study using standard techniques (Wilson-like fermion discretizations, RHMC algorithm). I presented a study of lattice field theory methods for *gauge-fixed* string -models relevant in Advantagement in Advantagement in Advantagement with the method in the method of the method of the<br>The method of the method with the method in the method of the method of the method of the method of the method
	- $\blacksquare$  Ma obearve good agreement with expectation at large  $a$ Non-perturbative definition of string theory?  $\blacktriangleright$  We observe good agreement with expectation at large  $g$ , and indications of non-perturbative physics;

Ongoing work on several open questions, which include the proper continuum limit. I we have a with a server and a provision which include the reasonable at the server with the server and the server and the server of the and indications of non-perturbative physics;  $\mathbf{r}$ 

- Future: different backgrounds/gauge-fixing/observables *...* Ongoing work on several open questions, which include the proper continuum limit. Future. Unierent backgrounds/gauge-iixing/observables ...
- Non-perturbative definition of string theory? For sure, suitable framework for first principle statements (proofs of AdS/CFT) and (potentially) very efficient tool in numerical holography.

Thanks for your attention.

# Extra-slides

### Boundary conditions

Fluctuations must vanish at the AdS boundary (two sides of the grid)

$$
\tilde{X}(t = -\infty, s) = 0 = \tilde{X}(t, s = +\infty)
$$

and be free to fluctuate elsewhere. Field redefinitions adopted in the continuum lead to exotic (unstable) boundary conditions.

So far we used periodic BC for all the fields (antiperiodic temporal BC for fermions). and evaluated finite volume effects  $\sim e^{-m L} \equiv e^{-M N}$ .

Most run are done at  $M$   $N = 4$   $(e^{-4} \simeq 0.02)$ , some at  $M$   $N=6$   $(e^{-6}\simeq 0.002).$ *g* = 50 ,<br>0;

Appear to play a role only in evaluating the coefficient of divergences. *< S*LAT *> /*(2



### A remark on numerics

The most difficult part of the algorithm is the *inversion* of the fermionic matrix

$$
|\mathrm{Pf}\, O_F| \equiv (\det O_F^\dagger O_F)^\frac{1}{4} \equiv \int d\zeta d\bar\zeta \, e^{-\int d^2\xi\, \bar\zeta\, \left(O_F^\dagger O_F\right)^{-\frac{1}{4}}\zeta} \; .
$$

The RHMC (Rational Hybrid Montecarlo) uses a rational approximation

$$
\overline{\zeta} \left( O_F^{\dagger} O_F \right)^{-\frac{1}{4}} \zeta = \alpha_0 \, \overline{\zeta} \, \zeta + \sum_{i=1}^P \overline{\zeta} \, \frac{\alpha_i}{O_F^{\dagger} O_F + \beta_i} \, \zeta
$$

with  $\alpha_i$  and  $\beta_i$  tuned by the range of eigenvalues of  $O_F$ .

Defining  $s_i \equiv \frac{1}{O_F^{\dagger} O_F + \beta_i}$  $\zeta$ , one solves

$$
(O_F^{\dagger}O_F+\beta_i)s_i=\zeta, \qquad i=1,\ldots,P.
$$

with a (multi-shift conjugate) solver for which

number of iterations  $\sim \lambda_{\rm min}^{-1}$ 

In our case the spectrum of  $O_F$  has very small eigenvalues.

And:

$$
O_F \,=\, \left(\begin{array}{c} \mathrm{i}\partial_t \\ \mathrm{i} \frac{z^M}{z^3} \end{array}\right)^{\hspace{-.15cm}1\overline{\partial}_t} \left(\partial_s - \tfrac{m}{2}\right)
$$

 $A \rightarrow \infty$  , and antisymmetry and antisymmetry and antisymmetry and antisymmetry and antisymmetry and antisymmetry

 $\mathcal{A}$  are linear in the linearization of  $\mathcal{A}$  $\overline{\mathsf{r}}$ ator (including dux.  $\Gamma_5$ -hermiticity and antisymmetry ho  $\Gamma$  bermiticity and antievementry be  $\Gamma_5$ -hermiticity and antisymmetry hold now for the full operator (including aux. fields)

$$
O_F^\dagger = \Gamma_5 \, O_F \, \Gamma_5 \,, \qquad \qquad O_F^T = - O_F
$$

 $\frac{1}{\sqrt{2}}$ Pfaffian is real,  $({\rm Pf}O_F)^2=\det O_F\geq 0,$  but not positive definite,  ${\rm Pf}O_F=\pm \det O_F.$  $F = \frac{F}{\sqrt{2\pi}}$  ,  $F = \frac{F}{\sqrt{2$ Pfaffian is real, (Pf*O<sup>F</sup>* )<sup>2</sup> = det *<sup>O</sup><sup>F</sup>* <sup>0</sup>, but not positive definite, Pf*O<sup>F</sup>* <sup>=</sup> *<sup>±</sup>* det *<sup>O</sup><sup>F</sup>* . Pfaffian is real,  $({\rm Pf}O_F)^2=\det O_F\geq 0$ , but not positive definite,  ${\rm Pf}O_F=\pm \det O_F$ .

**Spectrum of Contracting Contracts** Gain in computational costs: for large values of  $N$  (finer lattices) the algorithm for evaluating complex determinants is very inefficient. Now just a sign flip.  $T_{\rm eff}$  then for the reweighting is the reweighting then for the reweighting the reweighting the reweighting the reweighting theorem is the reweighting the reweighting the reweighting the reweighting the reweighting the <sup>h</sup>*O*ireweight <sup>=</sup> <sup>h</sup>*<sup>O</sup> <sup>e</sup>i*✓i✓=0 Yes, but gain in computational costs: for large values of *N* (finer lattices) the algorithm for evaluating complex determinants is very inefficient. Now just a sign flip.

$$
\langle \mathcal{O} \rangle_{\text{reweight}} = \frac{\langle \mathcal{O} e^{i\theta} \rangle_{\theta=0}}{\langle e^{i\theta} \rangle_{\theta=0}} \longrightarrow \langle \mathcal{O} \rangle_{\text{reweight}} = \frac{\langle \mathcal{O} w \rangle}{\langle w \rangle_{\sqrt{\det O_F}}}.
$$
\nwhere  $w = \pm 1$ , and  $\sqrt{\det O_F} = (\det O_F^{\dagger} O_F)^{\frac{1}{4}}$ .

In simpler models with four-fermion interactions, similar manipulations ensure a definite positive Pfaffian. There real, antisymmetric operator with doubly degenerate eigenvalues. Quartets  $(ia, ia, -ia, -ia)$ ,  $a \in \mathbb{R}$ . In simpler models with four-fermion interactions, similar manipulations ensure a definite positive Pfaffian. There real, antisymmetric operator with doubly degenerate eigenvalues: quartets  $(ia, ia, -ia, -ia)$  ,  $a \in \mathbb{R}$ .

[Catterall 2016, Catterall and Schaich 2016]

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## Spectrum of  $O_F$

From  $\Gamma_5$ -hermiticity and antisymmetry,

$$
\mathcal{P}(\lambda) = \det(O_F - \lambda \mathbb{1}) = \det(\Gamma_5 (O_F - \lambda \mathbb{1}) \Gamma_5)
$$
  
= 
$$
\det(O_F^{\dagger} + \lambda \mathbb{1}) = \det(O_F + \lambda^* \mathbb{1})^* = \mathcal{P}(-\lambda^*)^*
$$



Spectrum characterized by quartets  $\{\lambda, -\lambda^*, -\lambda, \lambda^*\}$ .

$$
\det O_F = \prod_i |\lambda_i|^2 |\lambda_i|^2 \longrightarrow \text{Pf}(O_F) = \pm \prod_i |\lambda_i|^2
$$

Choosing a starting configuration with positive Pfaffian, no sign change possible.

### Spectrum of  $O_F$

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$$
  
= 
$$
\det(O_F^{\dagger} + \lambda \mathbb{1}) = \det(O_F + \lambda^* \mathbb{1})^* = \mathcal{P}(-\lambda^*)^*
$$



For  $\lambda = \pm \lambda^*$ , no four-fold property: due to zero crossings, Pfaffian may change sign. For = *±* ⇤, no four-fold property: due to zero crossings, Pfaffian may change sign.

Purely imaginary eigenvalues correspond to Yukawa-terms, even those present in the original Lagrangian: no "suitable enough" choice of auxiliary fields.

## Previous study

### [McKeown Roiban, arXiv: 1308.4875]



 $\boldsymbol{g}$ 

### Parameters of the simulations

$\boldsymbol{g}$	$T/a \times L/a$	Lm	am	$\tau_{\rm int}^S$	$\tau_{\rm int}^{m_x}$	statistics [MDU]
$\bf 5$	$16\times 8$	$\overline{4}$	0.50000	0.8	2.2	900
	$20\times10$	$\overline{4}$	0.40000	0.9	2.6	900
	$24\times12$	$\overline{4}$	0.33333	$0.7\,$	$4.6\,$	900,1000
	$32\times16$	$\overline{4}$	0.25000	0.7	4.4	850,1000
	$48\times24$	$\overline{4}$	0.16667	1.1	3.0	92,265
10	$16 \times 8$	$\overline{4}$	0.50000	0.9	2.1	1000
	$20 \times 10$	$\overline{4}$	0.40000	0.9	2.1	1000
	$24\times12$	$\overline{4}$	0.33333	1.0	$2.5\,$	1000,1000
	$32\times16$	$\overline{4}$	0.25000	1.0	$2.7\,$	900,1000
	$48\times24$	$\overline{4}$	0.16667	1.1	3.9	594,564
20	$16 \times 8$	$\overline{4}$	0.50000	5.4	1.9	1000
	$20\times10$	$\overline{4}$	0.40000	9.9	1.8	1000
	$24\times12$	$\overline{4}$	0.33333	4.4	$2.0\,$	850
	$32\times16$	$\overline{4}$	0.25000	7.4	2.3	850,1000
	$48\times24$	$\overline{4}$	0.16667	8.4	3.6	264,580
30	$20\times10$	$\,6$	0.60000	$1.3\,$	$2.9\,$	950
	$24\times12$	$\sqrt{6}$	0.50000	1.3	2.4	950
	$32\times16$	$\,6$	0.37500	1.7	2.3	975
	$48\times24$	$\,6$	0.25000	1.5	$2.3\,$	533,652
	$16 \times 8$	$\overline{4}$	0.50000	1.4	1.9	1000
	$20\times10$	$\overline{4}$	0.40000	1.2	$2.7\,$	950
	$24\times12$	$\overline{4}$	0.33333	1.2	2.1	900
	$32\times16$	$\overline{4}$	0.25000	1.3	1.8	900,1000
	$48\times24$	$\overline{4}$	0.16667	1.3	4.3	150
50	$16 \times 8$	$\overline{4}$	0.50000	1.1	1.8	1000
	$20\times10$	$\overline{4}$	0.40000	1.2	1.8	1000
	$24\times12$	$\overline{4}$	0.33333	0.8	$2.0\,$	1000
	$32\times16$	$\overline{4}$	0.25000	1.3	$2.0\,$	900,1000
	$48\times24$	$\overline{4}$	0.16667	1.2	2.3	412
100	$16 \times 8$	$\overline{4}$	0.50000	1.4	2.7	1000
	$20\times10$	$\bf 4$	0.40000	1.4	4.2	1000
	$24\times12$	$\overline{4}$	0.33333	1.3	1.8	1000
	$32\times16$	$\overline{4}$	0.25000	1.3	$2.0\,$	950,1000
	$48\times24$	$\overline{4}$	0.16667	1.4	$2.4\,$	541

Table 1: Parameters of the simulations: the coupling  $g$ , the temporal  $(T)$  and spatial  $(L)$ extent of the lattice in units of the lattice spacing *a*, the line of constant physics fixed by *Lm* and the mass parameter  $M = am$ . The size of the statistics after thermalization is given in the last column in terms of Molecular Dynamic Units (MDU), which equals an HMC trajectory of length one. In the case of multiple replica the statistics for each replica is given separately. The auto-correlation times  $\tau$  of our main observables  $m_x$  and  $S$  are also given in the same units.

We proceed subtracting the continuum extrapolation of  $\frac{c}{2}$  multiplied by  $N^2$ : divergences appear to be completely subtracted, confirming their quadratic nature. Errors are small, and do not diverge for  $N \to \infty$ .

Flatness of data points indicates very small lattice artifacts.



We can thus extrapolate at infinite N to show the continuum limit. ifinite  $N$  to show the continuum limit.