General Aspects of Renormalization Group Flows in Diverse Dimensions

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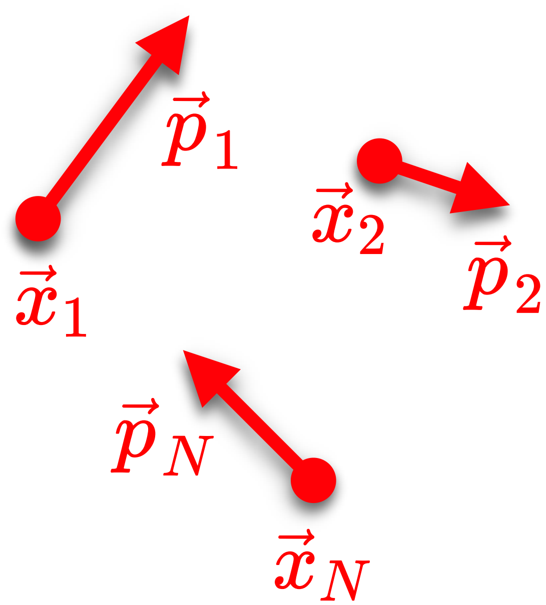
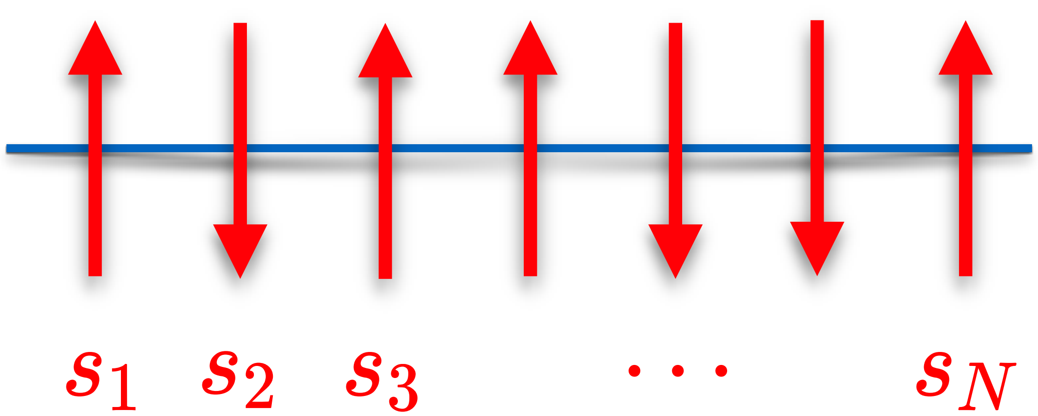
Degrees of Freedom in QM

In a finite quantum system there are several well-defined ways to quantify the number of degrees of freedom (d.o.f.):

- Number of arguments of the wave function (finite, $\sim N$)
- Dimension of the Hilbert space: $\dim \mathcal{H} = Z(T \to \infty)$
- Various measures of quantum entanglement



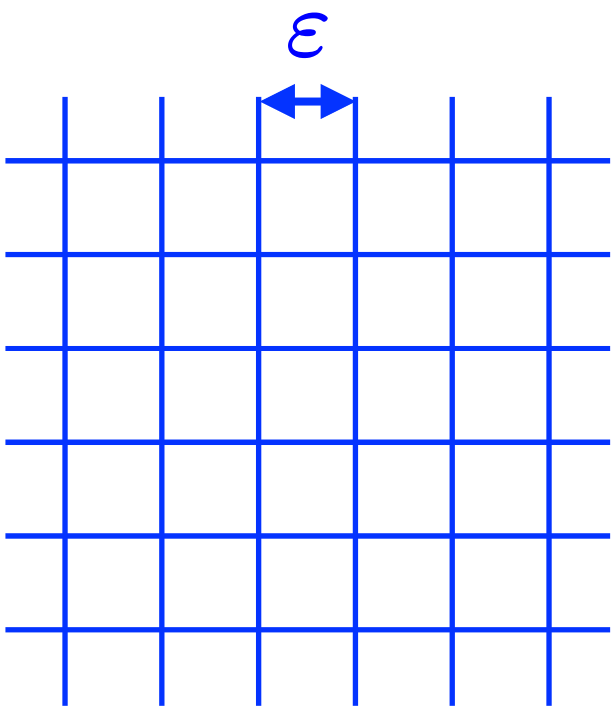


$\Psi(\vec{x}_1, \ldots, \vec{x}_N)$ $\Psi(s_1, \ldots, s_N)$

Degrees of Freedom in QFT

It is often useful to think of continuum QFTs as arising from finite systems with a short-distance (UV) cutoff ε. We are interested in physical properties of the long-distance continuum theory (universal), not cutoff artifacts (scheme-dependent). All simple ways to count degrees of freedom in finite systems diverge in the continuum limit $\varepsilon \to 0$:

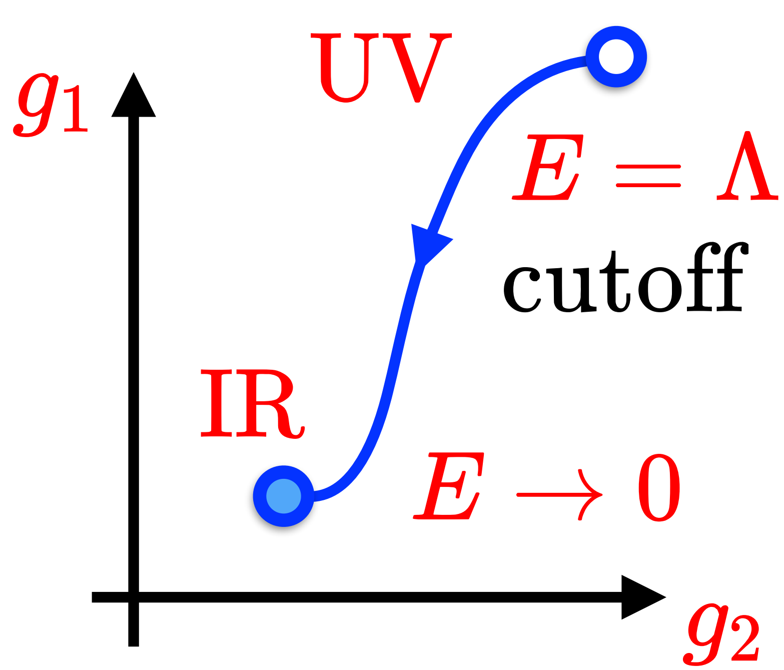
- Infinitely many microscopic d.o.f.
  Most of them are short-distance modes (depend on the cutoff, not universal).
- To the extent possible, we would like to forget about them (powerful idea).



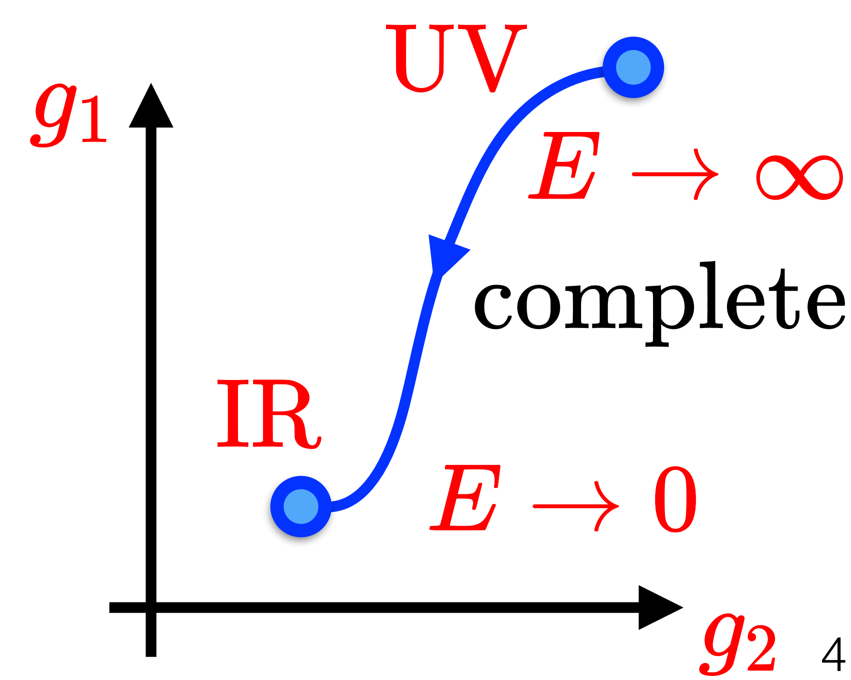
Renormalization Group (RG) Flows

The RG organizes the dynamics of a QFT by energy scale E:

- Restrict to energies below the UV cutoff: $E < \Lambda \sim \varepsilon^{-1}$
- RG flow = tracking the theory from the UV to the IR. We can think of this flow as taking place in the space $\{g_i\}$ of all couplings of the theory $\subset$ space of all theories (imprecise).
- Along the flow, heavy d.o.f. decouple from the long-distance physics (they are integrated out).



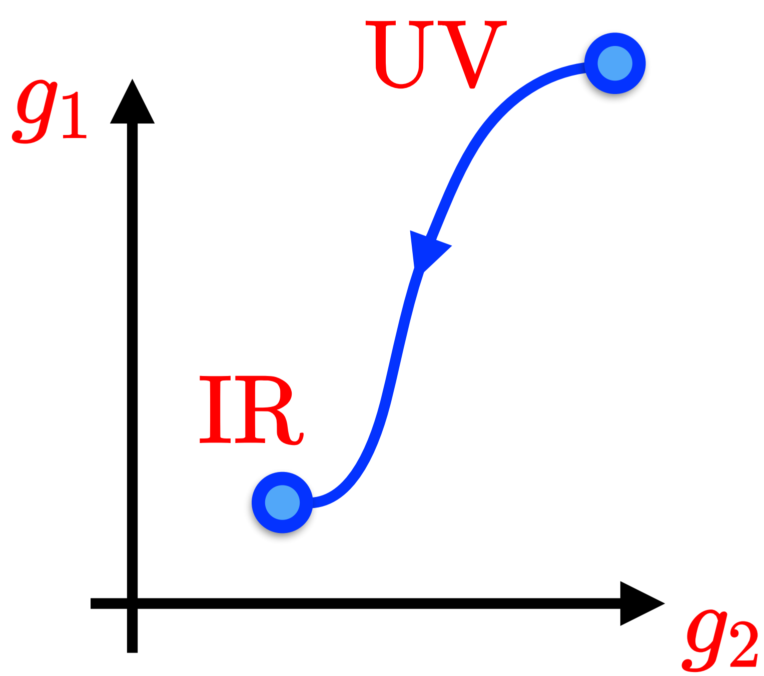
If we can take $\Lambda \to \infty$, we get a continuum theory at all distances; the RG flow explores arbitrarily high energies (UV complete).



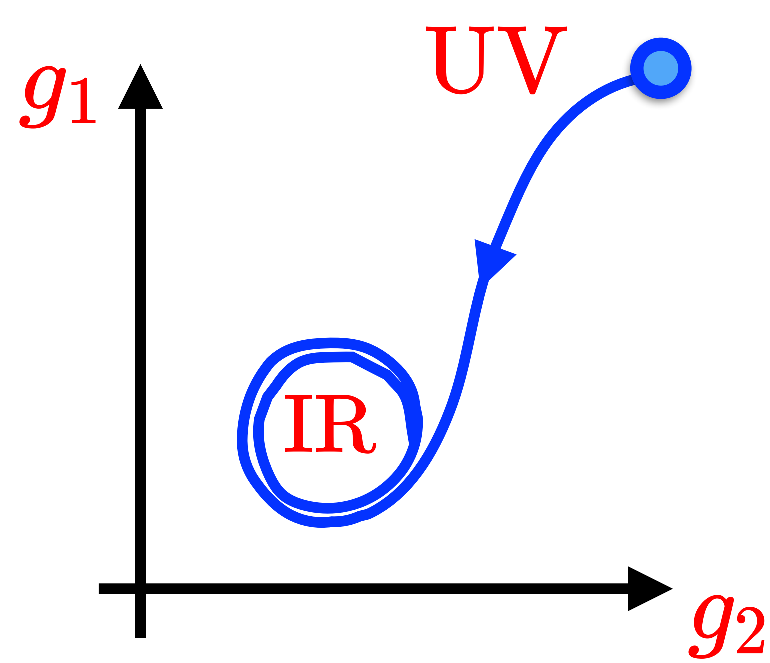
Asymptotics of the RG Flow

Understand the limit $E \to 0$ (also $E \to \infty$). We expect all mass scales to decouple there: emergent scale invariance.

- In principle various scale-invariant asymptotics possible. Fixed points are the most familiar. What about the others?
- Under favorable conditions, asymptotic scale invariance is further enhanced to the conformal group $SO(d,2)$. When does it happen? Does it imply new constraints on the flow?



fixed point



limit cycle

Can also imagine more exotic scale-invariant behavior (e.g. ergodic or turbulent) [Kogut, Wilson '74]

Counting Degrees of Freedom in QFT

- The qualitative picture of RG flow is rooted in the intuition that heavy d.o.f. decouple at long distances (irreversible).
- To make this precise, need a way to count d.o.f. that is well-defined in continuum QFT. In free theories, we could count the number of fields, but this is generally ill defined.
- Fruitful approach: look for a counting function (C-function):

1S.) Dimensionless function of scale, $C(E)$, which decreases monotonically from UV to IR, $C'(E) \geq 0$.

1W.) Asymptotic values $C_{\mathrm{UV}}, C_{\mathrm{IR}}$ such that $C_{\mathrm{UV}} \geq C_{\mathrm{IR}}$.

2.) In a unitary theory, we expect that $C(E), C_{\mathrm{UV}}, C_{\mathrm{IR}} \geq 0$.

1S and 1W sometimes called weak and strong C-theorems.

The C-Theorem in 2d

In 2d [Zamolodchikov '86] constructed a universal C-function:

$$C(x^2) \sim \langle T_{\mu\nu}(x) T_{\alpha\beta}(0) \rangle \geq 0 \qquad (x^2 \sim E^{-2})$$

$$C'(x^2) \sim -\langle T^\mu_\mu(x) T^\alpha_\alpha(0) \rangle \leq 0 \qquad \text{(strong C-theorem)}$$

- Uses Lorentz invariance, $\partial^\mu T_{\mu\nu} = 0$, reflection positivity.
- Scale invariance implies $C'(x^2) = 0$, and unitarity $T^\mu_\mu = 0$: CFT [Polchinski '88]. If $C(x^2) = 0$ then $T_{\mu\nu} = 0$: TQFT.
- Exactly marginal deformations don't change the C-function.
- Away from conformal fixed points $C'(x^2) < 0$, i.e. the C-function is strictly decreasing (rules out cyclic behavior).
- Can express $C(x^2)$ as a height function $C(g_i)$ on theory space: gradient potential for RG flow [Z; Friedan, Konechny '09]

The Many Faces of C at Fixed Points

A 2d CFT has Virasoro symmetry; central charge $c_{\text{Vir.}} \sim C$.

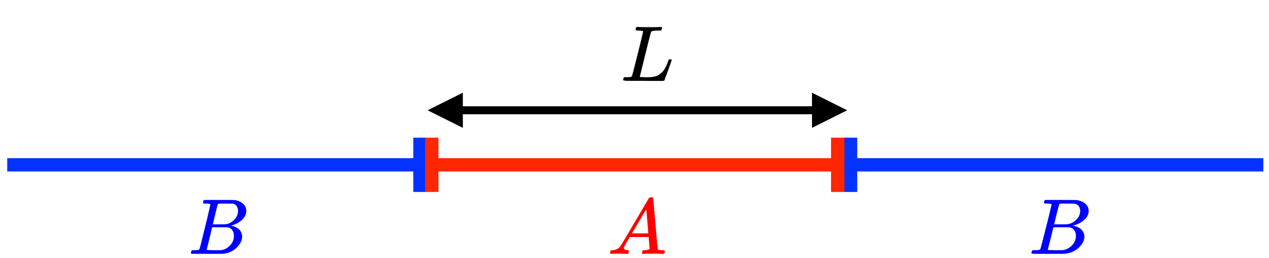
It shows up in many physical observables, not just $\langle T_{\mu\nu} T_{\alpha\beta} \rangle$:

- Finite-temperature free energy: $F(T) \sim c_{\text{Vir.}} T^2 L \quad (L \gg T)$
- Casimir energy, spectral asymptotics on S^1 [Cardy '86]
- Trace anomaly: activating a background metric $g_{\mu\nu}$ leads to

$$\langle T^{\mu}_{\mu}(x) \rangle_g \sim c_{\text{Vir.}} R[g]$$

- Sphere partition function: $\log Z_{S^2_r} \sim c_{\text{Vir.}} \log(\Lambda r) + (\text{const.})$
- Vacuum entanglement: $\rho = \text{Tr}_A \left(|0\rangle\langle 0| \right),\ S = -\text{Tr} \rho \log \rho$

$$\sim c_{\text{Vir.}} \log(\Lambda L) + (\text{const.})$$



$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

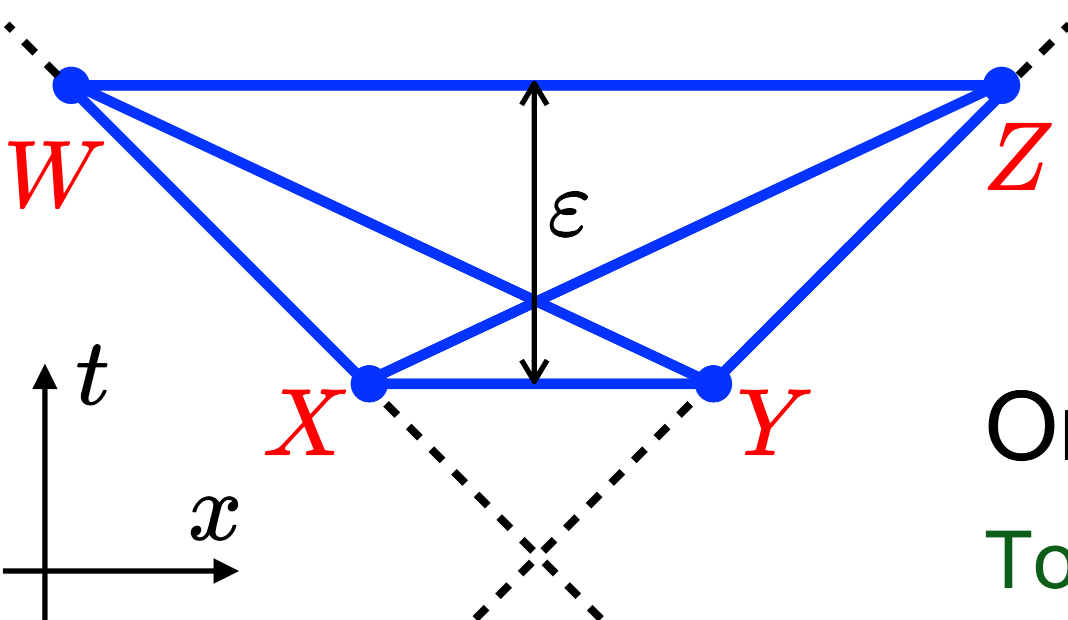
[Holzhey, Larsen, Wilczek '94; Calabrese, Cardy '04]

C-Theorem from Entanglement

$$S(L) \sim c_{\text{Vir.}} \log(\Lambda L) + (\text{const.}) \implies C(L) = LS'(L) \sim c_{\text{Vir.}}$$

The renormalized EE $C(L)$ [Liu, Mezei '12] defines a universal C-function along RG flows [Casini, Huerta '04]. Need unitarity, Lorentz invariance, some locality, e.g. deform Cauchy slice without changing S, strong subadditivity [Lieb, Ruskai '73]:

$$\underbrace{S(WX \cup XY)}_{S(WY)} + \underbrace{S(XY \cup YZ)}_{S(XZ)} \geq S(XY) + \underbrace{S(WX \cup XY \cup YZ)}_{S(WZ)}$$



As $\varepsilon \to 0$ all intervals coincide; nontrivial inequality at $\mathcal{O}(\varepsilon^2)$:

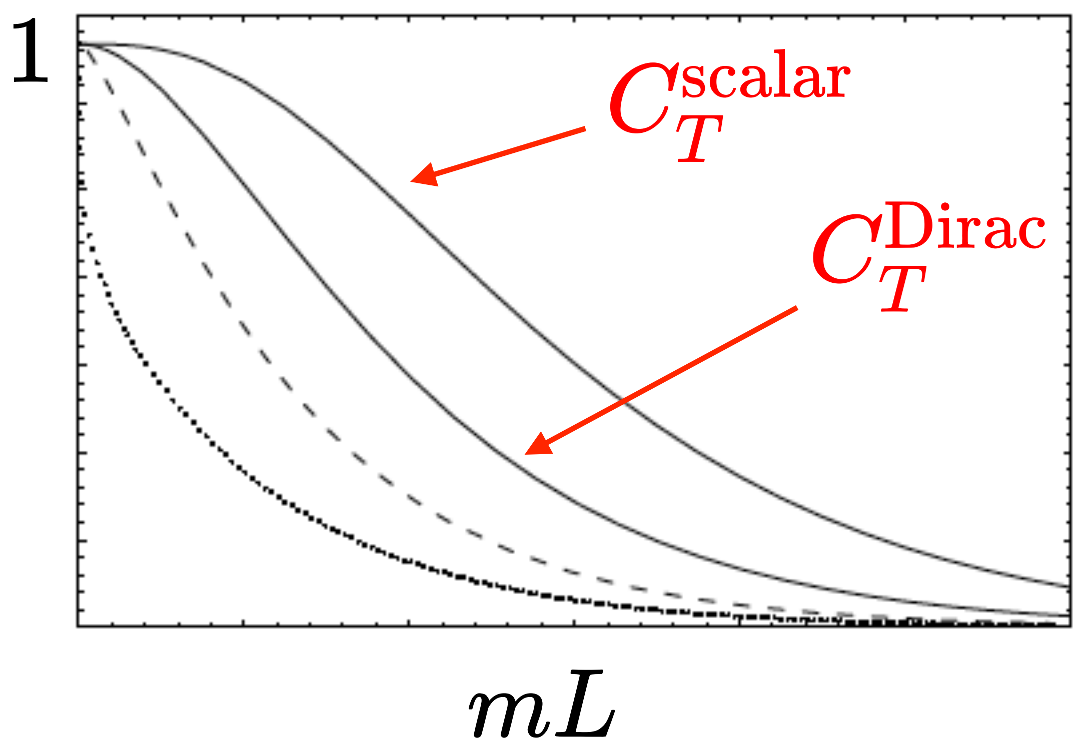
$$LS''(L) + S'(L) = C'(L) \leq 0$$

Or: use relative entropy [Casini, Teste, Torroba '16]. SSA also implies $C(L) \geq 0$.

Free Field Examples

The C-functions $C_T(L)$ (based on $\langle TT \rangle$) and $C_E(L)$ (from entanglement) are different. Compare them for a free real scalar and Dirac fermion of mass m [Casini, Huerta + Fosco'05].

[Casini, Huerta: cond-mat/0610375]



$C_E^{\text{Dirac}} =$ - - - - - - - - -

$C_E^{\text{scalar}} =$ ···············

Note: C_T is analytic near, and stationary at fixed points; C_E^{Dirac} stationary, not analytic in the UV ($L \ll m^{-1}$), C_E^{scalar} has cusp:

$$C_E^{\text{Dirac}}(L) = 1 - (mL)^2 \log^2(mL) + \cdots$$

$$C_E^{\text{scalar}}(L) = 1 + \frac{3}{2\log(mL)} + \cdots$$

Possible (any d) for sufficiently relevant perturbations [Klebanov, Nishioka, Pufu, Safdi '12; Nishioka '14; Lee, Lewkowycz, Perlmutter, Safdi '14; …].

C-Functions in Higher Dimensions?

The 2d results suggest that C-functions are easy to find. This is not the case. Several natural candidates (well defined and positive, e.g. in the deep UV and IR), are not C-functions:

- $C_T \sim \langle T_{\mu\nu}(x) T_{\alpha\beta}(0) \rangle \big|_{\text{spin-2}}$ increases in some SUSY RG flows, e.g. 4d SQCD [Anselmi, Freedman, Grisaru, Johansen '97], 3d WZ models [Nishioka, Yonekura '13] (also [Cardy '88; Cappelli, Friedan, Latorre '91]).
- $\sigma \sim F_{\text{therm.}} / V_{d-1} T^d$ can increase, e.g. 4d SQCD [Appelquist, Cohen, Schmaltz '99], 3d $O(N)$ models [Sachdev '93 + Chubukov, Ye '94]. Also, σ can depend on exactly marginal couplings, e.g. $\sigma_{d=4}^{\mathcal{N}=4 \text{ SYM}}(N, \lambda \to \infty) = (3/4)\sigma_{\text{free}}$ [Gubser, Klebanov, Peet '96].
- In principle C_T can depend on exactly marginal couplings; no solid examples ($N_Q^{\text{SUSY}} \geq 4 \Rightarrow C_T = \text{const.}$) [Nakayama '17].

Cardy's Conjecture in Even Dimensions

Trace anomaly in even d [Duff '77; ...; Deser, Schwimmer '93; ...]:

$$\langle T^\mu_\mu(x)\rangle_g \sim (-1)^{d/2-1}\, a\, \underbrace{\mathrm{Euler}_d[g]}_{\sim (R_{\mu\nu\rho\lambda})^{d/2}} + \sum_{i=1}^{N_B(d)} c_i\, \mathrm{Weyl}_i[g] + (\text{s.d.})$$

a-anomaly unique, resides in $\langle T_{\mu_1\nu_1}(x_1)\cdots T_{\mu_n\nu_n}(x_{n\geq d/2+1})\rangle$,

$$\log Z_{S^d(r)} \sim (-1)^{d/2-1} a \log(r\Lambda) + \sum_{n=0,2,\ldots}^{d} k_{d-n} \underbrace{(\Lambda r)^{d-n}}_{\sim \Lambda^{d-n}\int \sqrt{g}\,(R[g])^n}$$

Generally multiple c-anomalies; none in $d=2$ ($a=c_{\mathrm{Vir.}}$, $\mathrm{Euler}_2[g]\sim R[g]$).

[Cardy '88] conjectured that $a_{\mathrm{UV}} > a_{\mathrm{IR}}$ (a-theorem). We also expect that $a \geq 0$. Proofs in $d=2,4$; SUSY results in $d=6$.

A Universal Conjecture

Consider the CFT vacuum EE $S(r)$ across a sphere $S^{d-2}(r)$:

$$S(r) \sim k_{d-2}(\Lambda r)^{d-2} + k_{d-4}(\Lambda r)^{d-4} + \cdots + \begin{cases} (-1)^{d/2-1}\, s \log(\Lambda r) & \text{(even)} \\ (-1)^{(d-1)/2}\, s & \text{(odd)} \end{cases}$$

area term [Bombelli, Koul, Lee, Sorkin '86; Srednicki '93; …]

Expected divergences for a diff.-invariant regulator [Liu, Mezei '12] (proposal: mutual information [Casini, Huerta, Myers, Yale '15]).

Counterterms: $\int_{S^{d-2}(r)} \sqrt{h}\, K[h]^n \sim r^{d-2-n}$, in vacuum only odd n.

[Myers, Sinha '10] analyzed holographic RG flows in higher-curvature gravity (more generic than Einstein gravity). They always found $s_{UV} > s_{IR}$ and conjectured a universal C-theorem for renormalized EE [Liu, Mezei '12]. In even d, $s \sim a$ [Casini, Huerta, Myers '11], so it reduces to Cardy's a-conjecture.

A Universal Conjecture (cont.)

In odd d [Casini, Huerta, Myers '11] showed that $s \sim F$, the scheme-independent part of the partition function on $S^d(r)$:

$$\log Z_{S^d(r)} \sim (-1)^{(d-1)/2} F + \sum_{n=0,2,\ldots}^{d-1} k_{d-n} (\Lambda r)^{d-n}$$

Independently [Jafferis+Klebanov, Pufu, Safdi '11] conjectured that $F_{\rm UV} > F_{\rm IR}$ (F-theorem; proofs in $d=3$). In all known examples, $F \geq 0$; expected to be true in general (no proof).

- Uniform evidence for these conjectures in all dimensions: holography [Myers, Sinha '10; Casini, Huerta, Myers '11; …] and weakly relevant flows [Cardy '88; Klebanov, Pufu, Safdi '11; Giombi, Klebanov '14].
- The conjectured C-functions $s \sim a, F$ are invariant under exactly marginal deformations. Basic idea: $\langle \mathcal{O} \rangle_{S^d}^{\rm CFT} = 0$.

Example: Conformal Window of 3d QED

C-theorems constrain possible RG flows, and hence dynamics, e.g. they constrain phase diagrams (talk by [Seiberg]):

- 3d QED: $U(1)$ gauge theory, $2N_f$ 2-component fermions of charge $+1$, flavor symmetry $SU(2N_F) \times U(1)_{\text{top.}}$.
- For $N_F \geq N_F^*$, inside the conformal window; for $N_f < N_F^*$, could break $SU(2N_f) \to SU(N_f) \times SU(N_f) \times U(1)$, leaving $2N_f^2$ NG bosons and the dual photon in the IR.
- Can flow from the conformal window to the broken phase (e.g. add masses). The 3d F-theorem $F_{\text{conf.}} > F_{\text{broken}}$ then constrains $N_f^* \leq 5$ [Giombi, Klebanov, Tarnopolsky '15] (see [Grover '12]).
- Recent duality webs suggest that $N_f^* = 1$ [Seiberg, Senthil, Wang, Witten '16; Hsin, Seiberg '16], i.e. QED with $N_f = 1$ is a CFT.

The a-Theorem in 4d

- Early perturbative evidence [Cardy '88; Osborn '89, '91 + Jack '90].
- Strong evidence from SUSY: formula for a-anomaly in terms of 't Hooft anomalies [Anselmi, Freedman, Grisaru, Johansen '97]:

$$\mathcal{N} = 1 : \ a \sim 3\left(U(1)_R^3 \text{ anomaly}\right) - \left(U(1)_R\text{-(gravity)}^2 \text{ anomaly}\right)$$

Computable, if the $U(1)_R$ symmetry is known; often determined by a-maximization [Intriligator, Wecht '03; Kutasov, Parnachev, Sahakyan '03]: many checks of $\Delta a = a_{\text{UV}} - a_{IR} > 0$.

- Can show that $a \geq C_T \geq 0\,, C_T \sim \langle TT \rangle$. Non-trivial: a first appears in $\langle TTT \rangle$ (no obvious positivity). Related to positive energy flux at infinity (ANEC) [Hofman, Maldacena '08]; recent proofs: [Hofman et. al. '16; Faulkner et. al. '16; Hartman et. al. '16]

Proof of the 4d a-Theorem

The general proof [Komargodski, Schwimmer '11] (also [Komargodski '11; Luty, Polchinski, Rattazzi '12]) depends on several ingredients:

- Imagine starting with a $\mathrm{CFT}_{\mathrm{UV}}$ with a moduli space of vacua: conformal symmetry is spontaneously broken.
- In the deep IR, must find a nearly free, massless scalar NG boson: the dilaton φ. It very weakly interacts with itself (via irrelevant operators), and perhaps other d.o.f. in a $\mathrm{CFT}_{\mathrm{IR}}$.

$$\mathrm{CFT}_{\mathrm{UV}} \longrightarrow \mathrm{CFT}_{\mathrm{IR}} + \text{dilaton } \varphi \qquad \Delta a = a_{\mathrm{UV}} - (a_{\mathrm{IR}} + a_{\varphi})$$

- The a-anomaly must match between UV and IR: like 't Hooft anomaly matching [Schwimmer, Theisen '10]. The mismatch Δa is compensated by a Wess-Zumino like term in the dilaton Lagrangian (typical of anomaly matching with NG bosons).

Proof of the 4d a-Theorem (cont.)

$$\mathscr{L}_\varphi = \frac{1}{2}(\partial\varphi)^2 + \frac{\Delta a}{\varphi^4}(\partial\varphi)^4 + \mathcal{O}\left(\partial^6, \varphi\text{-CFT}_{\text{IR}} \text{ coupling}\right)$$

- Constrained by nonlinearly realized conformal symmetry acting on φ. There is a systematic procedure to classify all terms. The ∂^4-term is the Wess-Zumino term, whose coefficient Δa is fixed by the anomaly mismatch.
- Causality/unitarity force this coefficient to to be positive [Adams, Arkani-Hamed, Dubovsky, Nicolis, Rattazzi '06], proving $\Delta a > 0$.
- 4-dilaton scattering probes $\langle T^\mu_\mu(p_1)\cdots T^\mu_\mu(p_4)\rangle$ with $p_i^2 = 0$.
- Flows initiated by relevant couplings break conformal symmetry explicitly. Convert to spontaneous breaking by introducing φ as a background field (spurion/compensator).

Scale and Conformal Invariance

In the previous 4d discussion, we assumed that the UV, IR asymptotics are given by CFTs. Is this always true? In 2d, scale invariance implied $T^\mu_\mu = 0$ (sufficient) [Polchinski '88].

Not necessary in higher dimensions, due to improvements:

$$T'_{\mu\nu} = T_{\mu\nu} + \left(\partial_\mu\partial_\nu - \eta_{\mu\nu}\partial^2\right)\mathcal{O}\,, \quad T'^\mu_\mu = T^\mu_\mu + (1-d)\partial^2\mathcal{O}$$

CFT requires $T^\mu_\mu \sim \partial^2\mathcal{O}$. In 4d: check in perturbation theory [Osborn '89, '91 + Jack '90,'13; Luty, Polchinski, Rattazzi '12; Fortin, Grinstein, Stergiou '12]. Beyond:

$$\langle 0|T^\mu_\mu(p_1)\cdots T^\mu_\mu(p_n)|\Psi\rangle = 0\,, \quad p_i^2 = 0$$

[Luty, Polchinski, Rattazzi '12; Dymarsky, Komargodski, Schwimmer, Theisen '13]

Dilaton S-matrix trivial, so $\varphi\, T^\mu_\mu$ must vanish on shell. Thought to imply $T^\mu_\mu \sim \partial^2\mathcal{O}$. No such results in other dimensions; free Maxwell theory is a counterexample [El-Showk, Nakayama, Rychkov '11].

The F-Theorem in 3d

Recall: $\log Z_{S^3(r)} = k_3(\Lambda r)^3 + k_1(\Lambda r) - F$. In all known examples, $F \geq 0$. No proof, but in TQFT $F = -\log|S_{0,0}| \geq 0$ is the topological EE of [Kitaev, Preskill '05; Levin, Wen '05]: nonlocal, cannot be extracted from $T_{\mu\nu}$ correlators, e.g. $F_{U(1)_k} \sim \log k$.

Strong evidence for $F_{\rm UV} > F_{\rm IR}$ from $\mathcal{N}=2$ SUSY RG flows with a $U(1)_R$ symmetry: F can be computed exactly using localization [Kapustin, Willett, Yaakov '09; Jafferis '10; Hama, Hosomichi, Lee '10] and F-maximization [Jafferis '10; Jafferis, Klebanov, Pufu, Safdi '11; Closset, TD, Festuccia, Komargodski, Seiberg '12].

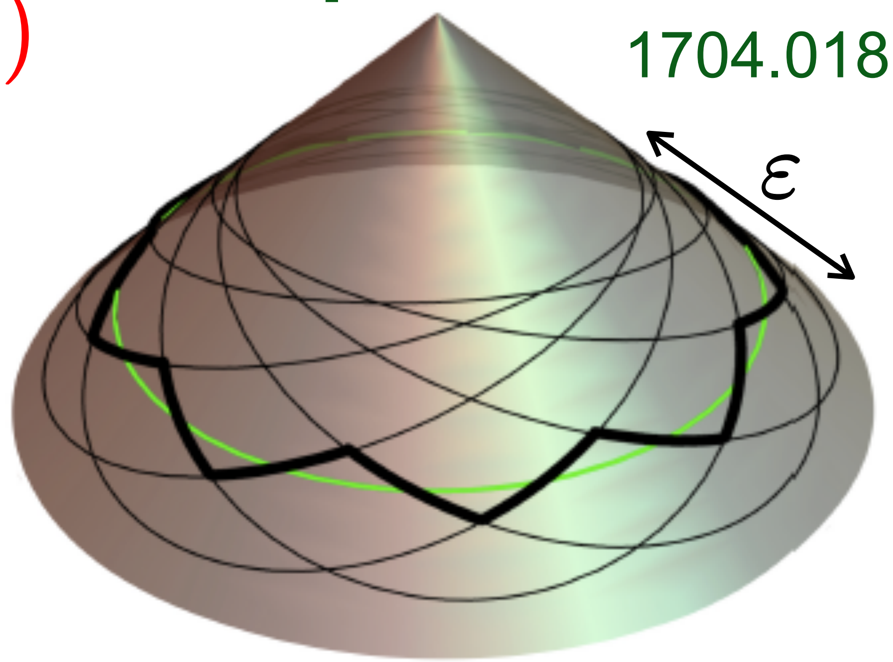
The F-theorem forces $F_{\rm Maxwell} = \infty$ (flow from free Maxwell in the UV to $U(1)_k$). A closely related fact: it is not a CFT.

The only known proofs use EE [Casini, Huerta '12; Casini, Huerta, Myers, Yale '15; Casini, Teste, Torroba '17] (gong show talk by [Teste]).

The F-Theorem in 3d (cont.)

Renormalized EE: $F(r) = rS'(r) - S(r)$

[Casini, Teste, Torroba: 1704.01870]



$S(r)$ is the bare EE across a circle.

To get an inequality for $S(r)$ from SSA, consider N uniformly spaced circles X_i (radius $\sqrt{rR}$) on null cone:

$$\sum_i S(X_i) \geq \underbrace{S(\cup_i X_i)}_{\sim S(R)} + S(\cup_{ij}(X_i \cap X_j)) + \cdots + \underbrace{S(\cap_i X_i)}_{\sim S(r)}$$

As $N \to \infty$, the EE of all jagged circles reduce to that of a round circle (varying radius between r, R). As $\varepsilon \to 0$:

$$S''(r) \leq 0 \qquad \Longrightarrow \qquad F'(r) = rS''(r) \leq 0$$

Can generalize to prove 4d a-theorem: regulate singularities by applying SSA to $S(r) - S_{\mathrm{UV}}(r)$ (Markov property: [CTT '17; Lashkari '17]).

Boundary RG Flows

So far we have only considered RG flows of QFTs in infinite flat space. We can also consider the theory in finite volume, with suitable boundary conditions, e.g. take a 2d CFT on an interval of length L. Thermal partition function [Affleck, Ludwig '91]

$$\log Z(T) \sim c_{\text{Vir.}} L\, T + \log g \qquad (L \gg T^{-1})$$

If the boundary is also conformal, then g is a constant. If we perturb the system by adding a relevant boundary operator (fixed bulk theory), then g evolves along the RG flow in such a way that $g_{\text{UV}} > g_{\text{IR}}$ [Affleck, Ludwig '91; Friedan, Konechny '03; Casini, Landea, Torroba '16]. Recent generalizations to higher dimensions and co-dimensions, e.g. [Jensen, O'Bannon '15] use dilaton effective actions to prove a boundary C-theorem in 3d.

RG Flows in $d > 4$ Dimensions

We do not know of any interacting CFTs in $d > 4$ without SUSY (why?), but there is a rich zoo of interacting SCFTs in $d = 5, 6$. Their existence is inferred from string theory, no Lagrangian description (talk by [Kim]). Can we understand their RG flows? Do these flows obey any C-theorems?

Status of the F-theorem in d = 5:

- [Jafferis, Pufu '12] verified $F_{\rm UV} > F_{IR}$ for some flows between SCFTs in the UV and IR (localization on S^5).
- No general proof, even for SUSY-preserving flows.
- No argument that $F \geq 0$ at (S)CFT fixed points.
- $F_{\rm Maxwell} = -\infty$ [Giombi, Klebanov, Tarnopolsky '15] seems to ruin positivity away from fixed points (many SUSY examples).

The a-Theorem in Six Dimensions

[Maxfield, Sethi '12; Elvang, Freedman, Hung, Kiermaier, Myers, Theisen '12] showed:

$$\mathscr{L}_\varphi = \frac{1}{2}(\partial\varphi)^2 + \frac{b}{\varphi^3}\,(\partial\varphi)^4 + \frac{\Delta a}{\varphi^6}(\partial\varphi)^6 + \mathcal{O}\left(\partial^8, \varphi\text{-CFT}_{\text{IR}} \text{ coupling}\right)$$

- $b > 0$, but not in general related to Δa. The ∂^6 Wess-Zumino term is schematic (interactions of ≥ 4 dilatons).
- No known constraint on low-energy actions gives $\Delta a \geq 0$. Intuitively, this is because b dominates at low energies.
- Focus on (1,0) SUSY RG flows starting from a UV SCFT:
  - no SUSY-preserving relevant operators [Louis, Lüst '15; Cordova, TD, Intriligator '15 + '16]
  - instead, consider RG flows onto the moduli space of vacua, where the dilaton is a physical NG boson.

The a-Theorem in Six Dimensions (cont.)

$$\mathscr{L}_\varphi = \frac{1}{2}(\partial\varphi)^2 + \frac{b}{\varphi^3}(\partial\varphi)^4 + \frac{\Delta a}{\varphi^6}(\partial\varphi)^6 + \cdots$$

- On tensor branches φ resides in a tensor multiplet; SUSY constraints imply $\Delta a \sim b^2 > 0$ (a-theorem) [Cordova, TD, Intriligator '15]
- This relation was first noticed in maximally supersymmetric (2,0) theories [Maxfield, Sethi '12; Elvang et. al. '12; Cordova, TD, Yin '15].
- Turning on supergravity background fields shows that b , Δa are related to Green-Schwarz terms for R-symmetry and gravitational 't Hooft anomalies: formula for a in terms of anomalies (more a-theorem checks [Heckman, Rudelius '15]).
- No general proof that $a \geq 0$ in 6d, but one can show it for SUSY theories with tensor branches [Cordova, TD, Intriligator - in progress]

Two Interesting Proposals

- Recall: $\sigma \sim F_{\text{therm.}}/V_{d-1}T^d$ violates $\sigma_{\text{UV}} > \sigma_{\text{IR}}$ for some RG flows (not a general C-function). All counterexamples involve an interacting UV CFT; conjectured to hold if the UV theory is free [Appelquist, Cohen, Schmaltz '99]. If true, it gives a tighter bound on the conformal window of 3d QED:

$$N_f^* \leq 2 \quad (\sigma\text{-conjecture}) \qquad \text{vs.} \qquad N_f^* \leq 5 \quad (F\text{-theorem})$$

- [Gukov '15 + '16] has investigated global aspects of RG flows, e.g. using arguments from Morse theory he conjectured that if the RG flow is a gradient flow, then

$$\mu_{\text{UV}} > \mu_{\text{IR}} \qquad \mu = \#\,(\text{relevant operators})$$

  Counterexamples with dangerously irrelevant operators in $d > 2$ suggest obstructions to gradient flow.

Instead of Conclusions

	2d	3d	4d	5d	6d
$C_{\text{UV}} > C_{\text{IR}}$	✓	✓	✓	SUSY Checks	**SUSY Proof**
$C_{\text{UV, IR}} > 0$	✓		✓	✗	**SUSY Proof**
$C'(E) > 0$	✓	✓	✓		
$C(E) > 0$	✓		✓		
$\text{SFT} \Rightarrow \text{CFT}$	✓		✓		
gradient flow	✓		**P.T.**		

I expect that some entries will be filled in by Strings 2018!

Thank You for Your Attention!