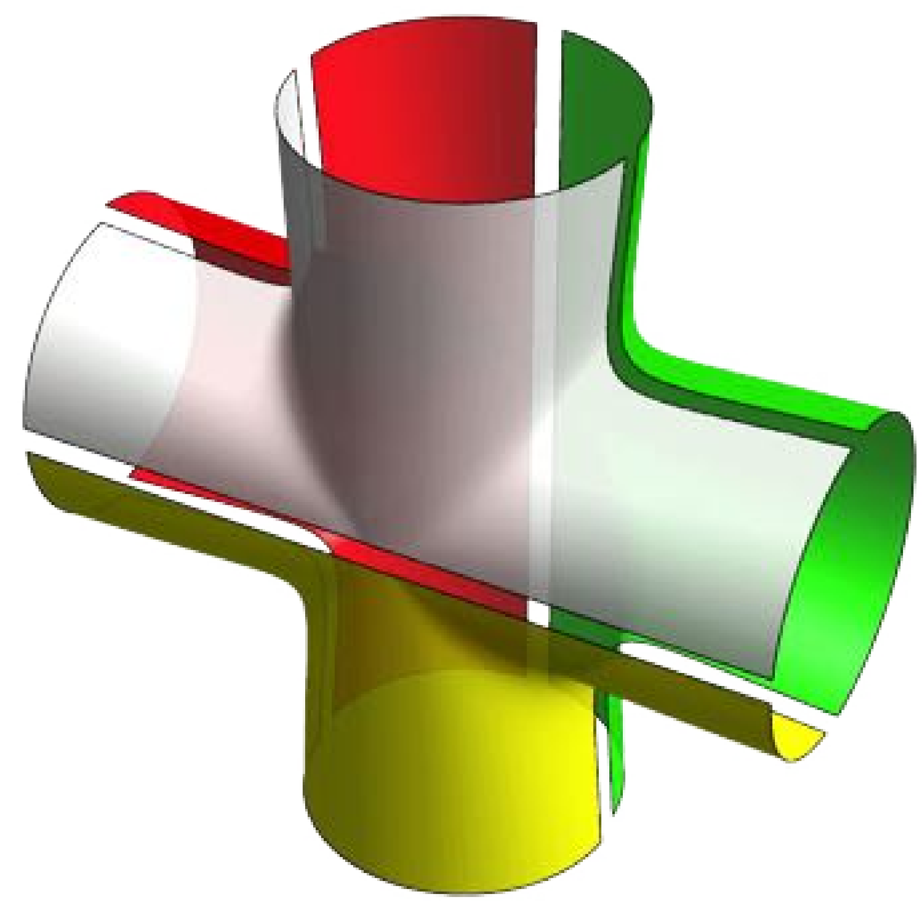
Divide and Conquer - An Integrability Status Report

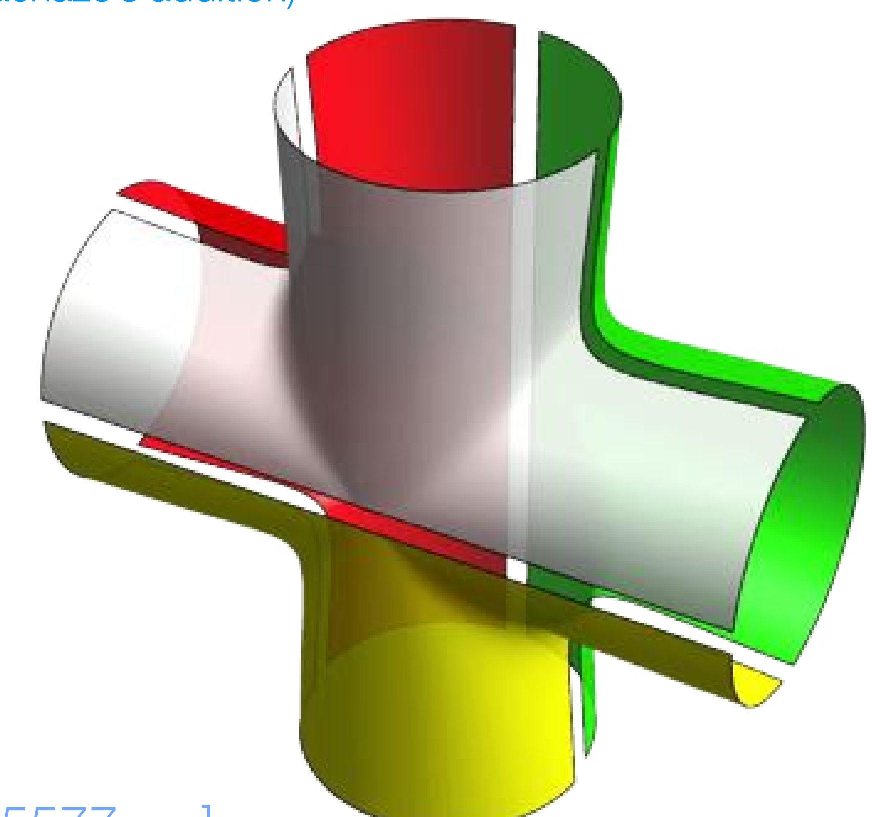
Pedro Vieira
Perimeter Institute & ICTP-SAIFR



Punch-line and an highlight

- In AdS_5, string amplitudes with complex topologies can be cut into **rectangular**, **pentagonal** or **hexagonal** patches which can be *Bootstrapped using Integrability at any 't Hooft coupling*.
- Amplitudes are given as infinite sums and integrals arising from stitching back these patches.
- Sometimes we can re-sum (part of) these sums/integrals (often finding hints of yet to be understood structures).
- Comparisons with weak and strong coupling computations work (so far) *and* they are key in developing new integrability tools themselves. "*Shut up, calculate and contemplate*"

(Freddy Cachazo's addition)



[…, Komatsu, Fleury "Hexagonalization of Correlation Functions" 1611.05577, …]

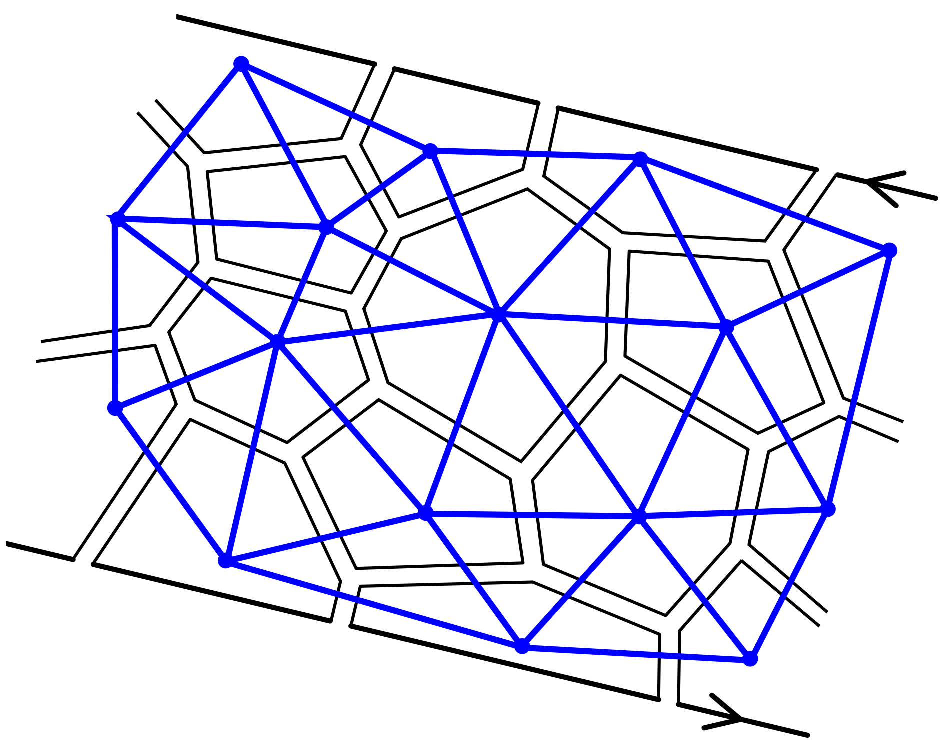
Outline

- 2D and Integrability
- Spectrum (i.e. cylinder)
- Beyond the Spectrum (i.e. other topologies)
- Open problems

Start in 2D

- Strings are two dimensional.
- 4D large N gauge theories are *also* string theories when properly thought of.
    - Correlation functions of n single trace operators = n closed strings
    - Flux tubes = open strings

$$\text{string tension} = \sqrt{\lambda}$$
$$\text{string coupling} = 1/N$$



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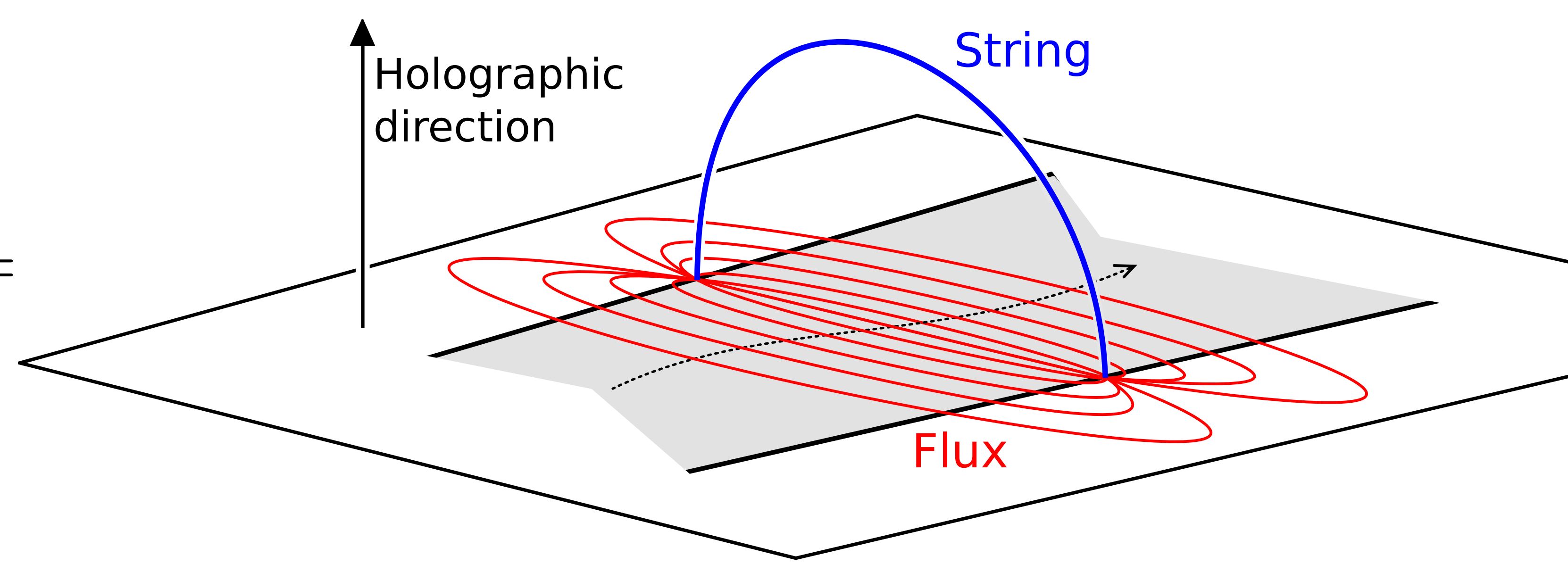
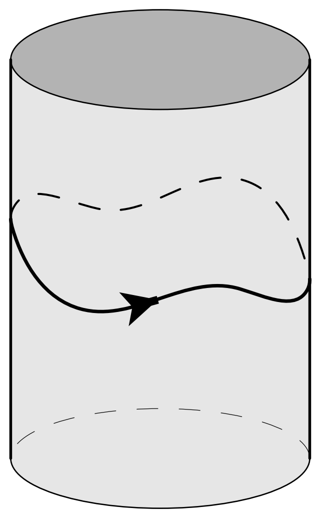
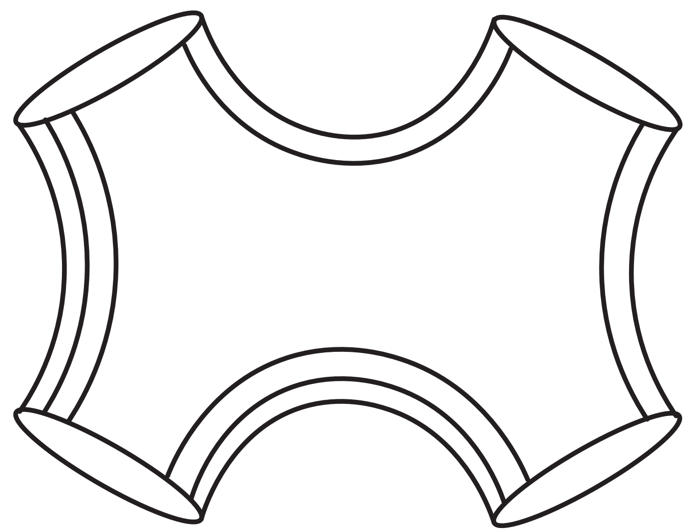


A Zoo of 2D Possibilities

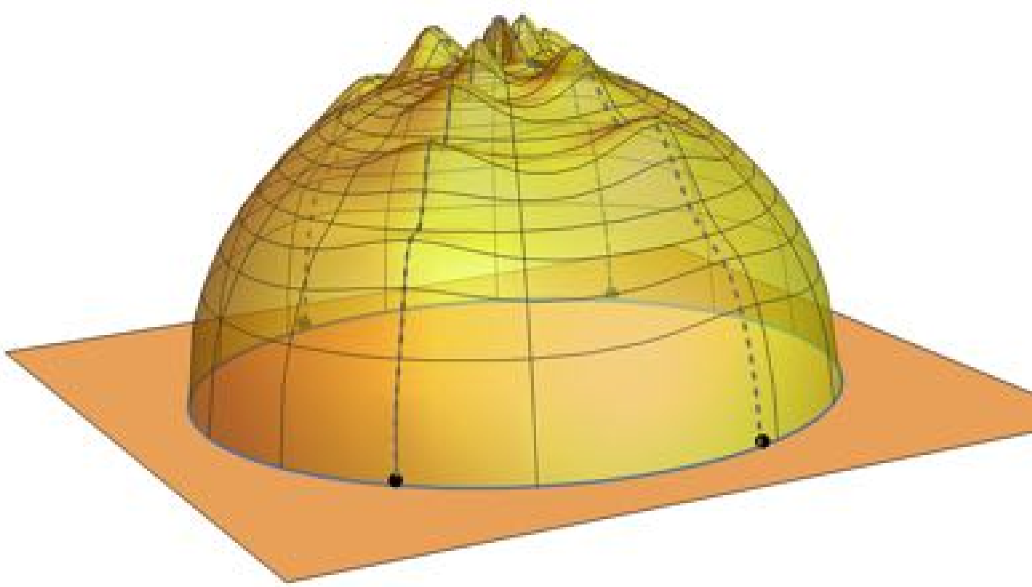
Cylinder
[Beisert et al review 2009]



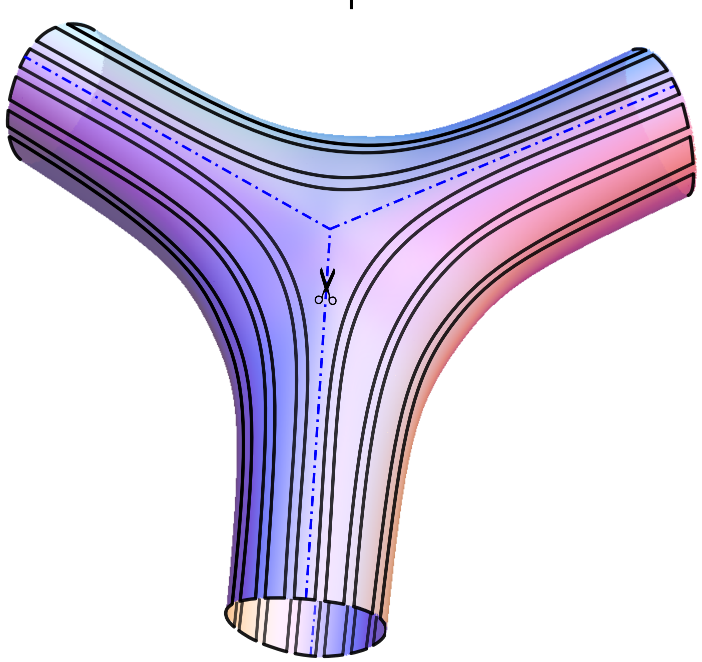
Sphere with Four Punctures



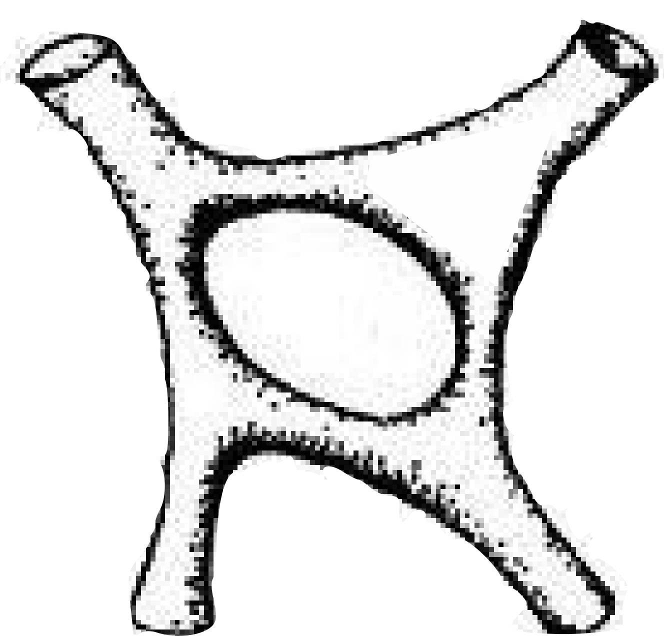
Disk with Circular Boundary
[Giombi, Roiban, Tseytlin 2017]



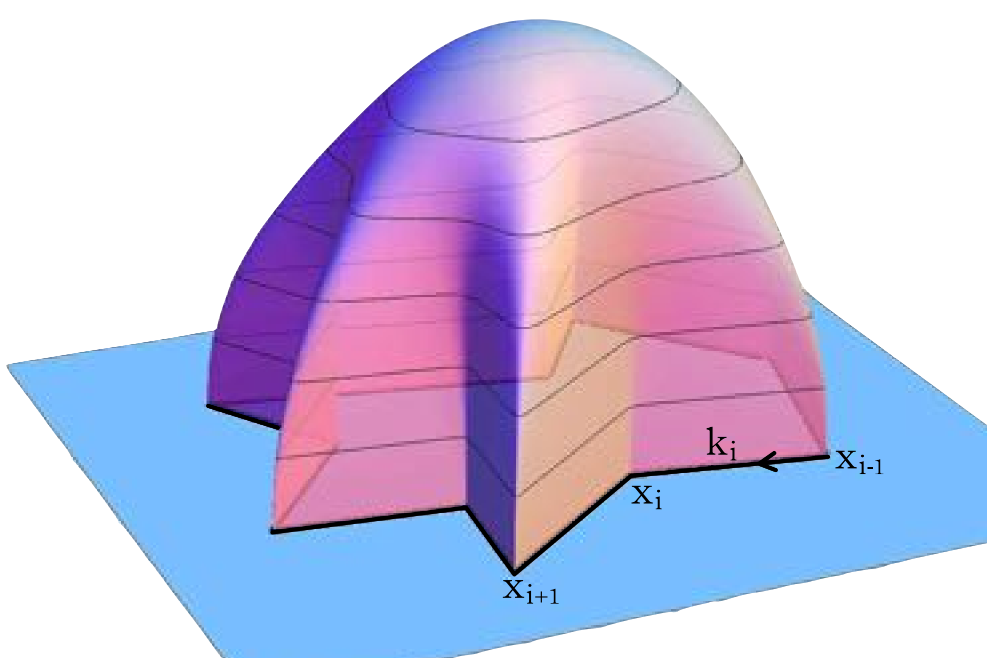
Pair of pants



Sphere with Four Punctures
and one Handle



Disk with Null Polygonal Boundary
[Alday, Maldacena 2007,…]

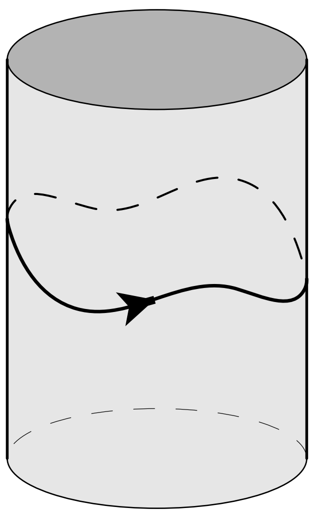


Standard 2D QFT
(in finite volume)

2D QFT on funny topologies

A Zoo of 2D Possibilities

Cylinder
[Beisert et al review 2009]



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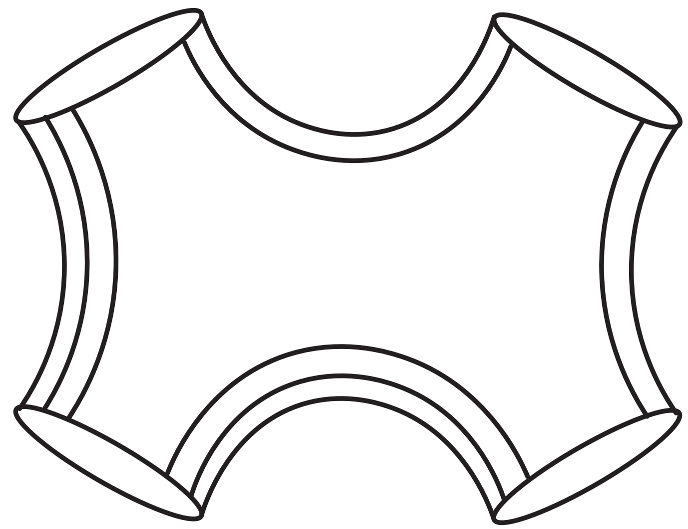

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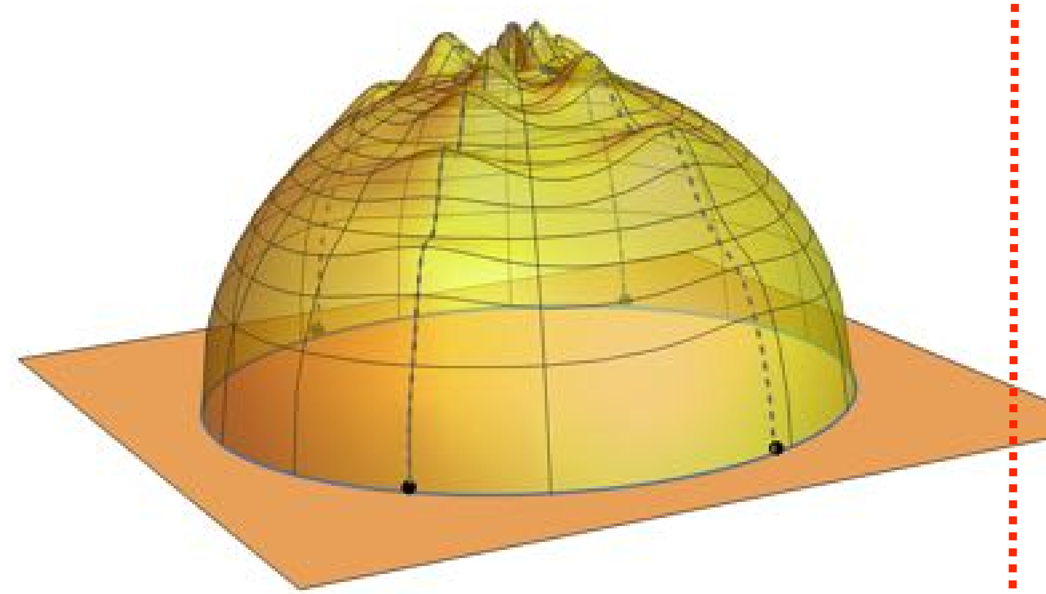

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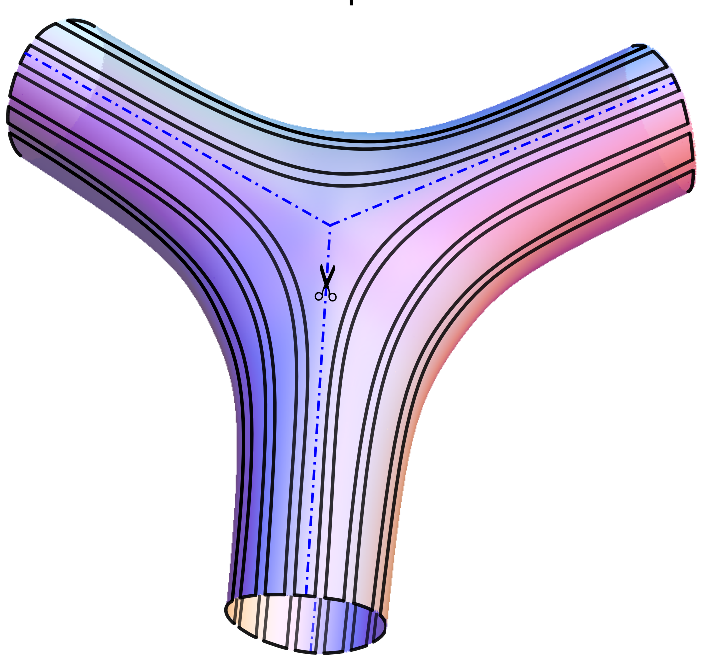

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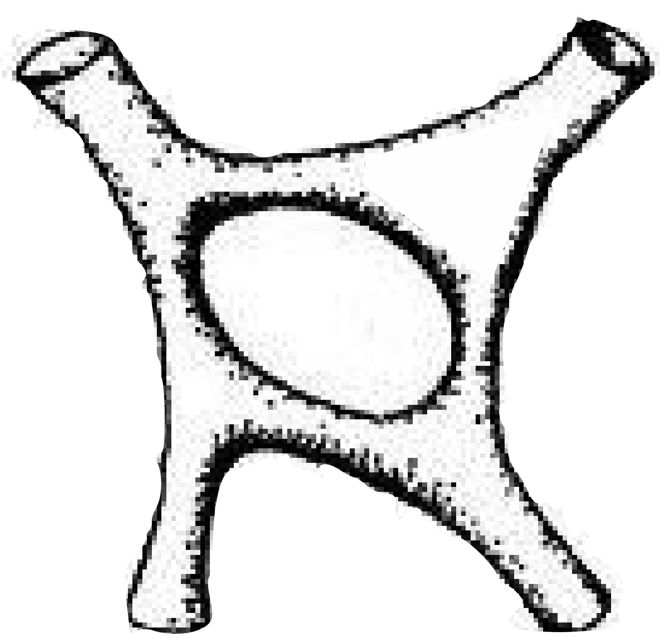

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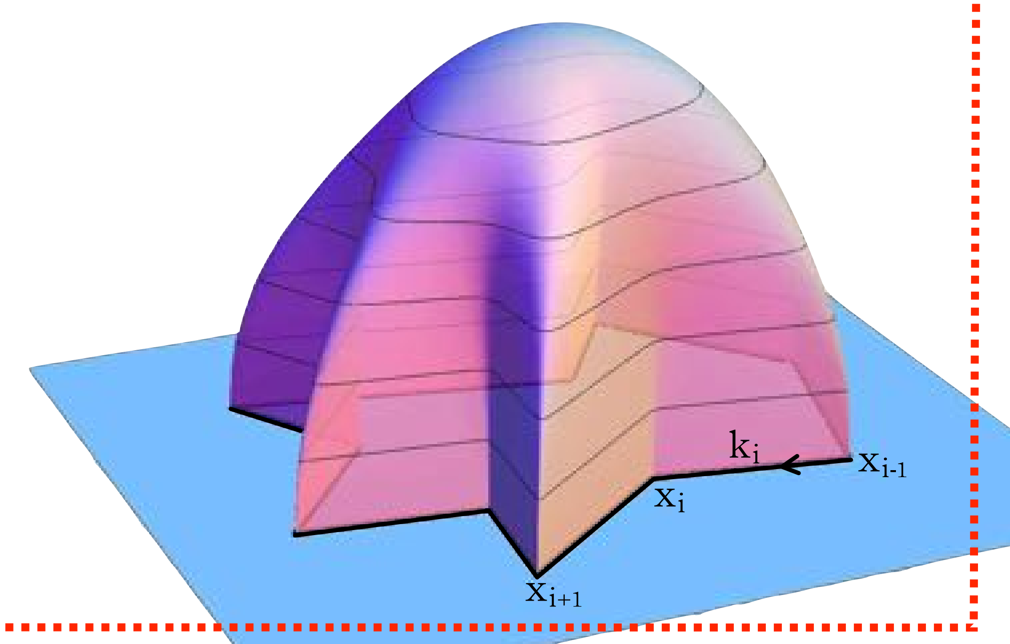
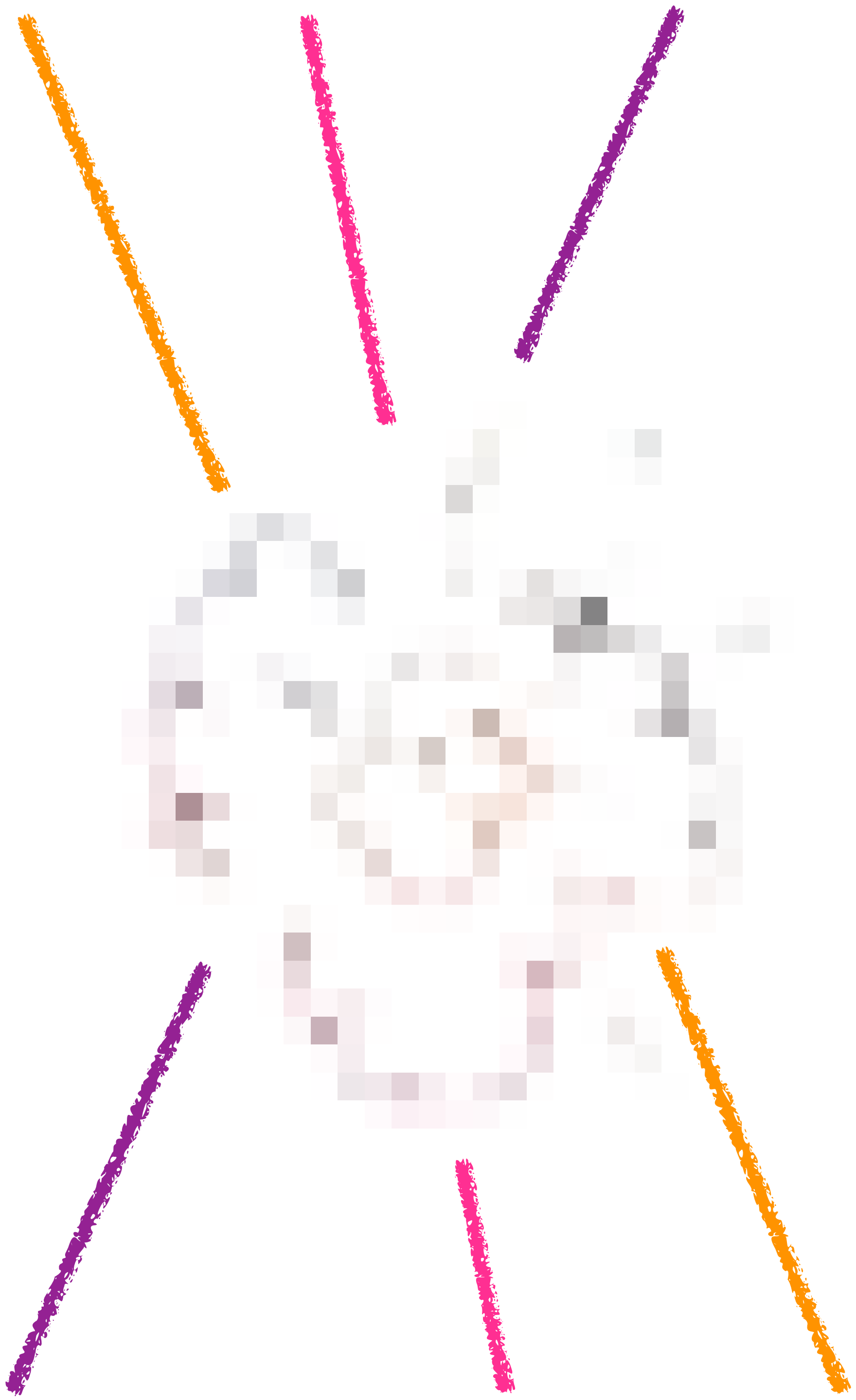


Integrability



$$P_{\text{final}} = \begin{cases} \{p_1', p_2', p_3'\} = \{p_1, p_2, p_3\} \\ \text{or} \\ \text{something else} \end{cases}$$

Integrability Smoking Gun

Generic case

Oliver: So Walter, roughly speaking what are the chances that the world is going to be destroyed?
Walter: It is 50%. If you have something that can happen and something that won't necessarily happen, its gonna either happen or not happen and so the best is 1 in 2…
Oliver: I'm not sure that's how probability works Walter…

$$P_{\text{initial}} = \{p_1, p_2, p_3\}$$

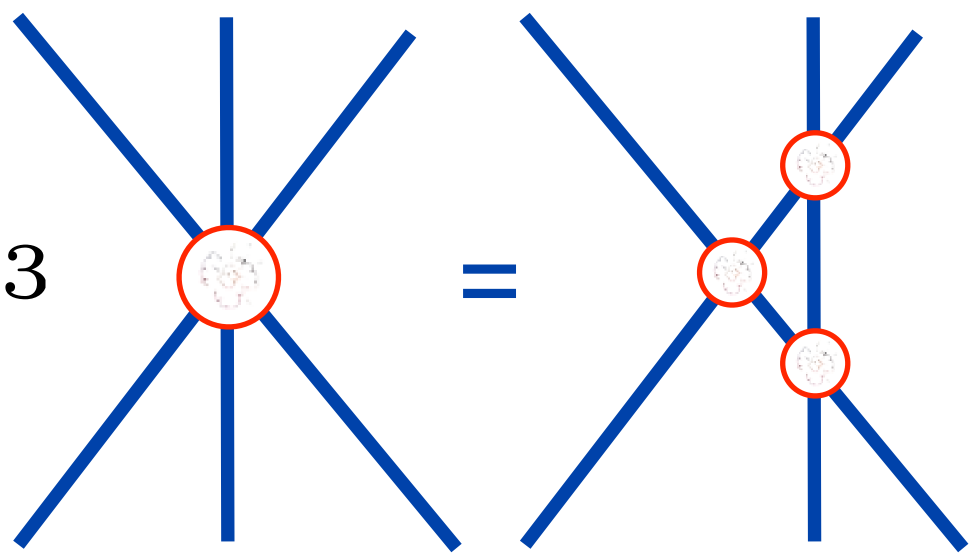
[BHs at the LHC]

Integrability

In 2D $\quad Q_1 = \sum p_j \ , \ Q_2 = \sum p_j^2 \ , \ \Rightarrow \ \{p_1, p_2\} = \{p'_1, p'_2\}$

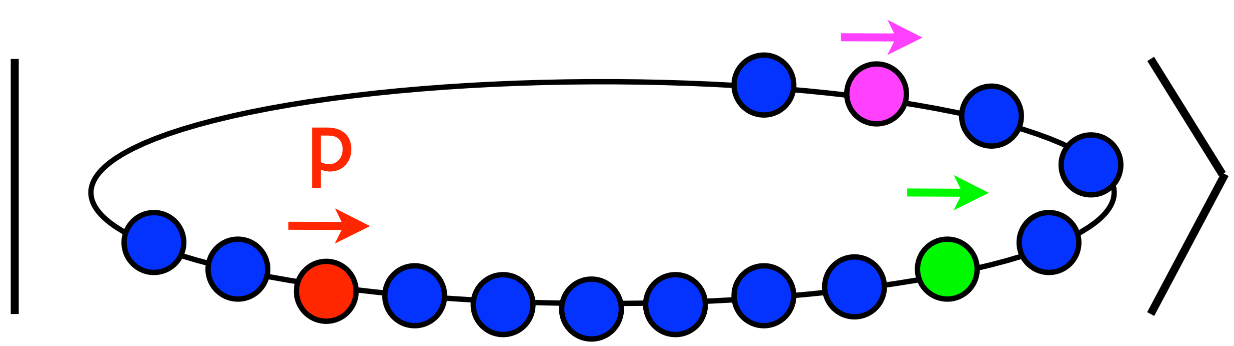
***Integrability* : If** $\quad \exists Q_3 = \sum p_j^3 \ \Rightarrow \ \{p_1, p_2, p_3\} = \{p'_1, p'_2, p'_3\}$

$$e^{i\alpha Q_3}$$



Integrable Spin Chains at Weak Coupling, **Integrable** Classical Ripples at Strong Coupling

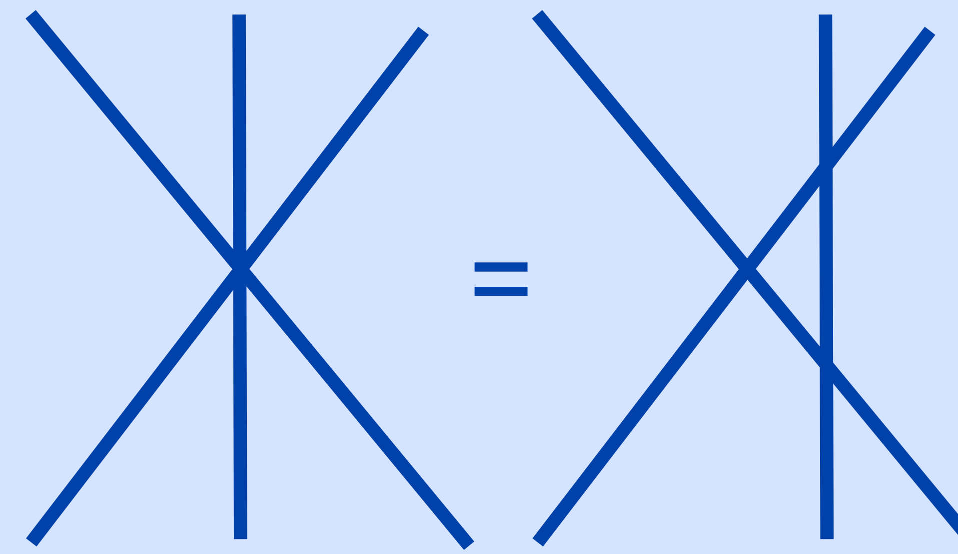
composite operator in the gauge theory:



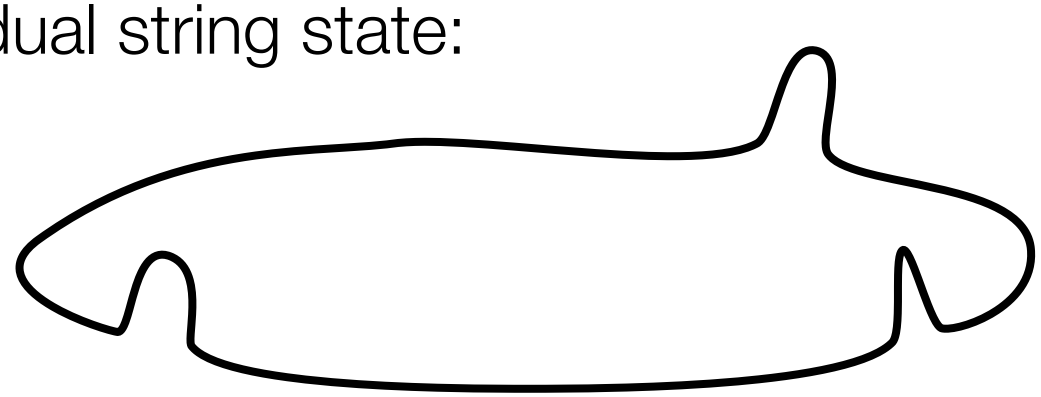
Integrable Spin Chain

[Minahan, Zarembo; Beisert, Staudacher]

Integrability persists at any coupling
(true but not proved)



dual string state:

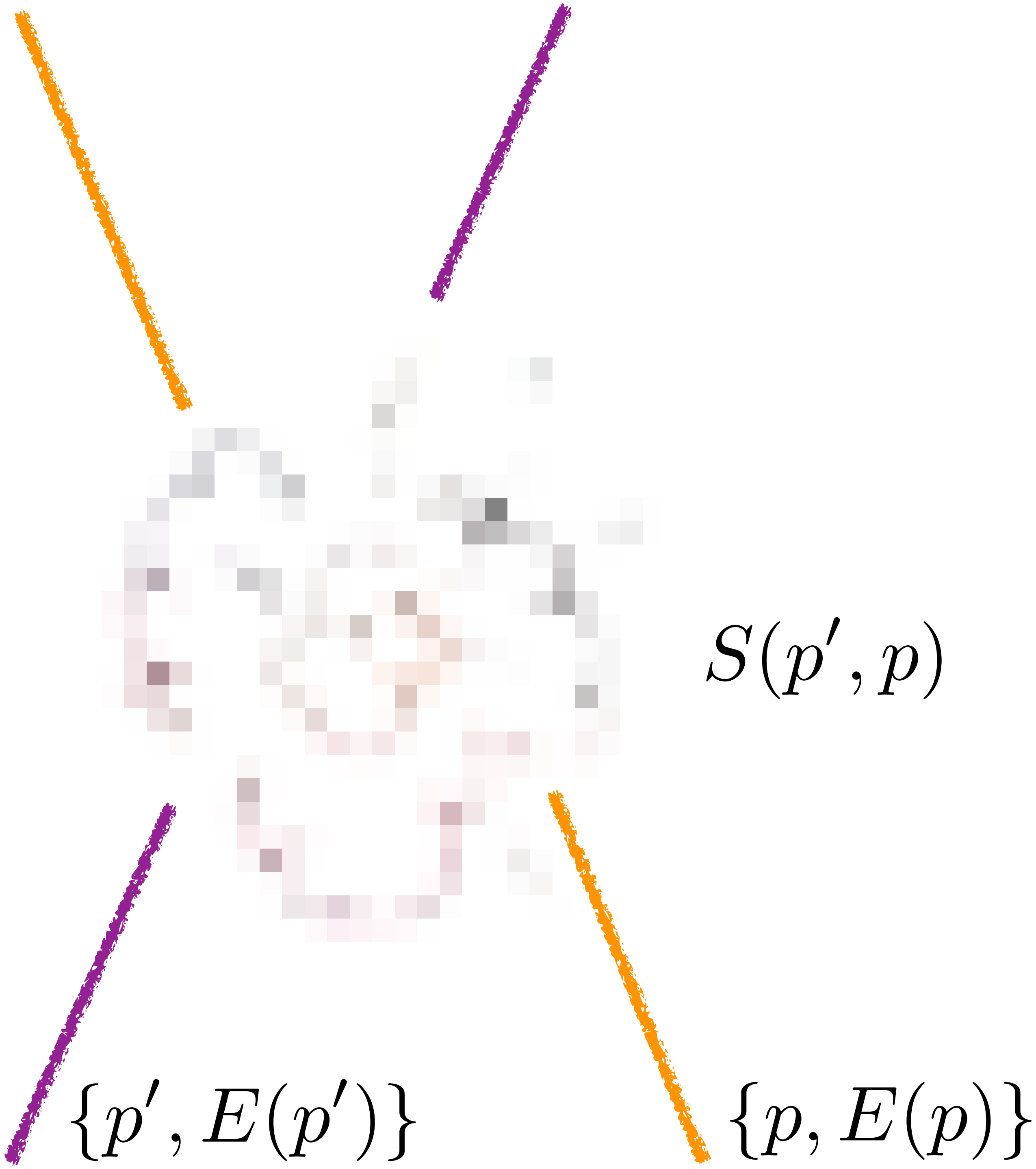


Integrable Classical String

[Benna, Polchinski, Roiban]

$\mathcal{N}$=4 SYM

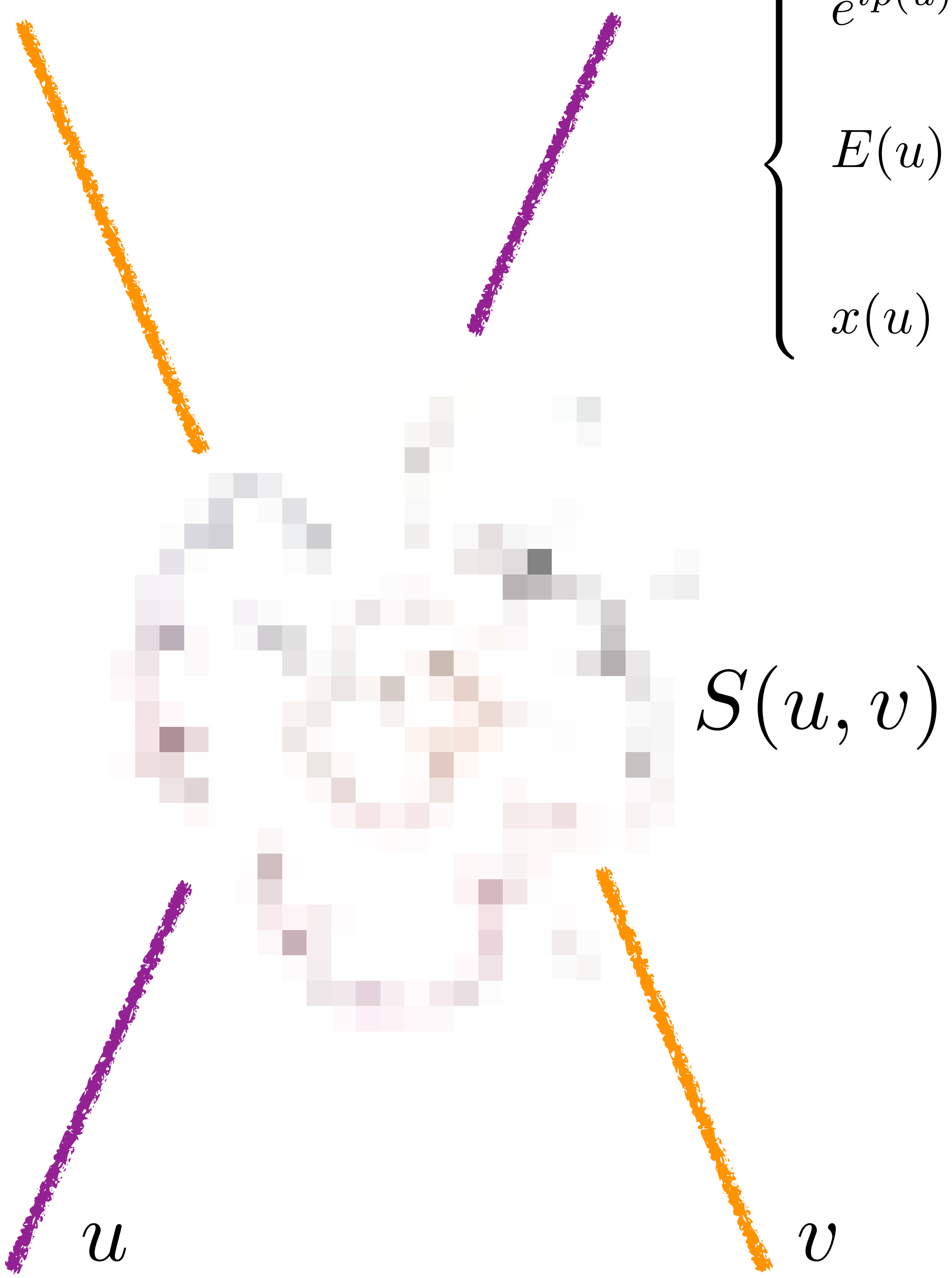
Unusual and rich 2D particle theory



$$E(p) = \sqrt{1 + 16g^2 \sin^2 \frac{p}{2}}$$

HALF SPIN-CHAIN MAGNON,
HALF RELATIVISTIC PARTICLE

Rapidity u

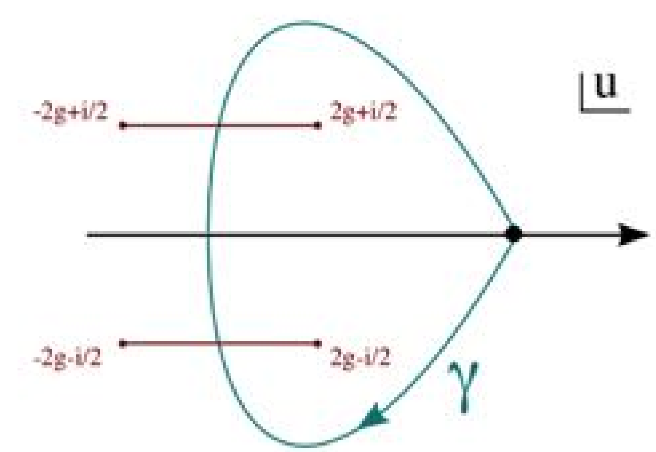


$$\begin{cases} e^{ip(u)} = \frac{x(u+i/2)}{x(u-i/2)} \\ E(u) = \frac{2ig}{x(u+i/2)} - \frac{2ig}{x(u-i/2)} \quad \Leftrightarrow E(p) = \sqrt{1+16g^2\sin^2\frac{p}{2}} \\ x(u) = \frac{u+\sqrt{u^2-4g^2}}{2g} \end{cases}$$

Crossing particle into anti-particle is a path in u:

A **mirror** transformation - or Wick rotation - is half that.

Both are non-perturbative

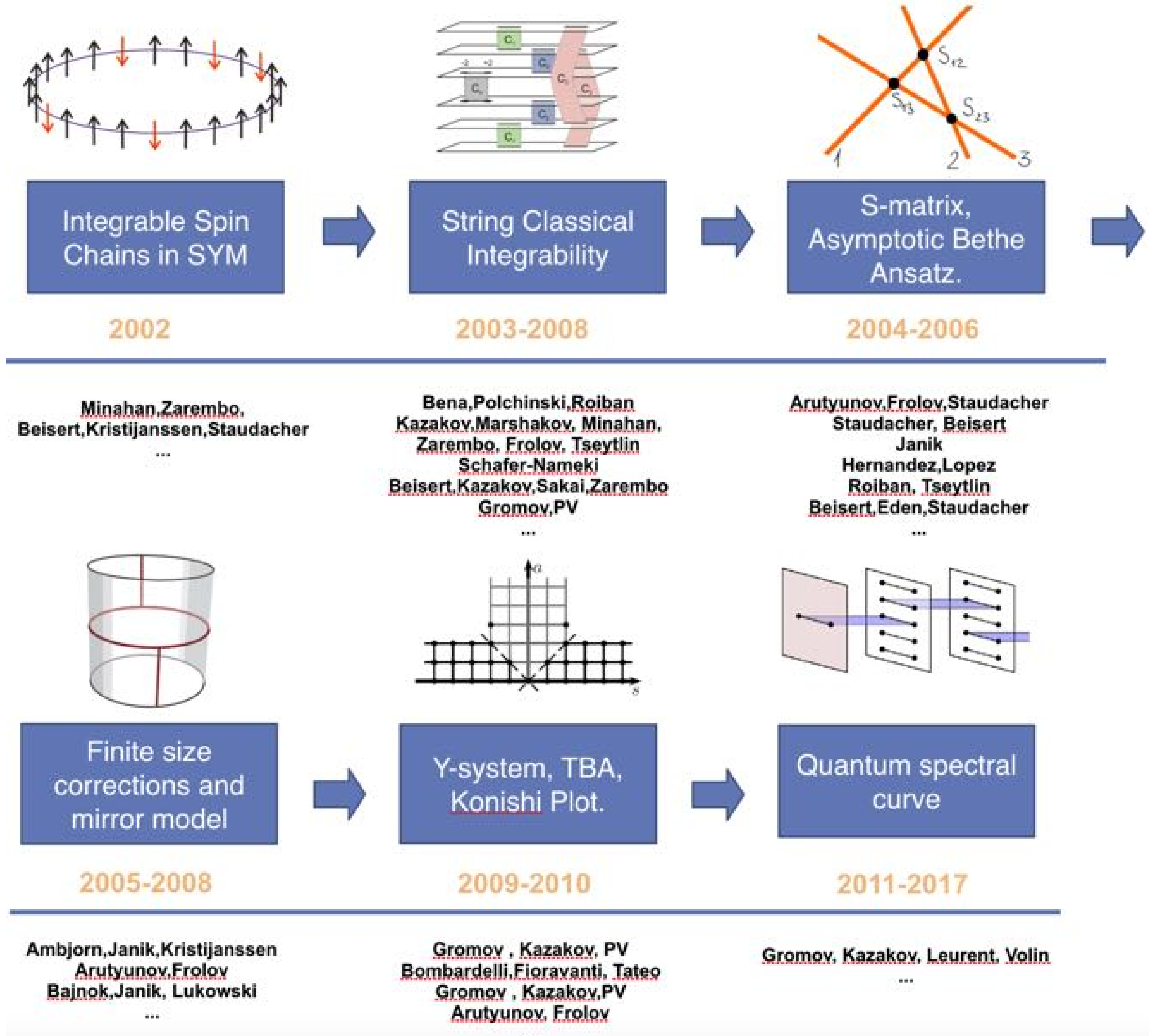


Analogue of usual hyperbolic rapidity

$$\begin{cases} p(\theta) = m\sinh(\theta) \\ E(\theta) = m\cosh(\theta) \end{cases} \Leftrightarrow E(p) = \sqrt{m^2+p^2}$$

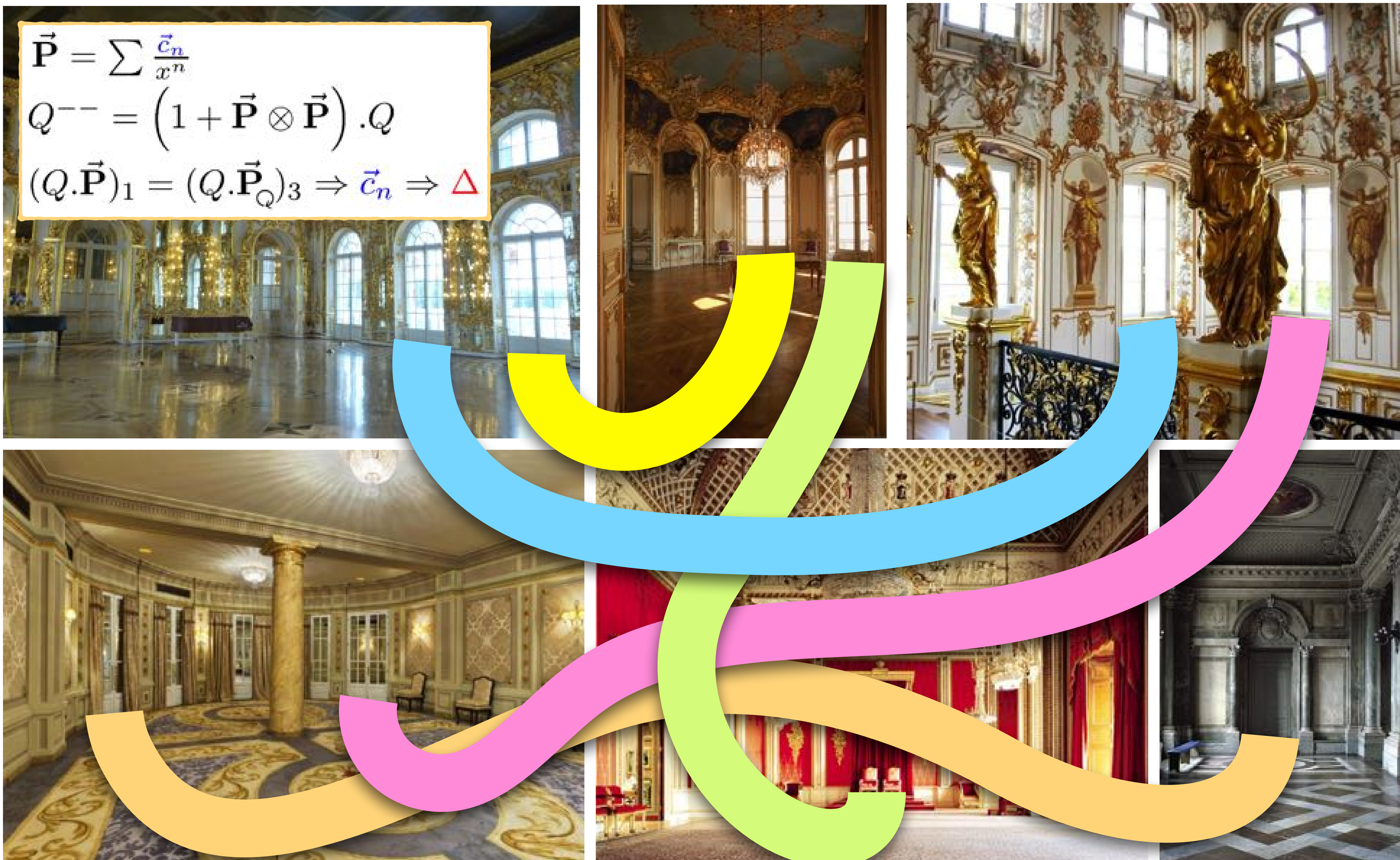
Crossing here is just translation of rapidity by **iπ**.
A **mirror** transformation - or Wick rotation - is half that.

The Planar Spectrum of a Gauge Theory



Quantum Spectral Curve

[Gromov, Kazakov, Leurent, Volin]



Cute spectrum plots

[Gromov, Levkovich-Maslyuuk, Sizov, 2015]

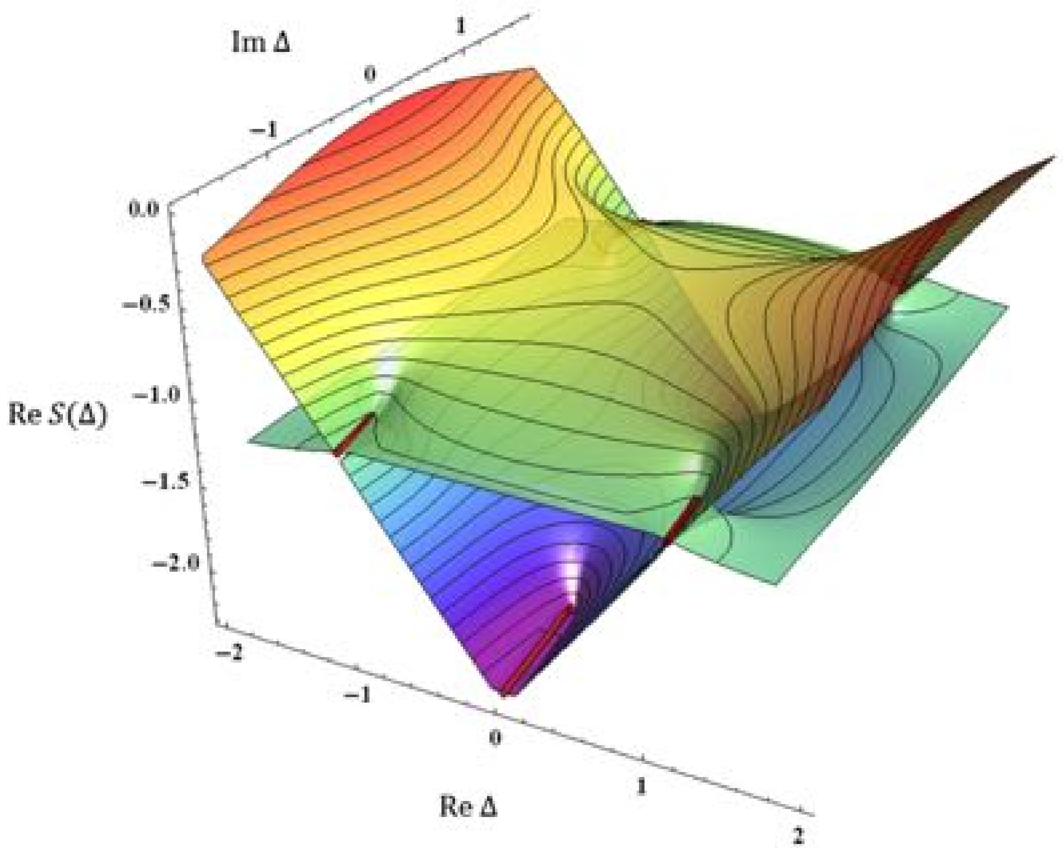


Figure 1: Riemann surface of the function $S(\Delta)$ for twist-2 operators. Plot of the real part of $S(\Delta)$ for complex values of Δ, generated from about 2200 numerical data points for $\lambda \approx 6.3$. We have mapped two Riemann sheets of this function. The thick red lines show the position of cuts. The upper sheet corresponds to physical values of the spin. Going through a cut we arrive at another sheet containing yet more cuts.

[Gromov, Kazakov, Korchemsky, Negro, Sizov 2017]

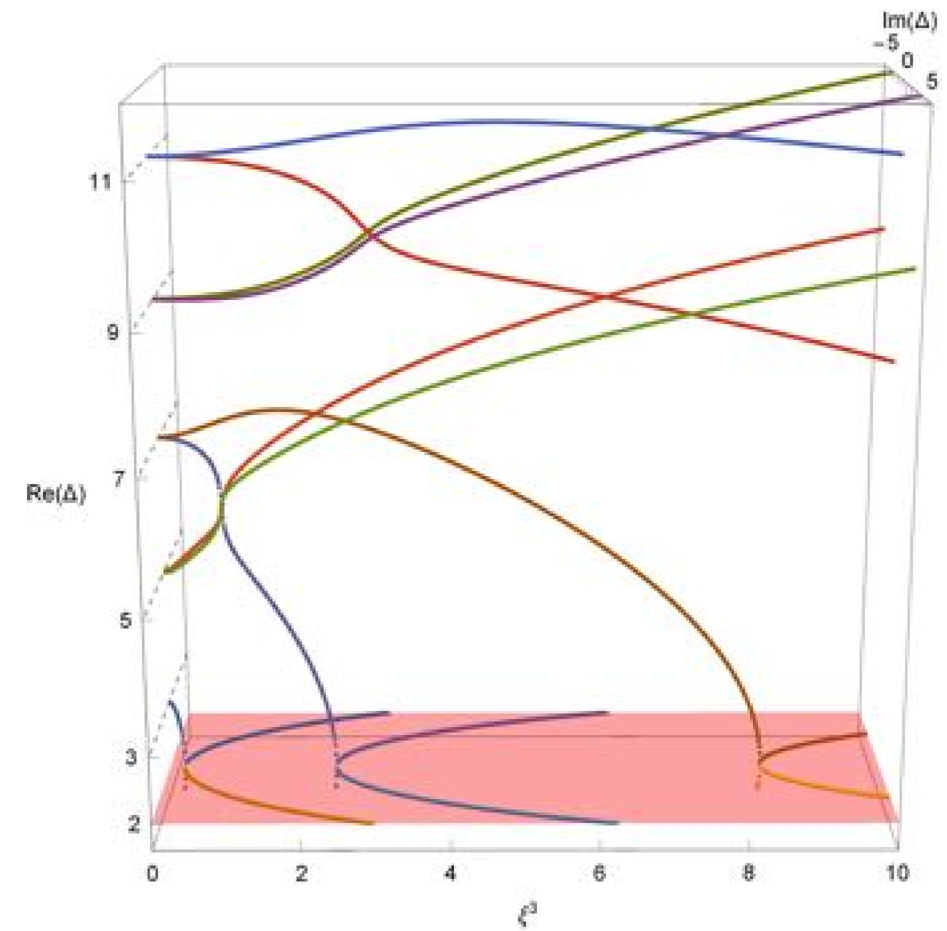
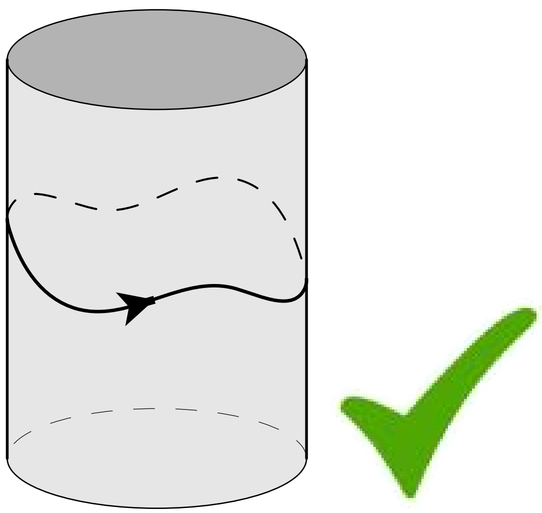


Figure 8. Real and imaginary part of the scaling dimension of the nine lowest lying states with $J = 3$. The curve that starts at $\Delta(0) = 3$ corresponds to the operator $\mathrm{tr}(\phi_1^3)$. The pair of states that start at $\Delta(0) = 3 + 2k$ with $k = 1, 2, 3, 4$ correspond to the operators of the form (1.2) (or rather to their linear combinations diagonalizing the dilatation operator).

N=4 SYM with extreme imaginary twists
[Gurdogan, Kazakov 2015]

That is it about the spectrum

Cylinder
[Beisert et al review 2009]



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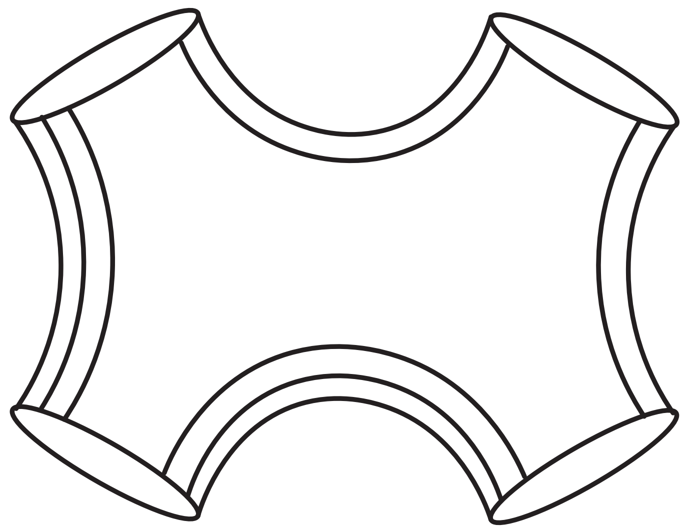

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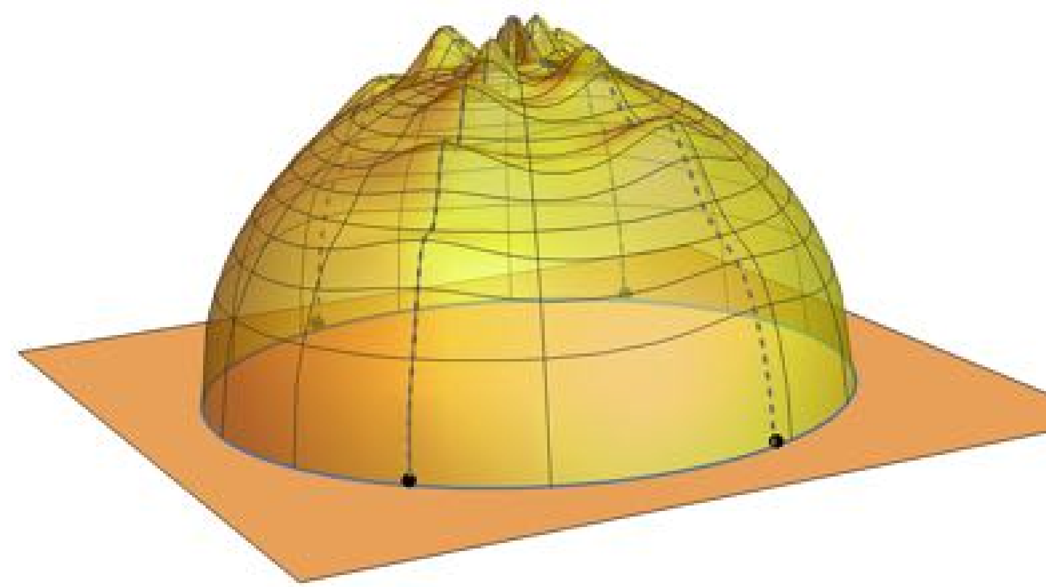
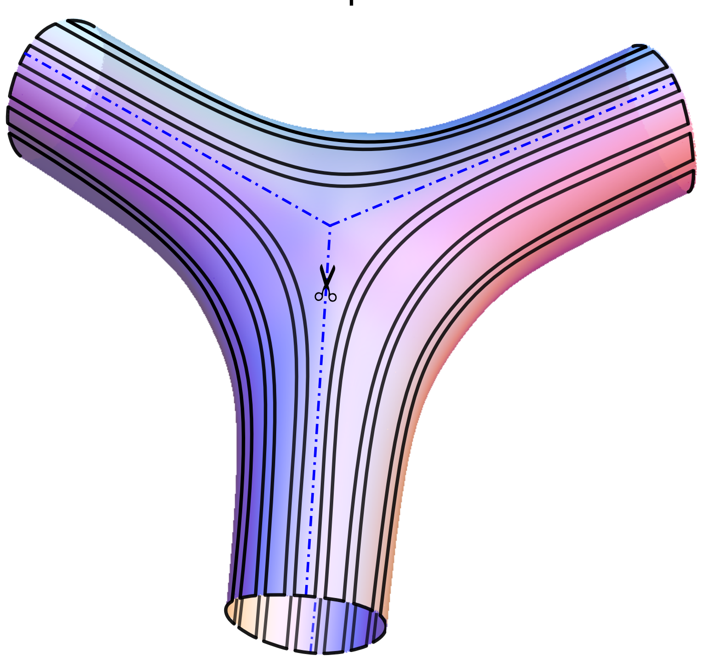


Pair of pants



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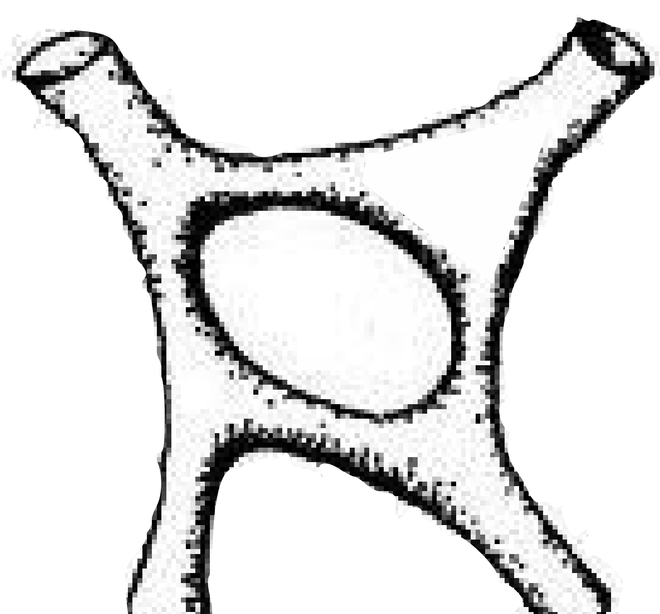

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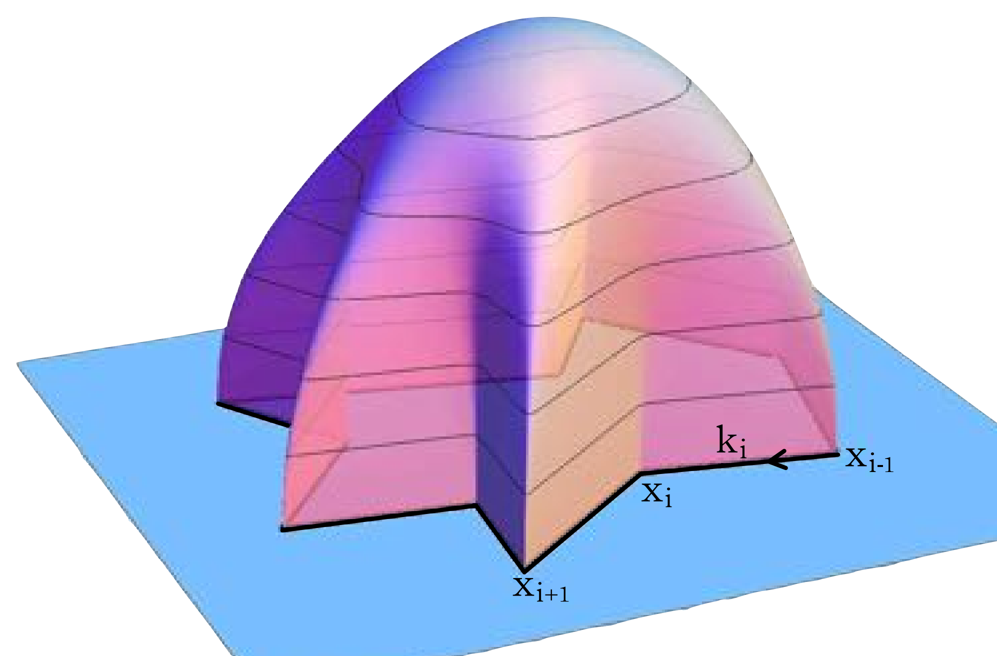
[Alday, Maldacena 2007,…]



... with hindsight, the spectrum was in The Book.

(The Bethe ansatz story; the more sophisticated quantum spectral curve story is definitely new)

The rest is less obvious as it involves dealing with Integrable theories on spaces of various topologies.

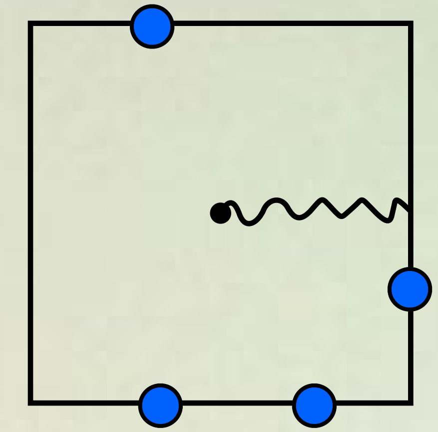
Same wonderful 't Hooft world-sheet fabric tailored into different topologies

As such, we should be able to tame any physical observable with a good large N limit - as well as any 1/N correction to it.

What can we do?

- Local operators are *not* the most natural thing in a string theory. After all, in quantum gravity (2d word-sheet gravity in this case) we have no local observables. We have S-matrices. They were key in the spectrum solution.

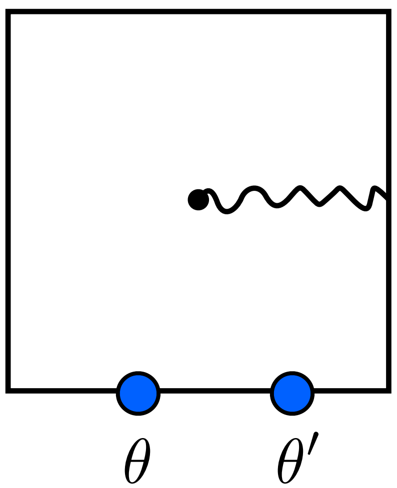
[*"Form factors of branch-point twist fields in quantum integrable models and entanglement entropy"*, Cardy, Castro-Alvaredo, Doyon, 2007]



Creative Patchwork

Spoiler:
Pentagons control scattering amplitudes and Wilson loops.
Hexagons govern correlation functions.

Free Relativistic Massive Boson



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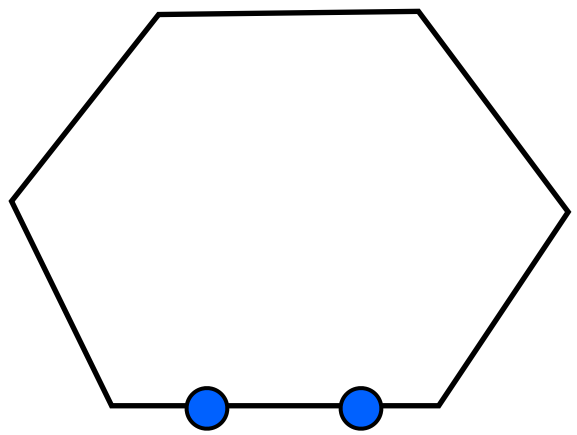
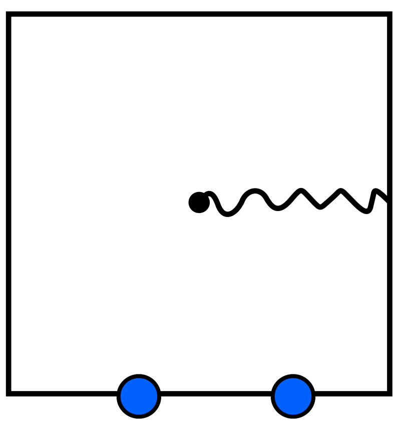


$$= f_n(\theta, \theta') \propto \frac{1}{\sinh \frac{i\pi + \theta - \theta'}{2n} \sinh \frac{i\pi - \theta + \theta'}{2n}}$$

Free Relativistic Massive Boson



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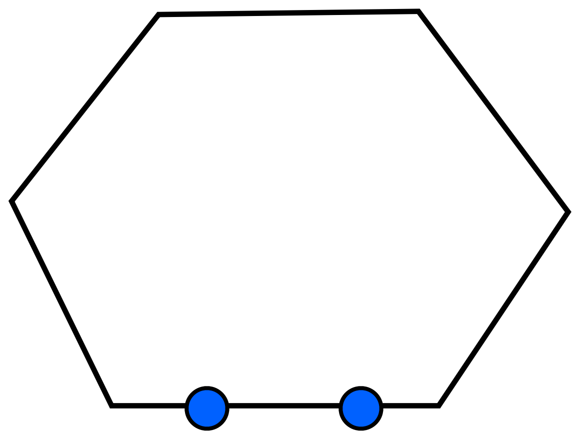

$$= f_n(\theta, \theta') \propto \frac{1}{\sinh \frac{i\pi+\theta-\theta'}{2n} \sinh \frac{i\pi-\theta+\theta'}{2n}}$$

$$\frac{f_n(\theta, \theta')}{f_n(\theta', \theta)} = 1\,, \qquad f_n(\theta + i\pi, \theta') \sim \frac{1}{\theta' - \theta} \times 1\,, \qquad f_n(\theta, \theta' - i\pi) \sim \frac{1}{\theta' - \theta} \times 1 \qquad f_n(\theta + in\frac{\pi}{2}, \theta') = f_n(\theta', \theta)$$

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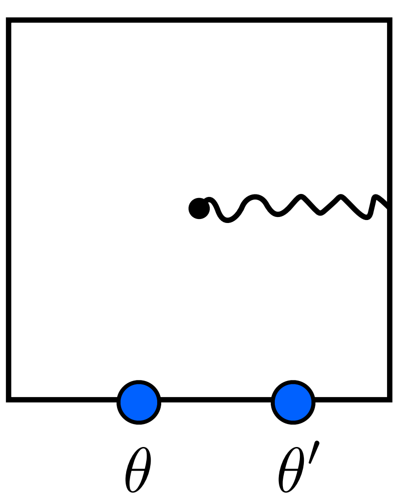

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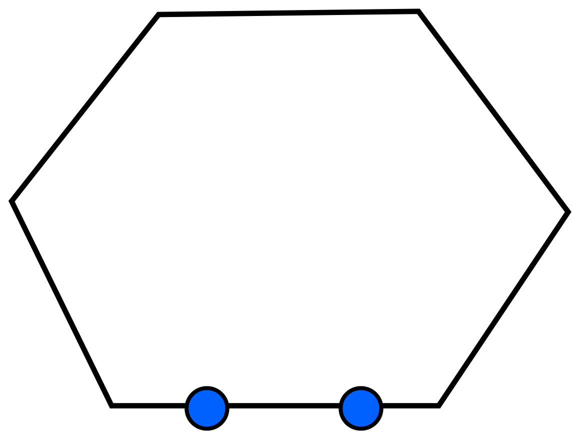


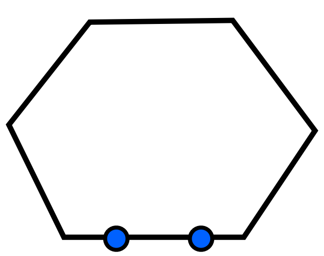
$$= f_n(\theta,\theta') \propto \frac{1}{\sinh\frac{i\pi+\theta-\theta'}{2n}\,\sinh\frac{i\pi-\theta+\theta'}{2n}}$$

S-matrix

Form factor without these two particles
(which leaves zero particles in this 2pt case)

for local operators n=4

$$\frac{f_n(\theta,\theta')}{f_n(\theta',\theta)} = 1\,, \qquad f_n(\theta+i\pi,\theta') \sim \frac{1}{\theta'-\theta}\times 1\,, \qquad f_n(\theta,\theta'-i\pi) \sim \frac{1}{\theta'-\theta}\times 1 \qquad f_n(\theta+in\tfrac{\pi}{2},\theta') = f_n(\theta',\theta)$$



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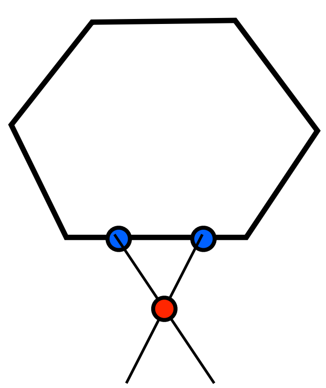

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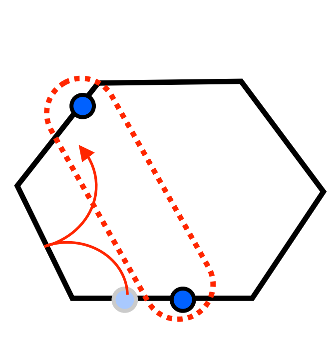

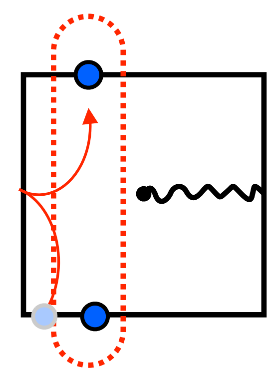

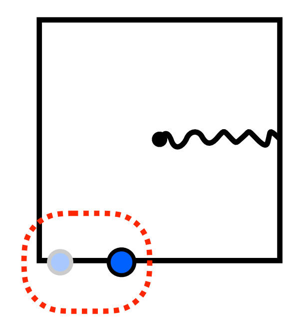

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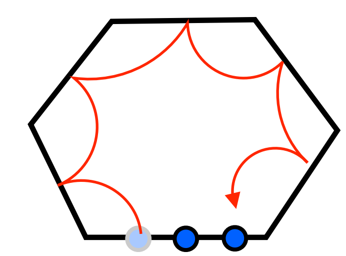

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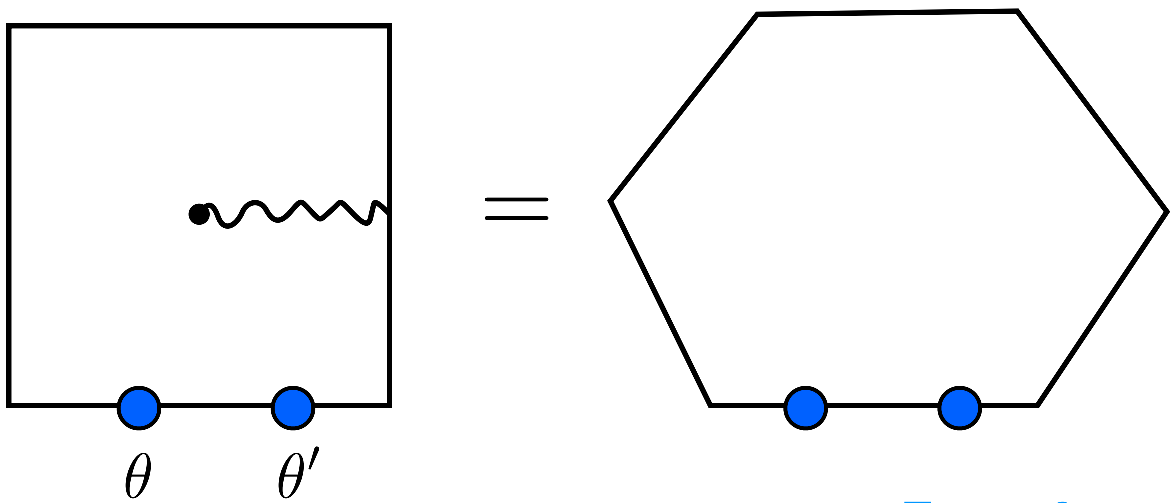

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Form factor without these two particles
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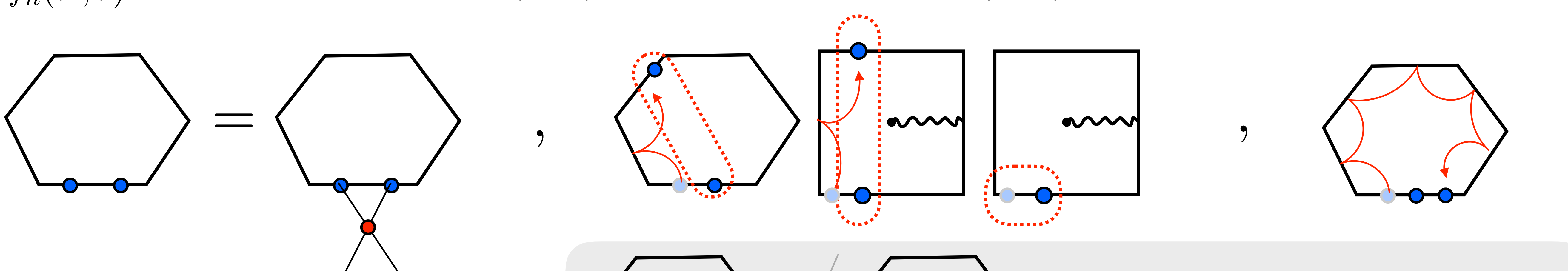
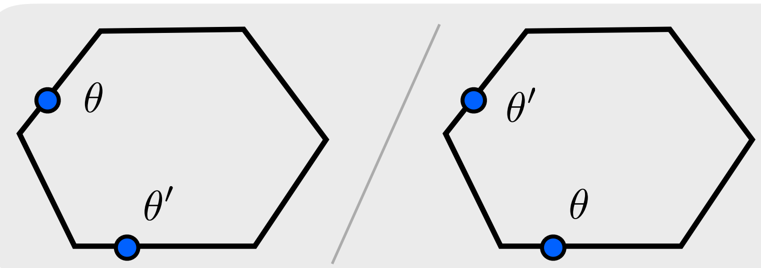
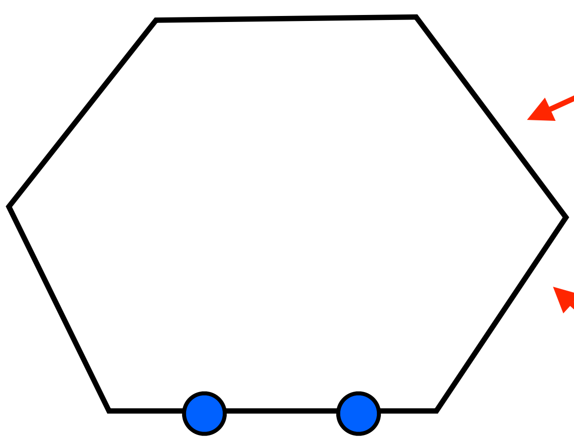




$$= \frac{f_n(\theta+i\pi,\theta')}{f_n(\theta'+i\pi,\theta)} = \text{something ugly}$$

N=4 SYM one can bootstrap **two** cases:

[Basso, Komatsu, PV]



[Bereinstein, Maldacena, Nastase]

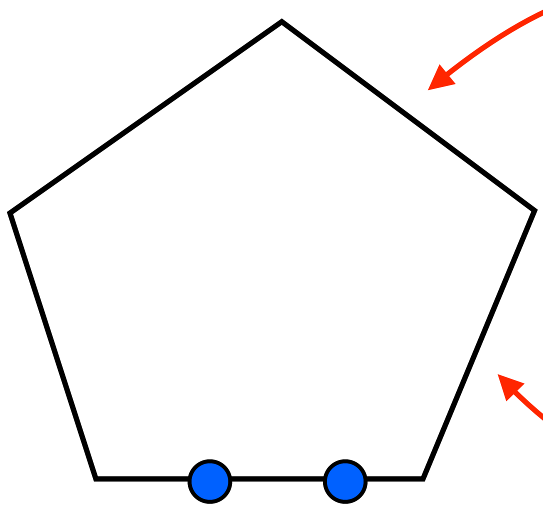
BMN Vacuum

$\mathrm{Tr}(Z^J)$

✓

Mirror BMN Vacuum

[Basso, Sever, PV]



✓

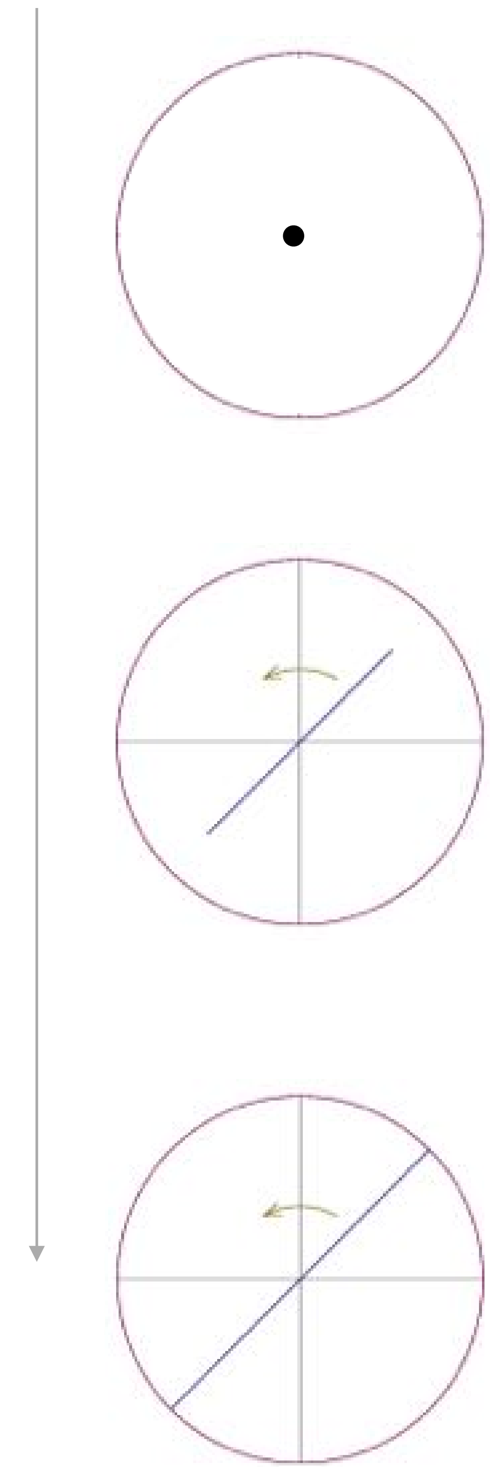
[Gubser, Klebanov, Polyakov]

GKP Vacuum

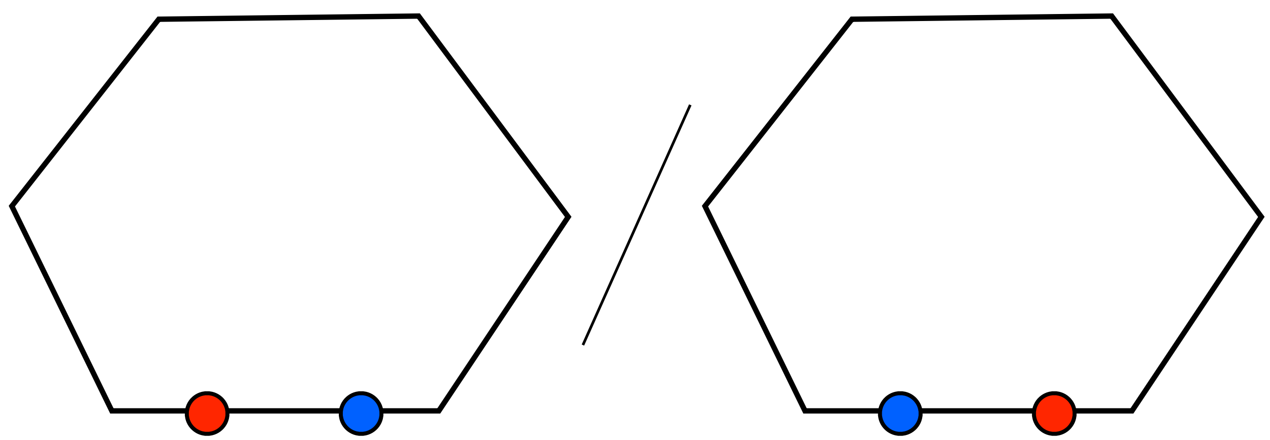
$\mathrm{Tr}(ZD^S Z)$

Global AdS:

Increase spin, decrease R-charge

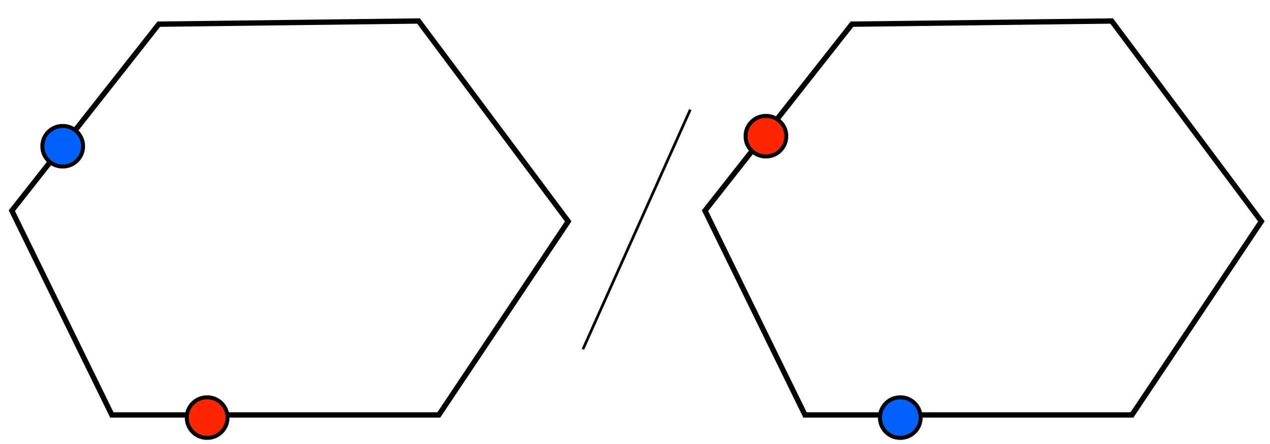


Fundamental relation



$$= S({\color{red}u}, {\color{blue}v})$$

Usual Watson equation



$$= S({\color{red}u}, {\color{blue}v})$$

Mysterious Fundamental Relation

(Holds both for the GKP pentagons and for the BMN hexagons)

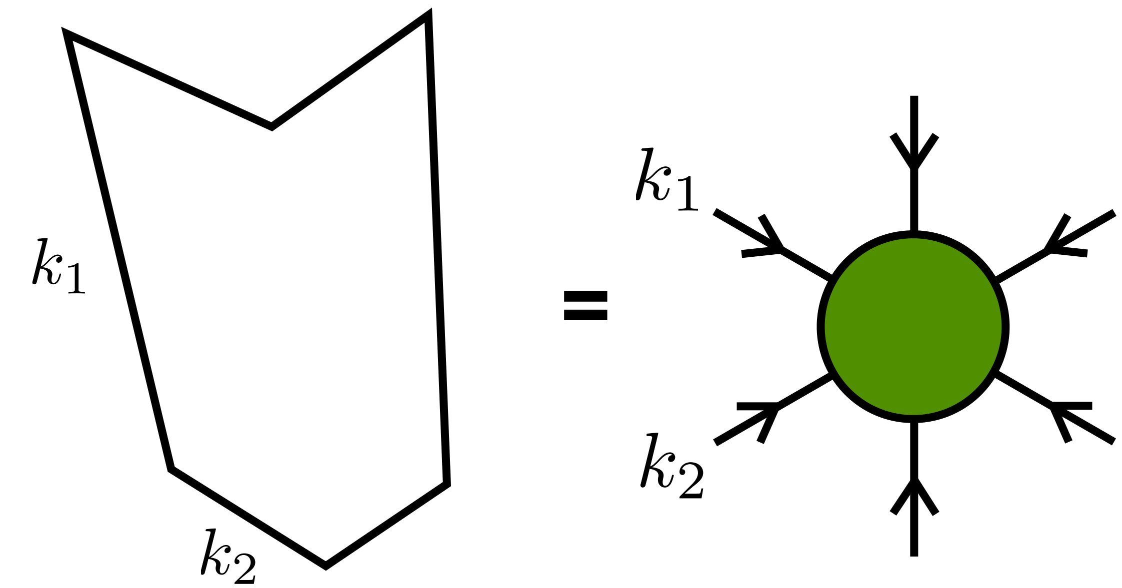
Why?

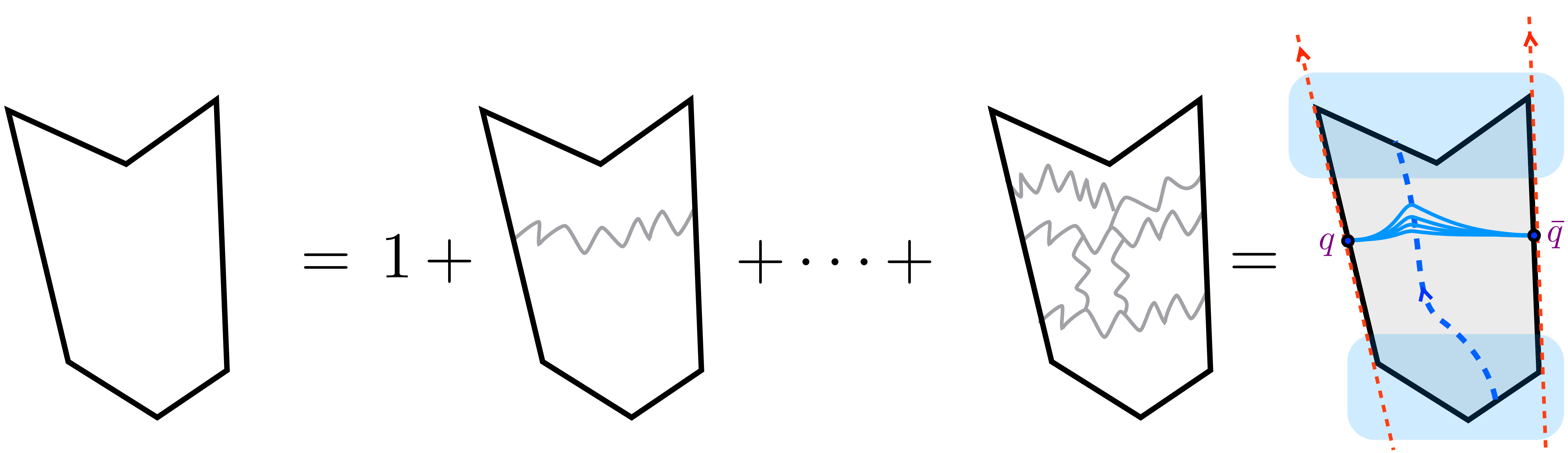
No idea

Null Wilson Loops and Scattering Amplitudes

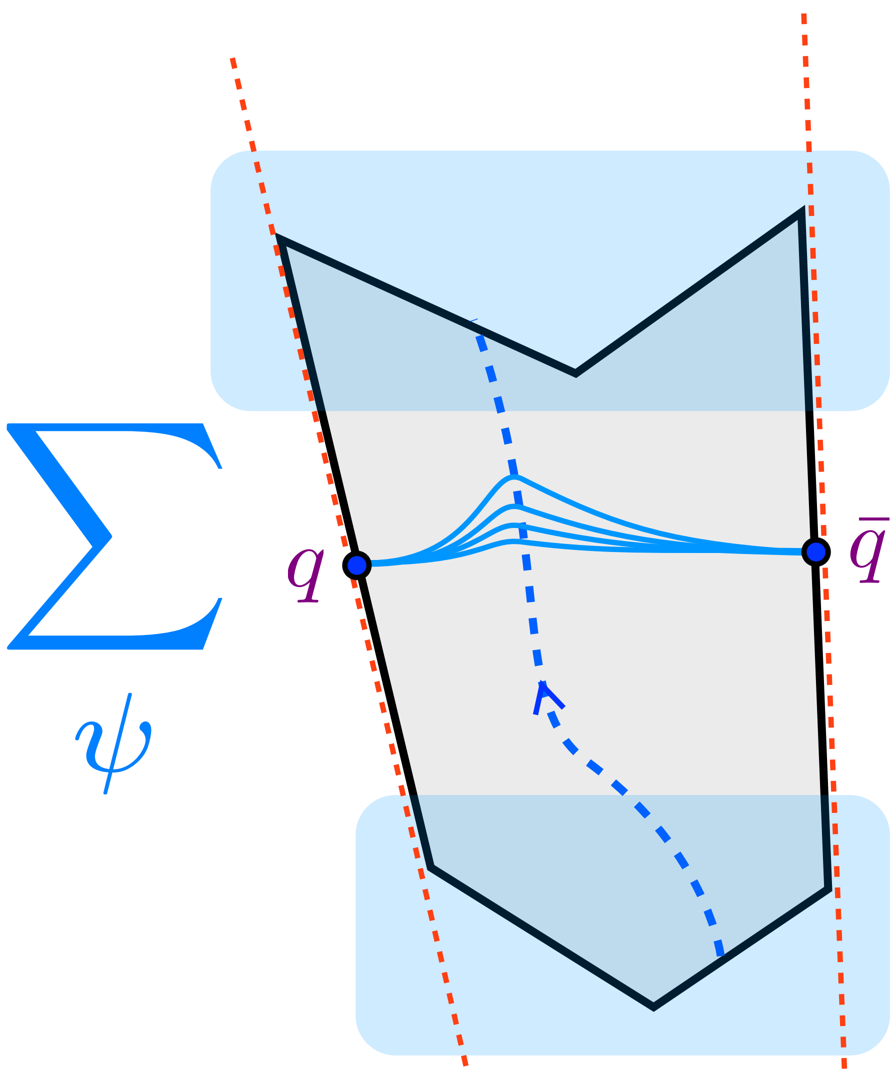
- In planar N=4 SYM
  WL = Scattering Amplitudes

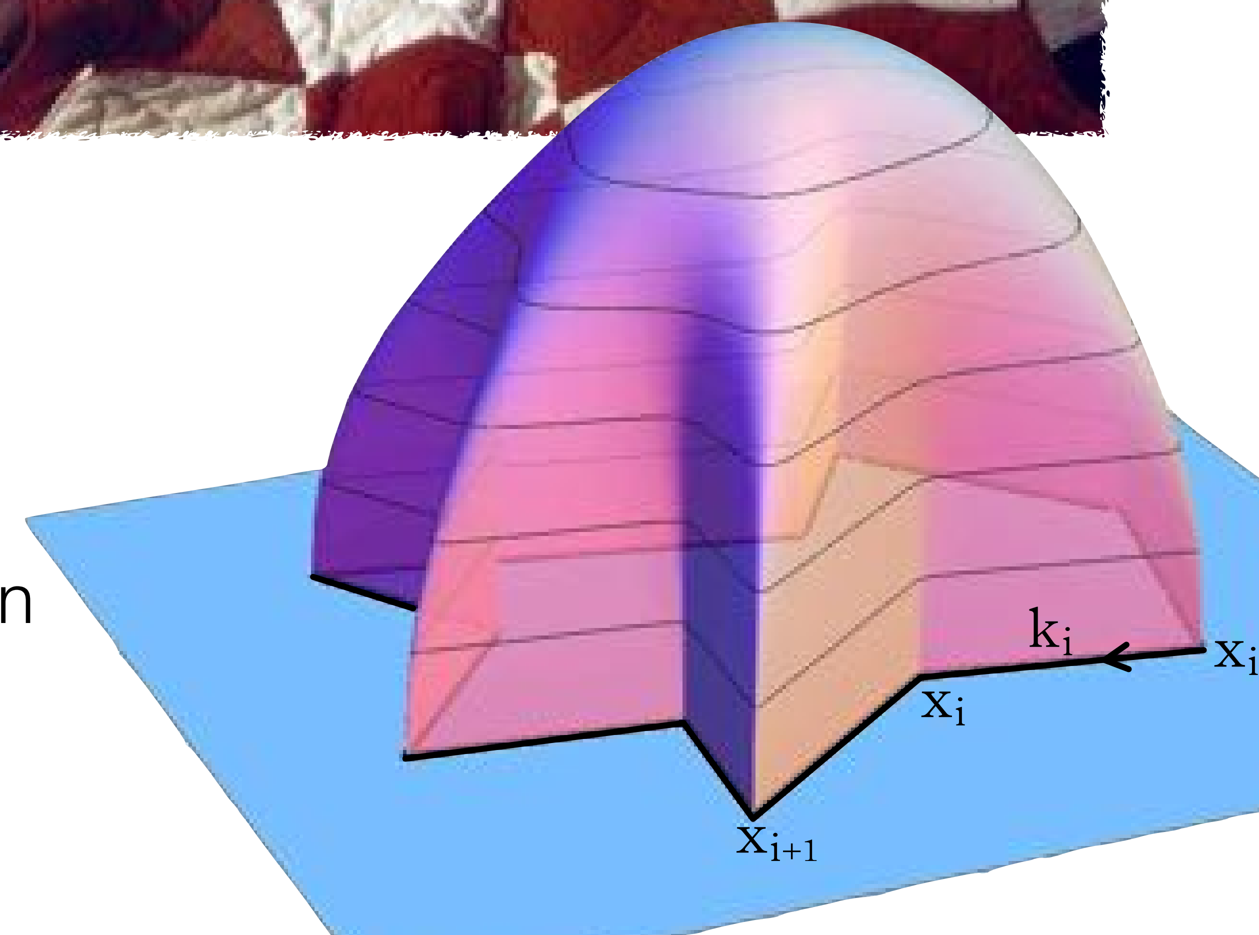
  [Alday, Maldacena; Drummond, Korchemsky, Sokatchev; Brandhuber, Heslop, Travaglini; Drummond, Henn, Korchemsky, Sokatchev; Berkovits, Maldacena]





Amplitudes = Sum over Flux Tube states
= Open String Partition Function

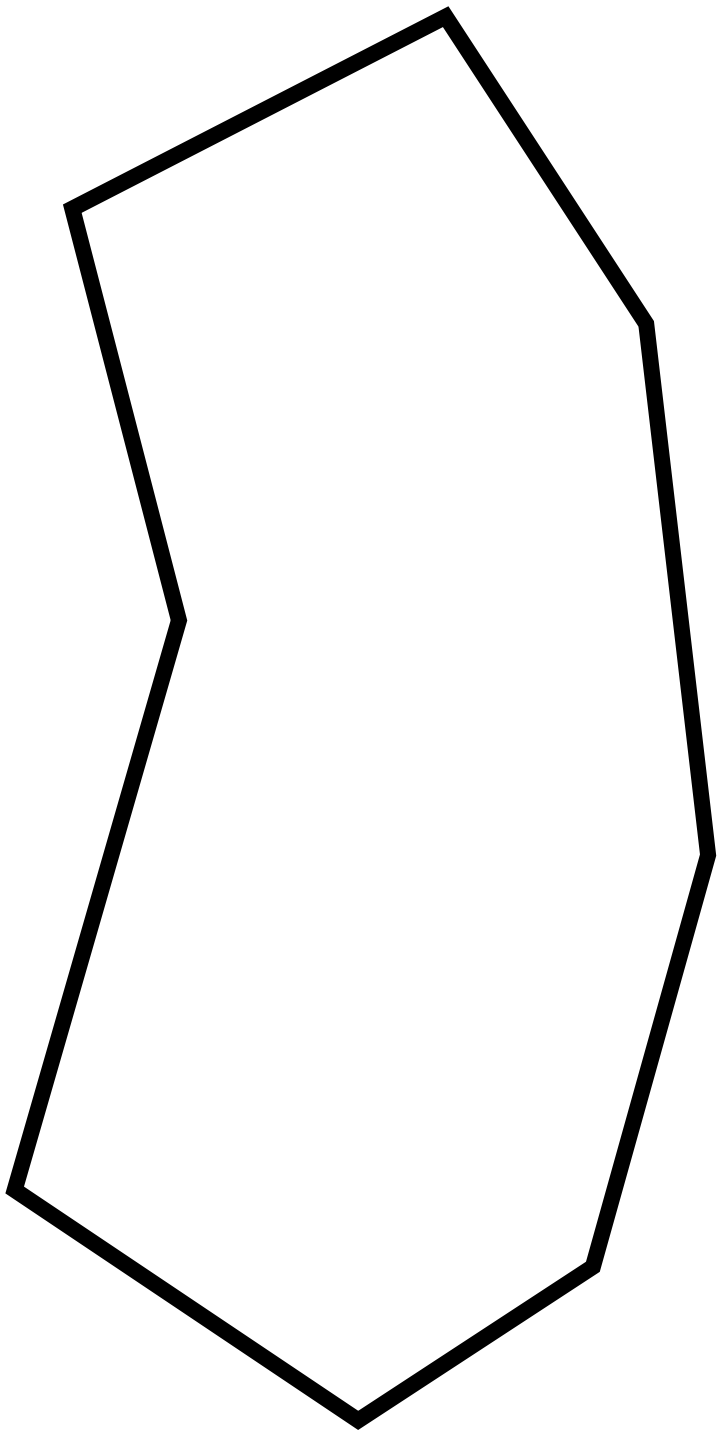




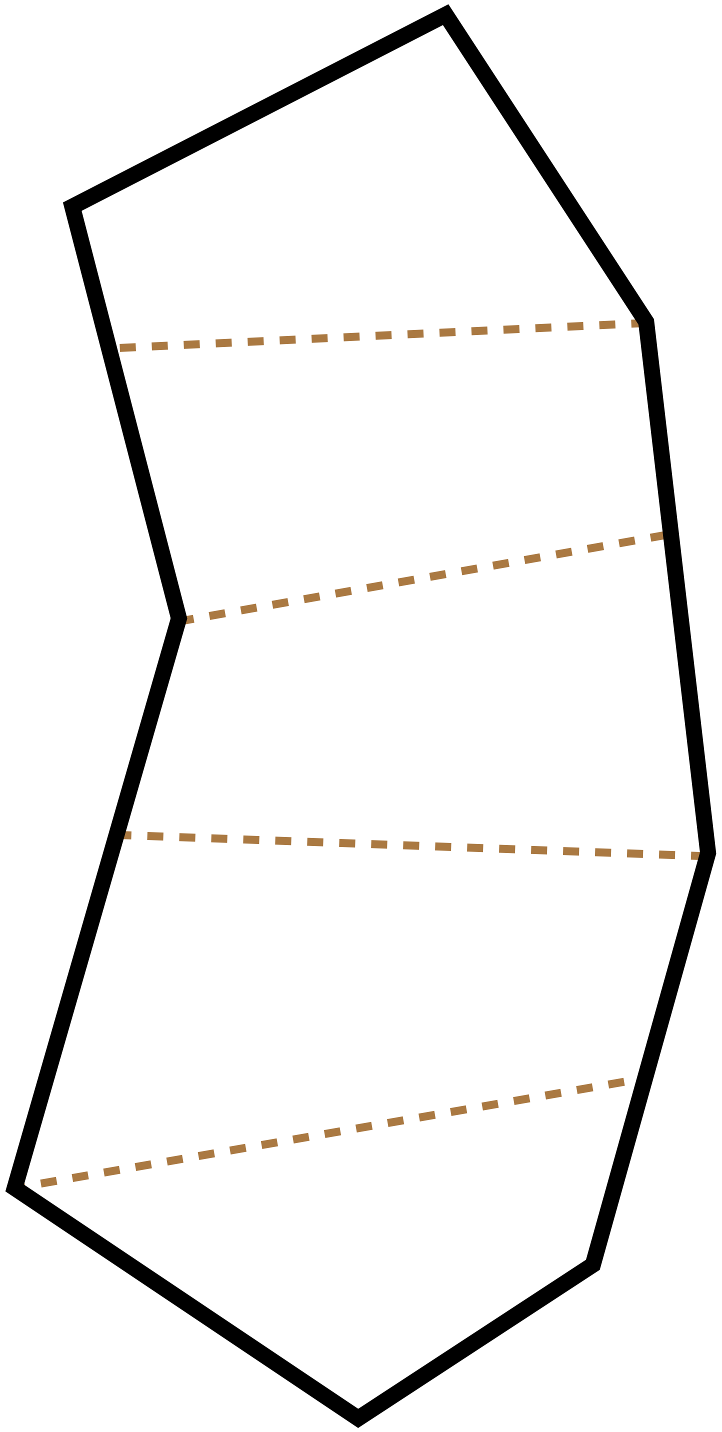
Basic idea

1. Use the spectrum to describe the propagation
2. Tesselate the flux tube world-sheet as quilt to tame the null polygonal geometry

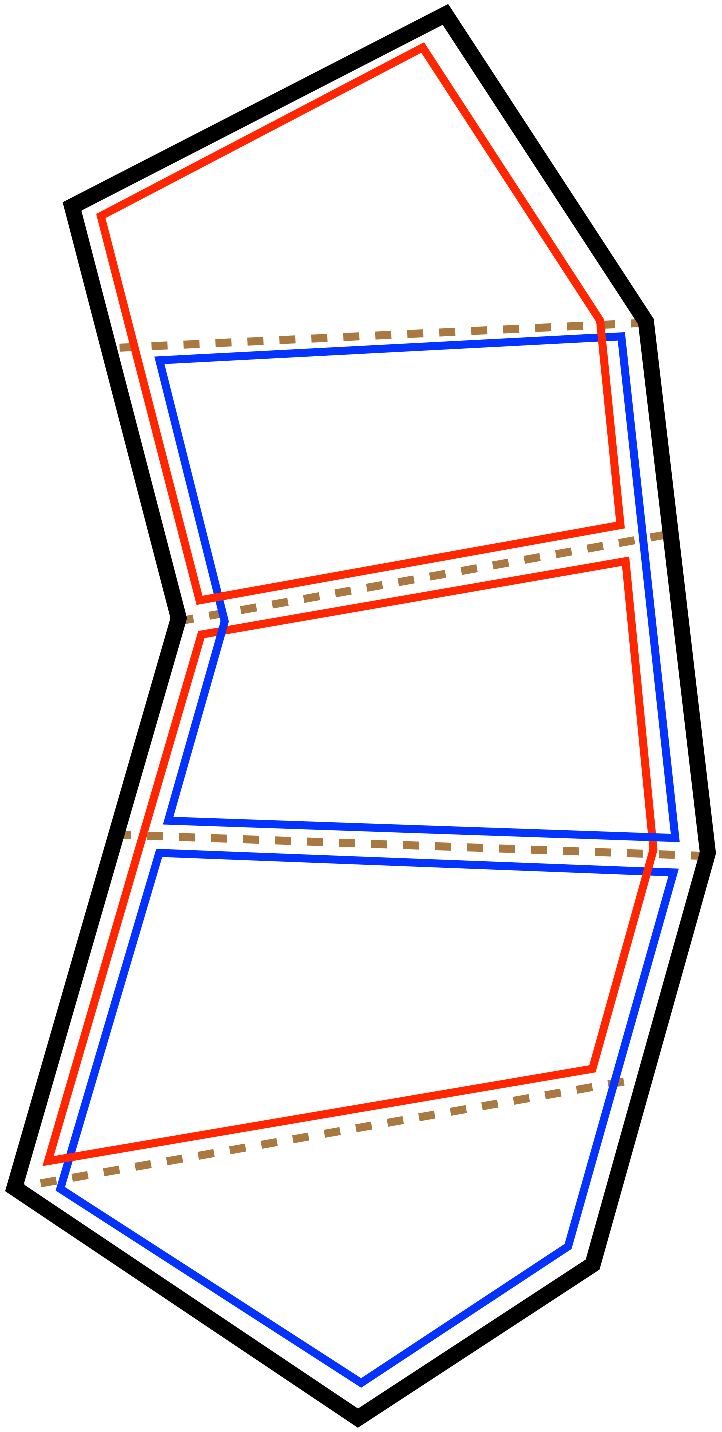
4D Amplitudes as 2D Flux Tube Gas



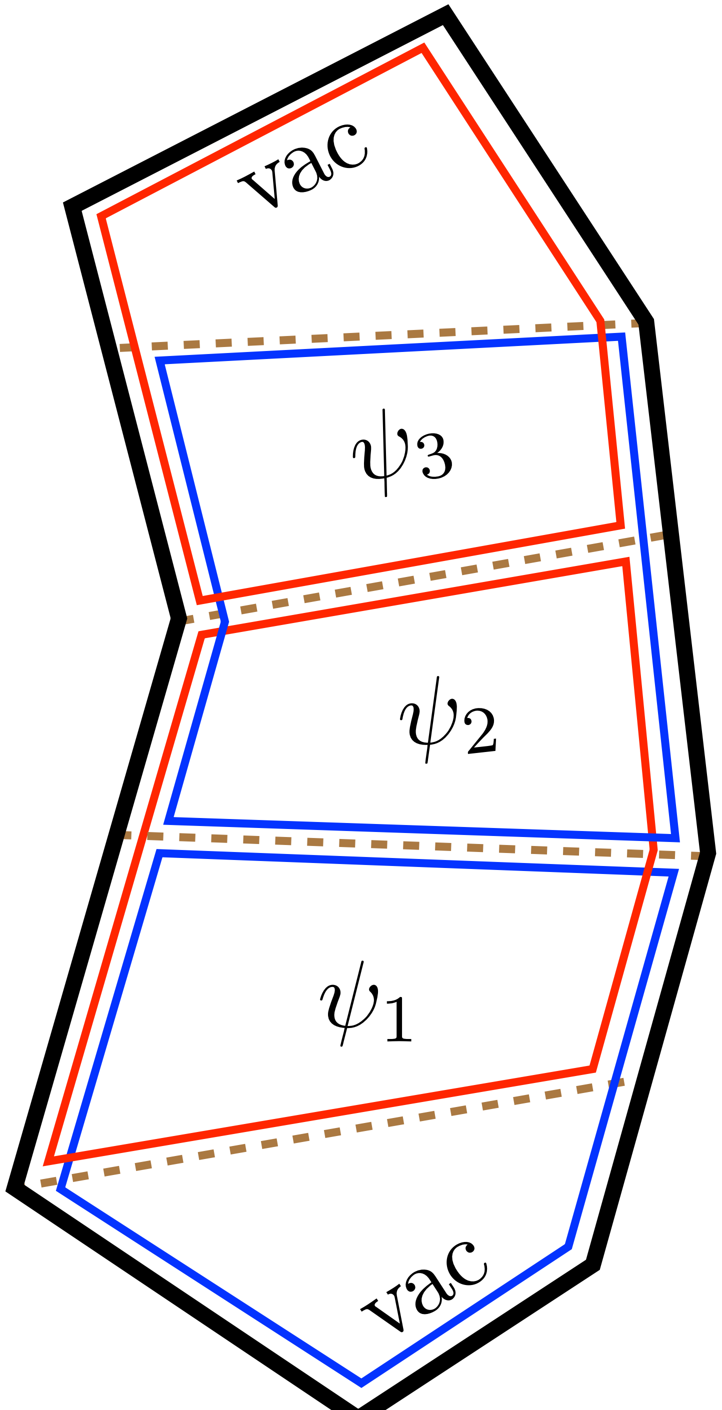
4D Amplitudes as 2D Flux Tube Gas



4D Amplitudes as 2D Flux Tube Gas



4D Amplitudes as 2D Flux Tube Gas



energy

momentum

angular momentum

pentagon transition

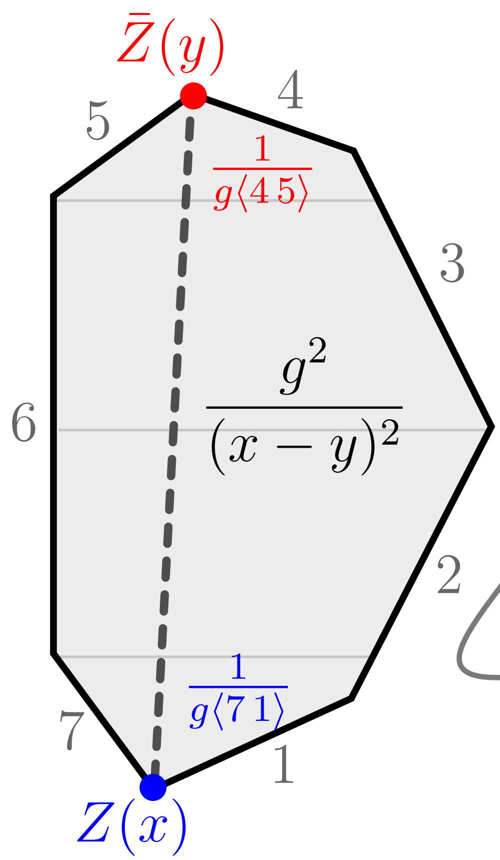
$$= \sum_{\psi_i} \left[\prod_i e^{-E_i\tau_i + ip_i\sigma_i + im_i\phi_i} \right] P(0|\psi_1)P(\psi_1|\psi_2)P(\psi_2|\psi_3)P(\psi_3|0)$$

geometry

Tree Level Example

[Caron-Huot;Mason,Skinner]

$$\mathcal{R}^{(7145)}_{\text{tree}} =$$



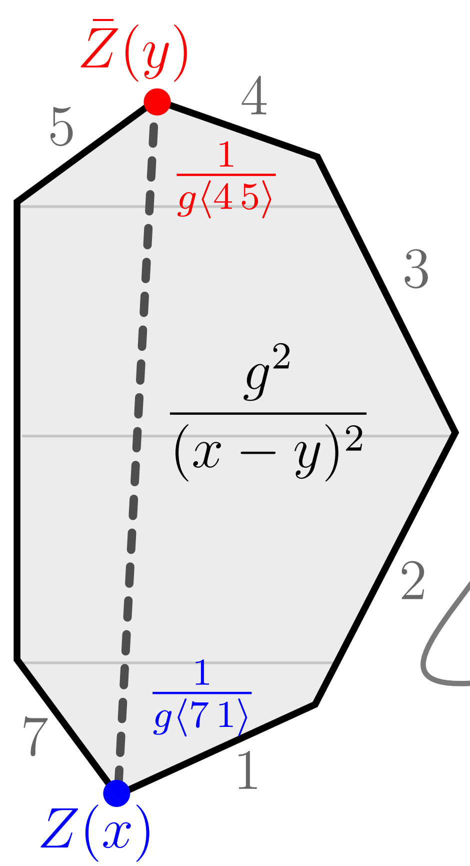
$$= \frac{1}{\langle 7\,1\rangle (x-y)^2 \langle 4\,5\rangle}$$

From your favorite tree-level generator: BCFW, Nima's form whatever…

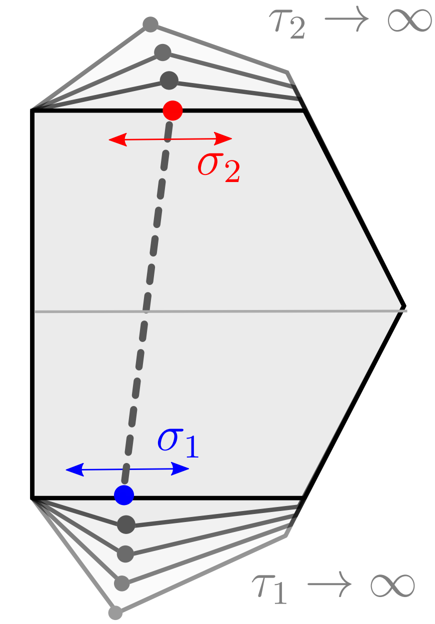
Tree Level Example

[Caron-Huot;Mason,Skinner]

$\mathcal{R}^{(7145)}_{\text{tree}} =$



$$= \frac{1}{\langle 7\,1\rangle (x-y)^2 \langle 4\,5\rangle} = e^{-\tau_1-\tau_2} \times \frac{1}{e^{\sigma_1-\sigma_2}+e^{\sigma_2-\sigma_1}+e^{\sigma_1+\sigma_2}} + \cdots =$$



Express it using the OPE coordinates

From your favorite tree-level generator: BCFW, Nima's form whatever…

Tree Level Example

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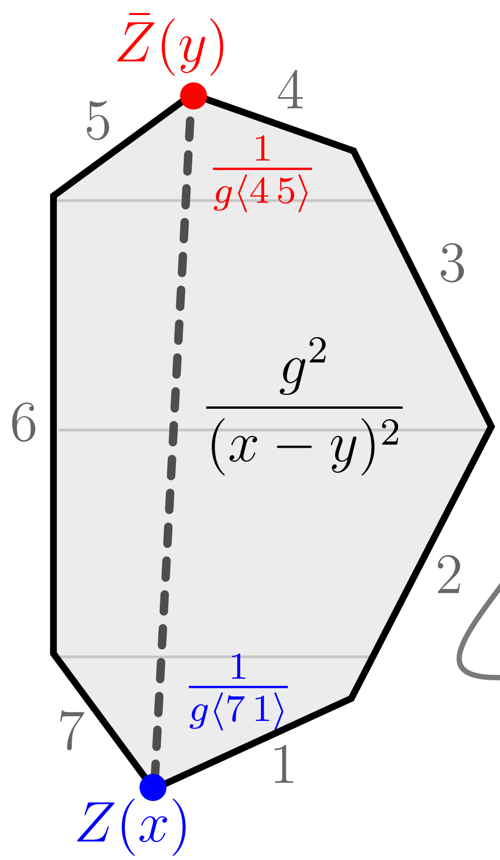
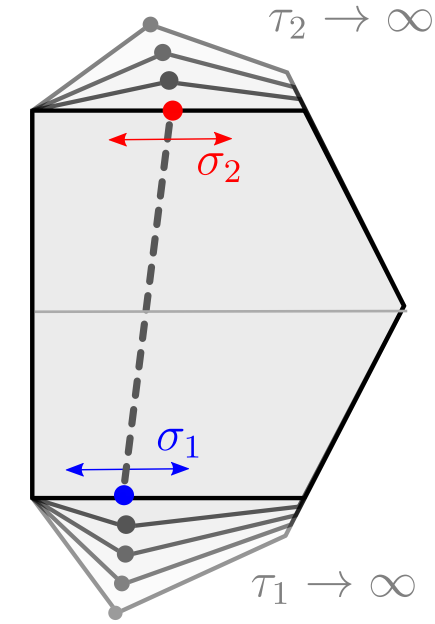


$$= \frac{1}{\langle 7\,1\rangle (x-y)^2 \langle 4\,5\rangle} = e^{-\tau_1-\tau_2} \times \frac{1}{e^{\sigma_1-\sigma_2}+e^{\sigma_2-\sigma_1}+e^{\sigma_1+\sigma_2}} + \cdots =$$



Express it using the OPE coordinates

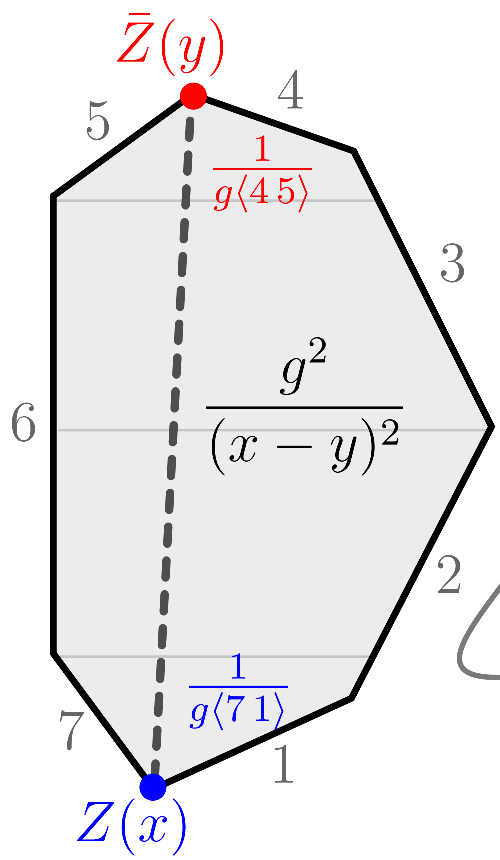
From your favorite tree-level generator: BCFW, Nima's form whatever…

$$\int \frac{dp_1 dp_2}{16\pi^2} e^{-ip_1\sigma_1 + ip_2\sigma_2} \Gamma\left(\frac{1}{2} - \frac{ip_1}{2}\right) \Gamma\left(\frac{1}{2} + \frac{ip_2}{2}\right) \Gamma\left(\frac{ip_1}{2} - \frac{ip_2}{2}\right)$$

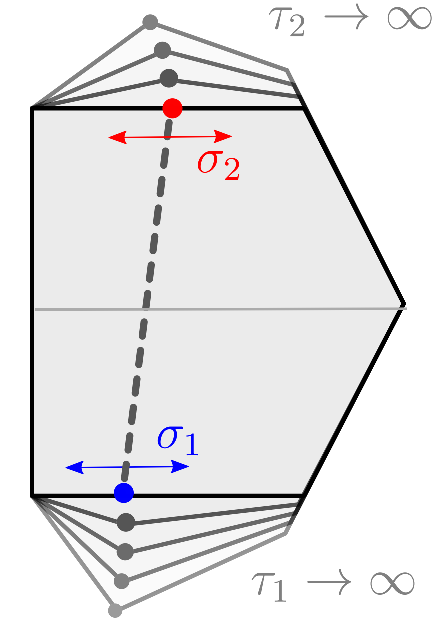
Tree Level Example

[Caron-Huot;Mason,Skinner]

$\mathcal{R}^{(7145)}_{\text{tree}} = $



$$= \frac{1}{\langle 7\,1\rangle (x-y)^2 \langle 4\,5\rangle} = e^{-\tau_1-\tau_2} \times \frac{1}{e^{\sigma_1-\sigma_2}+e^{\sigma_2-\sigma_1}+e^{\sigma_1+\sigma_2}} + \cdots =$$



Express it using the OPE coordinates

From your favorite tree-level generator: BCFW, Nima's form whatever…

$$\int \frac{dp_1 dp_2}{16\pi^2} e^{-ip_1\sigma_1+ip_2\sigma_2} \Gamma\left(\frac{1}{2}-\frac{ip_1}{2}\right)\Gamma\left(\frac{1}{2}+\frac{ip_2}{2}\right)\Gamma\left(\frac{ip_1}{2}-\frac{ip_2}{2}\right)$$

From Integrability, a *totally* different computation yields

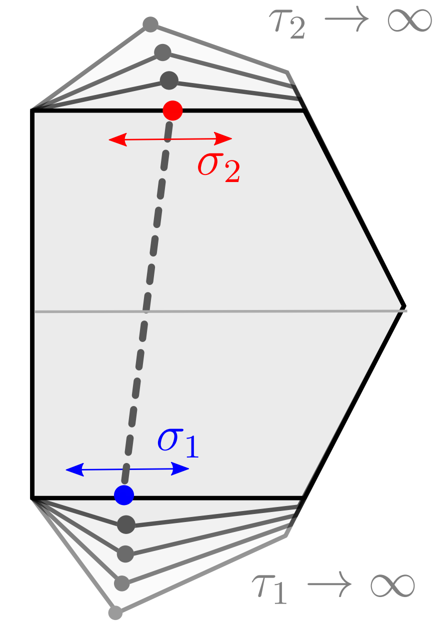
$$S(p_1,p_2) = \frac{\Gamma\left(\frac{1}{2}-\frac{ip_1}{2}\right)\Gamma\left(\frac{1}{2}+\frac{ip_2}{2}\right)\Gamma\left(\frac{ip_1}{2}-\frac{ip_2}{2}\right)}{\Gamma\left(\frac{1}{2}-\frac{ip_2}{2}\right)\Gamma\left(\frac{1}{2}+\frac{ip_1}{2}\right)\Gamma\left(\frac{ip_2}{2}-\frac{ip_1}{2}\right)}$$

Tree Level Example

$\mathcal{R}^{(7145)}_{\text{tree}} =$

Extended test to many loops…

$5\rangle$ $= e^{-\tau_1-\tau_2} \times \frac{1}{e^{\sigma_1-\sigma_2}+e^{\sigma_2-\sigma_1}+e^{\sigma_1+\sigma_2}} + \cdots =$



Express it using the OPE coordinates

e tree-level
, Nima's

$ip_2\sigma_2\,\Gamma\left(\frac{1}{2}-\frac{ip_1}{2}\right)\Gamma\left(\frac{1}{2}+\frac{ip_2}{2}\right)\Gamma\left(\frac{ip_1}{2}-\frac{ip_2}{2}\right)$

From Integrability, a *totally* different computation yields

$$S(p_1,p_2) = \frac{\Gamma\left(\frac{1}{2}-\frac{ip_1}{2}\right)\Gamma\left(\frac{1}{2}+\frac{ip_2}{2}\right)\Gamma\left(\frac{ip_1}{2}-\frac{ip_2}{2}\right)}{\Gamma\left(\frac{1}{2}-\frac{ip_2}{2}\right)\Gamma\left(\frac{1}{2}+\frac{ip_1}{2}\right)\Gamma\left(\frac{ip_2}{2}-\frac{ip_1}{2}\right)}$$

Weak Coupling @ many loops

$$\mathcal{W}_{\text{hex}} = 1 + e^{-\tau}\left(e^{i\phi} + e^{-i\phi}\right)\mathcal{A} + e^{-2\tau}\left(e^{2i\phi} + e^{-2i\phi}\right)\mathcal{B} + e^{-2\tau}\mathcal{C} + \mathcal{O}(e^{-3\tau})$$

$$\begin{aligned}
\mathcal{A} \;=\;& g^2\Big[e^\sigma(2\sigma-1)+\dots\Big] + g^4\Big[e^\sigma(4-4\sigma)\tau + e^\sigma\left(-\frac{2\pi^2\sigma}{3} - 4\sigma + 6\right)+\dots\Big] + g^6\Big[e^\sigma(4\sigma-6)\tau^2 + e^\sigma\left(-4\sigma^2 + \frac{8\pi^2\sigma}{3} + 24\sigma - \frac{5\pi^2}{3} - 36\right)\tau + e^\sigma\Big(-6\sigma^2 + \frac{22\pi^4\sigma}{45} + \frac{5\pi^2\sigma}{3} + 36\sigma \\
&+ 4\zeta(3) - \pi^2 - 60\Big)+\dots\Big] + g^8\Big[e^\sigma\left(\frac{16}{3} - \frac{8\sigma}{3}\right)\tau^3 + e^\sigma\left(8\sigma^2 - 4\pi^2\sigma - 48\sigma - 8\zeta(3) + \frac{14\pi^2}{3} + 80\right)\tau^2 + e^\sigma\Big(-\frac{8\sigma^3}{3} + 4\pi^2\sigma^2 + 48\sigma^2 - \frac{12\pi^4\sigma}{5} - \frac{52\pi^2\sigma}{3} - 240\sigma - 24\zeta(3) + \frac{4\pi^4}{3} \\
&+ \frac{52\pi^2}{3} + 400\Big)\tau + e^\sigma\Big(-\frac{16\sigma^3}{3} + \frac{14\pi^2\sigma^2}{3} + 80\sigma^2 - \frac{146\pi^6\sigma}{315} - \frac{4\pi^4\sigma}{3} - \frac{52\pi^2\sigma}{3} - 400\sigma - 8\sigma^2\zeta(3) - 16\sigma\zeta(3)^2 + 24\sigma\zeta(3) - 40\zeta(5) - \frac{4\pi^2\zeta(3)}{3} - 48\zeta(3) + \frac{71\pi^4}{90} + \frac{40\pi^2}{3} + 700\Big)+\dots\Big] + \mathcal{O}(g^{10})
\end{aligned}$$

$$\begin{aligned}
\mathcal{B} \;=\;& g^2\Big[e^{2\sigma}\left(-\sigma - \frac{1}{4}\right)+\dots\Big] + g^4\Big[e^{2\sigma}\left(3\sigma - \frac{1}{2}\right)\tau + e^{2\sigma}\left(2\sigma^2 + \frac{\pi^2\sigma}{3} + \frac{\sigma}{2} + \frac{\pi^2}{6} - \frac{3}{8}\right)+\dots\Big] + g^6\Big[e^{2\sigma}\left(-\frac{9\sigma}{2} + \frac{21}{8}\right)\tau^2 + e^{2\sigma}\left(-\frac{7\sigma^2}{2} - 2\pi^2\sigma - \frac{9\sigma}{2} - \frac{\pi^2}{8} + \frac{27}{4}\right)\tau + e^{2\sigma}\Big(-\frac{4}{3}\pi^2\sigma^2 \\
&- \frac{27\sigma^2}{8} - \frac{11\pi^4\sigma}{45} - \frac{13\pi^2\sigma}{24} - \frac{3\sigma}{4} - \frac{5\zeta(3)}{2} - \frac{11\pi^4}{90} - \frac{\pi^2}{16} + \frac{105}{16}\Big)+\dots\Big] + g^8\Big[e^{2\sigma}\left(\frac{9\sigma}{2} - \frac{9}{2}\right)\tau^3 + e^{2\sigma}\tau^2\Big(\frac{3\sigma^2}{2} + \frac{9\pi^2\sigma}{2} + \frac{35\sigma}{2} + 5\zeta(3) - \frac{13\pi^2}{8} - \frac{113}{4}\Big) + e^{2\sigma}\tau\Big(-\frac{7\sigma^3}{2} + \frac{7\pi^2\sigma^2}{2} + \frac{29\sigma^2}{2} \\
&+ \frac{9\pi^4\sigma}{5} + \frac{13\pi^2\sigma}{4} + \frac{91\sigma}{4} + \frac{37\zeta(3)}{2} + \frac{17\pi^4}{90} - \frac{13\pi^2}{6} - \frac{629}{8}\Big) + e^{2\sigma}\Big(-\frac{4\sigma^4}{3} - \frac{5\sigma^3}{6} + \frac{6\pi^4\sigma^2}{5} + \frac{41\pi^2\sigma^2}{24} + \frac{79\sigma^2}{4} + \frac{73\pi^6\sigma}{315} + \frac{47\pi^4\sigma}{90} + \frac{3\pi^2\sigma}{2} + \frac{21\sigma}{8} + 5\sigma^2\zeta(3) + 8\sigma\zeta(3)^2 \\
&- \frac{5\sigma\zeta(3)}{2} + 25\zeta(5) + 4\zeta(3)^2 + \frac{5\pi^2\zeta(3)}{6} + \frac{39\zeta(3)}{2} + \frac{73\pi^6}{630} + \frac{121\pi^4}{1440} - \frac{17\pi^2}{24} - \frac{5815}{64}\Big)+\dots\Big] + \mathcal{O}(g^{10})
\end{aligned}$$

$$\begin{aligned}
\mathcal{C} =\;& g^2\Big[4\sigma - 2e^{2\sigma}+\dots\Big] + g^4\Big[8e^{2\sigma}\sigma\tau + e^{2\sigma}\left(4\sigma^2 + \frac{\pi^2}{3} + \frac{7}{2}\right) - \frac{4\pi^2\sigma}{3}+\dots\Big] + g^6\Big[e^{2\sigma}\left(-8\sigma^2 - 8\sigma - \frac{2\pi^2}{3} + 8\right)\tau^2 + e^{2\sigma}\tau\left(-8\sigma^2 - \frac{16\pi^2\sigma}{3} - 6\sigma + 8\zeta(3) - \frac{2\pi^2}{3}\right) + \frac{44\pi^4\sigma}{45} \\
&+ e^{2\sigma}\Big(-\frac{10}{3}\pi^2\sigma^2 - 4\sigma^2 - \frac{2\pi^2\sigma}{3} + 12\sigma - 8\sigma\zeta(3) + 8\zeta(3) - \frac{3\pi^4}{10} + \frac{\pi^2}{3} - \frac{141}{4}\Big)+\dots\Big] + g^8\Big[e^{2\sigma}\tau^3\left(\frac{32\sigma^3}{9} + \frac{32\sigma^2}{3} + \frac{8\pi^2\sigma}{9} - \frac{32\sigma}{3} - 16\zeta(3) + \frac{8\pi^2}{9}\right) + e^{2\sigma}\tau^2\Big(-\frac{32\sigma^3}{3} \\
&+ \frac{32\pi^2\sigma^2}{3} + 28\sigma^2 + \frac{16\pi^2\sigma}{3} + 16\sigma - 16\sigma\zeta(3) - 32\zeta(3) + \frac{14\pi^4}{15} - 3\pi^2 - \frac{87}{2}\Big) + e^{2\sigma}\tau\Big(-\frac{8}{9}\pi^2 - \frac{80\sigma^3}{3} + \frac{32\pi^2\sigma^2}{3} + 48\sigma^2 + \frac{226\pi^4\sigma}{45} - \frac{16\pi^2\sigma}{3} + 22\sigma + 16\sigma^2\zeta(3) \\
&- 64\zeta(5) - \frac{16\pi^2\zeta(3)}{3} - 24\zeta(3) + \frac{14\pi^4}{15} + 4\pi^2 + 8\Big) + e^{2\sigma}\Big(-\frac{8\pi^2\sigma^3}{9} - \frac{64\sigma^3}{3} + \frac{10\pi^4\sigma^2}{3} + 5\pi^2\sigma^2 + \frac{137\sigma^2}{2} + \frac{22\pi^4\sigma}{45} - 8\pi^2\sigma - 168\sigma + 16\sigma^3\zeta(3) - 32\sigma^2\zeta(3) \\
&+ 64\sigma\zeta(5) + \frac{16}{3}\pi^2\sigma\zeta(3) + 56\sigma\zeta(3) - 64\zeta(5) - \frac{16\pi^2\zeta(3)}{3} - 48\zeta(3) + \frac{296\pi^6}{945} - \frac{\pi^4}{20} + \frac{25\pi^2}{12} + \frac{3217}{8}\Big) - 32\sigma\zeta(3)^2 - \frac{292\pi^6\sigma}{315} + \dots\Big] + \mathcal{O}(g^{10})
\end{aligned}$$

This data was used intensively by Dixon et al in the so called Hexagon program

[Dixon,Drummond,Henn],[Dixon,Duhr,Pennington,Von Hippel],[Dixon, Drummond, Duhr,Pennington],[Dixon,Von Hippel],…

With some Steinmann technology, this is no longer needed (up to 5 loops)! Integrability derivation?

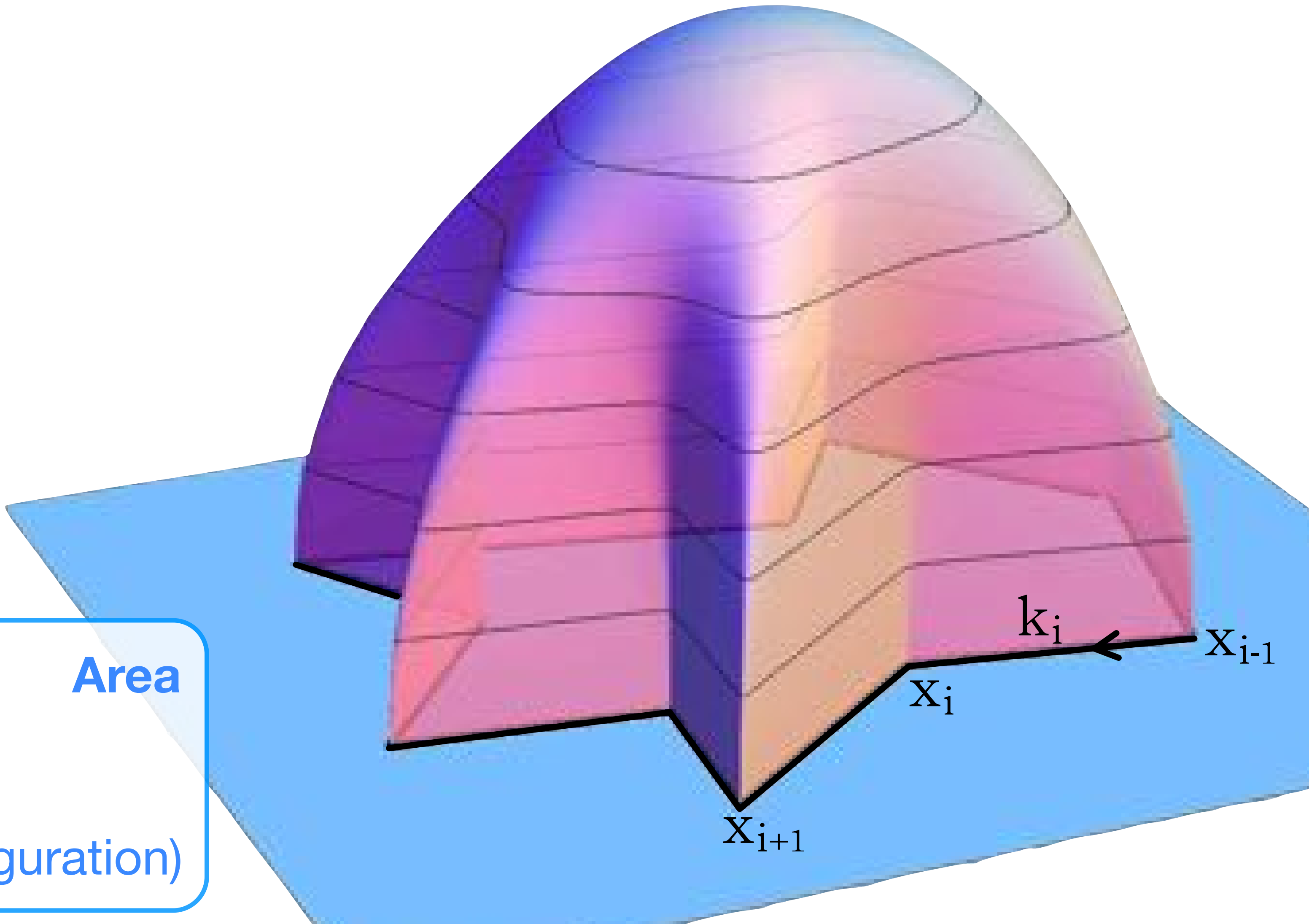
[Caron-Huot,Dixon,Von Hippel 2017]

Strong Coupling. The Emergence of Strings

$$\mathcal{W}^{\text{string}} \simeq \exp\left(-\frac{\sqrt{\lambda}}{2\pi} Y Y_c\right) = 1 - \frac{\sqrt{\lambda}}{2\pi}\left(e^{i\phi}+e^{-i\phi}\right)\int_{\mathbb{R}} \frac{d\theta}{\pi\cosh^2(2\theta)} e^{-\sqrt{2}\tau\cosh\theta + i\sqrt{2}\sigma\sinh\theta}$$

$$+\frac{\sqrt{\lambda}}{2\pi}\int_{\mathbb{R}+i0}\frac{d\theta}{\pi\sinh^2(2\theta)} e^{-2\tau\cosh\theta+2i\sigma\sinh\theta} + \ldots .$$

Direct computation of the Area.
(using classical Integrability of the string sigma model)
Purely Geometrical Problem.



Area

[Alday,Maldacena] (4pt),
[Alday,Maldacena] (special case of 8pt),
[Alday,Gaiotto,Maldacena] (6pt)
[Alday,Maldacena,Sever,PV] (general configuration)

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$$+\frac{\sqrt{\lambda}}{2\pi}\int_{\mathbb{R}+i0} \frac{d\theta}{\pi\sinh^2(2\theta)} e^{-2\tau\cosh\theta+2i\sigma\sinh\theta} + \dots .$$

Direct computation of the Area.
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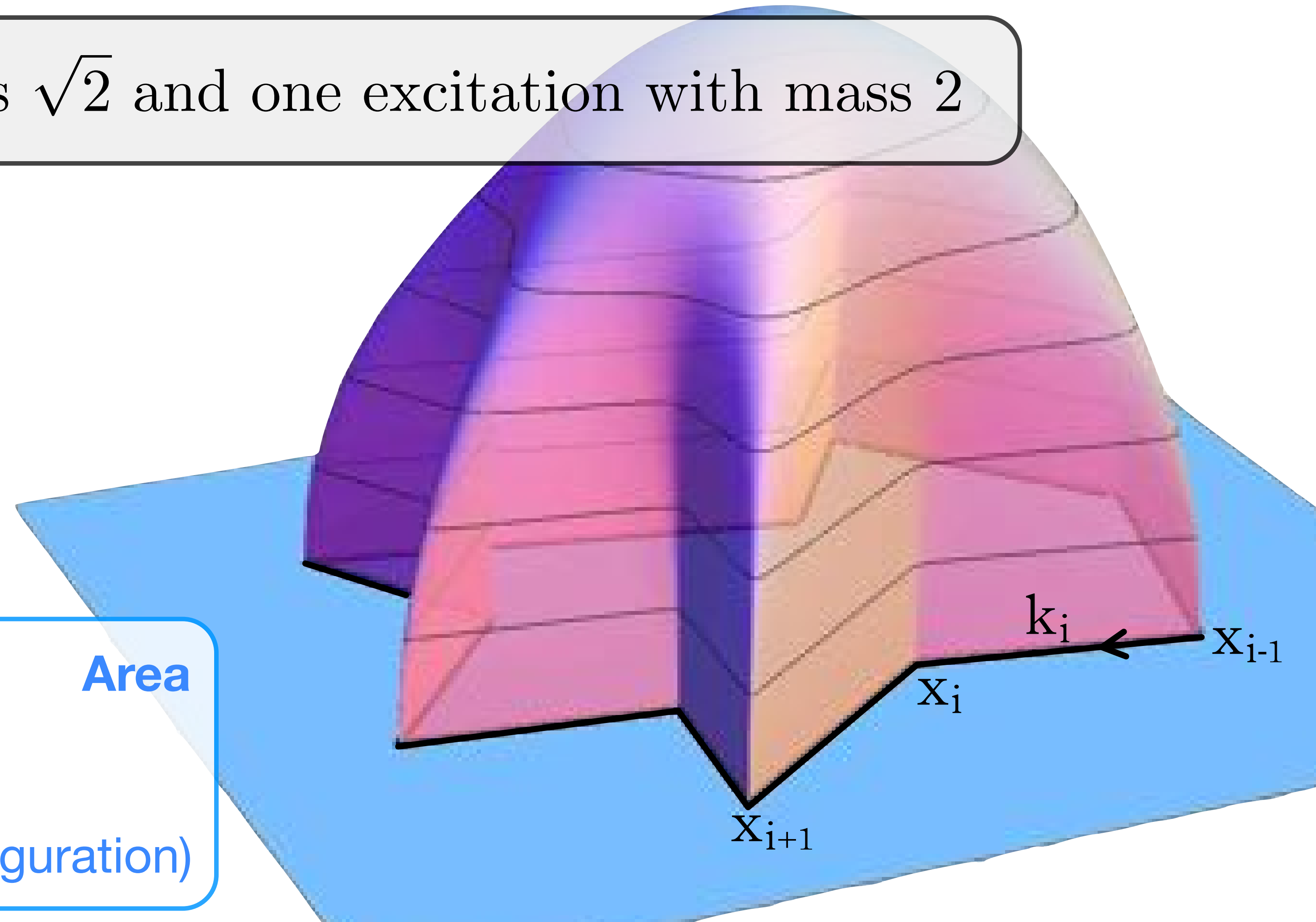
Purely Geometrical Problem.

We see two excitations with mass $\sqrt{2}$ and one excitation with mass 2



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Strong Coupling. The E

$$\mathcal{W}^{\text{string}} \simeq \exp\left(-\frac{\sqrt{\lambda}}{2\pi} YY_c\right) = 1 - \frac{\sqrt{\lambda}}{2\pi}(e^{i\phi} + \ldots$$

$$+\frac{\sqrt{\lambda}}{2\pi}$$

Direct computation of the Area.
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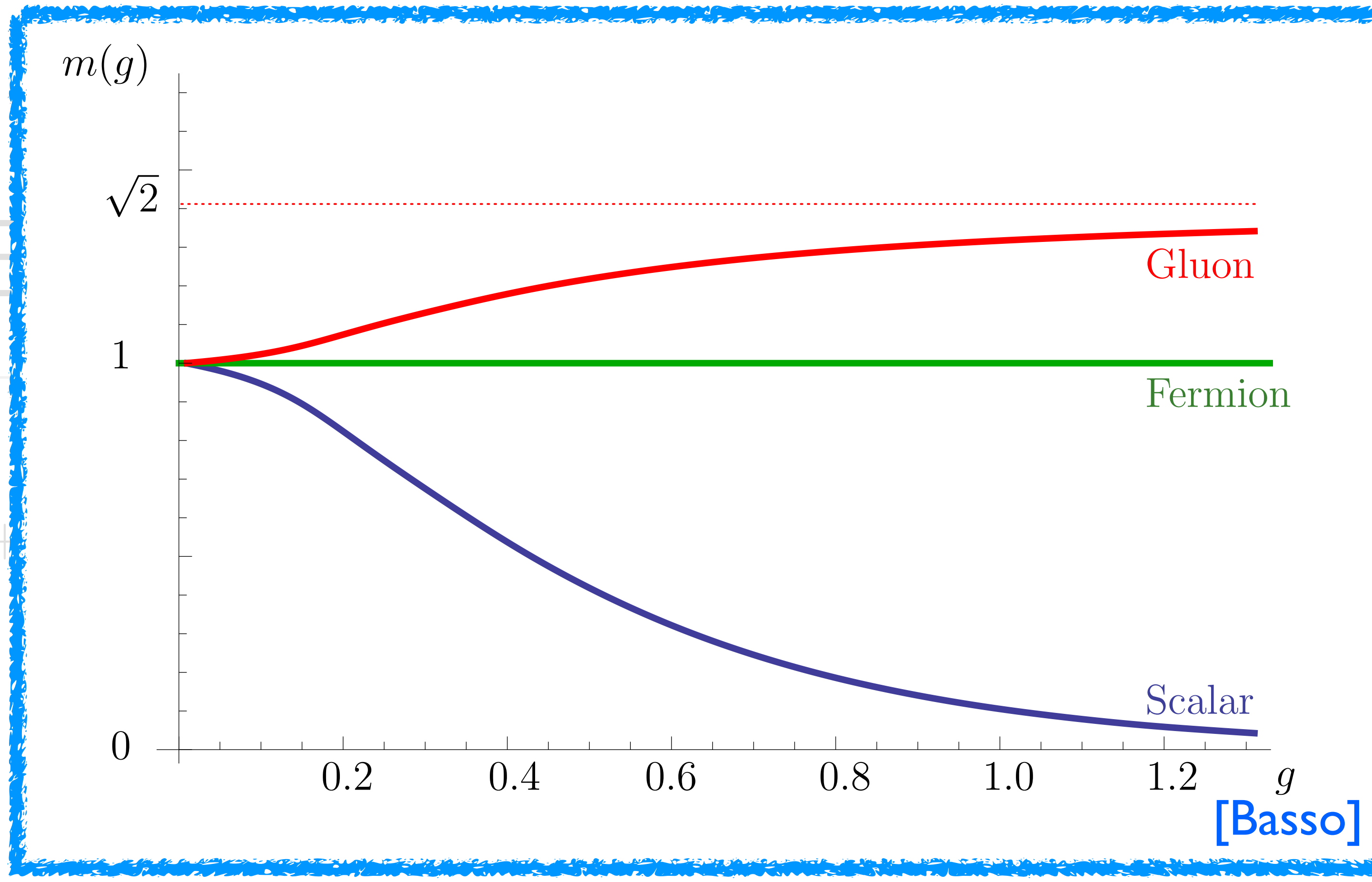
Purely Geometrical Problem.



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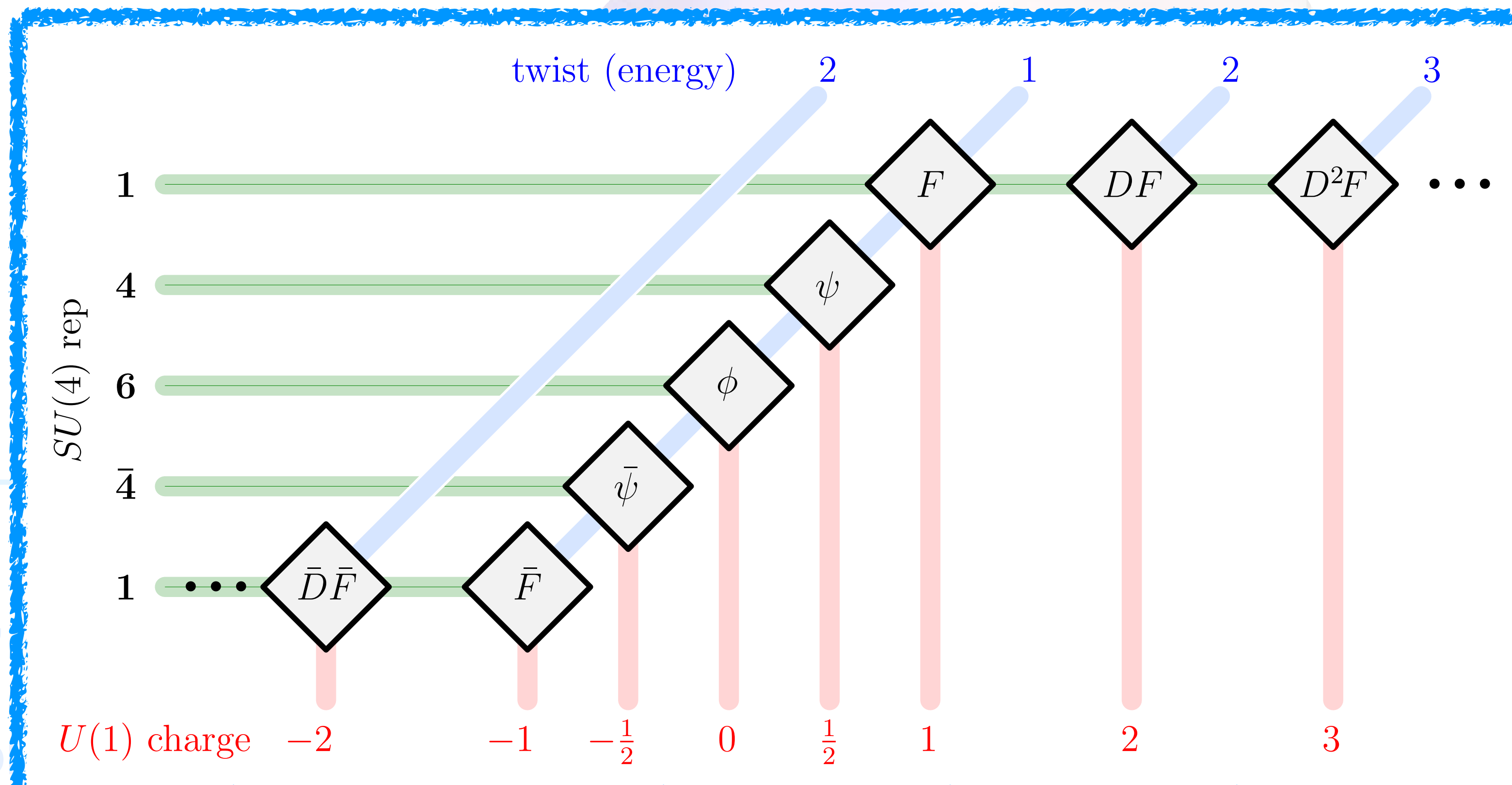

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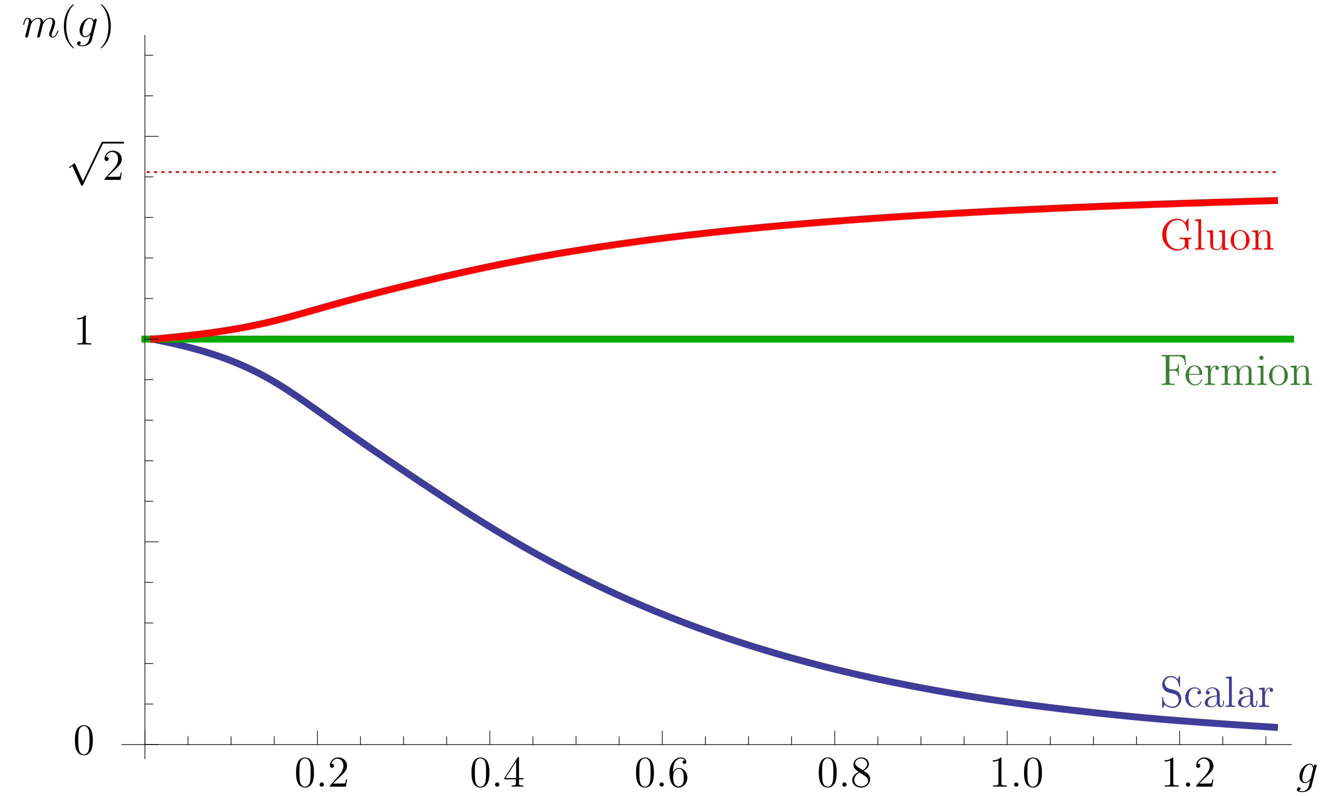
[Alday,Maldacena,Sever,PV] (general co

Strong Coupling. The E

The two lightest modes are the transverse excitations of the flux tube. The sum over multi-particle gluons exponentiates at strong coupling yielding precisely the corresponding terms in the Y-system.



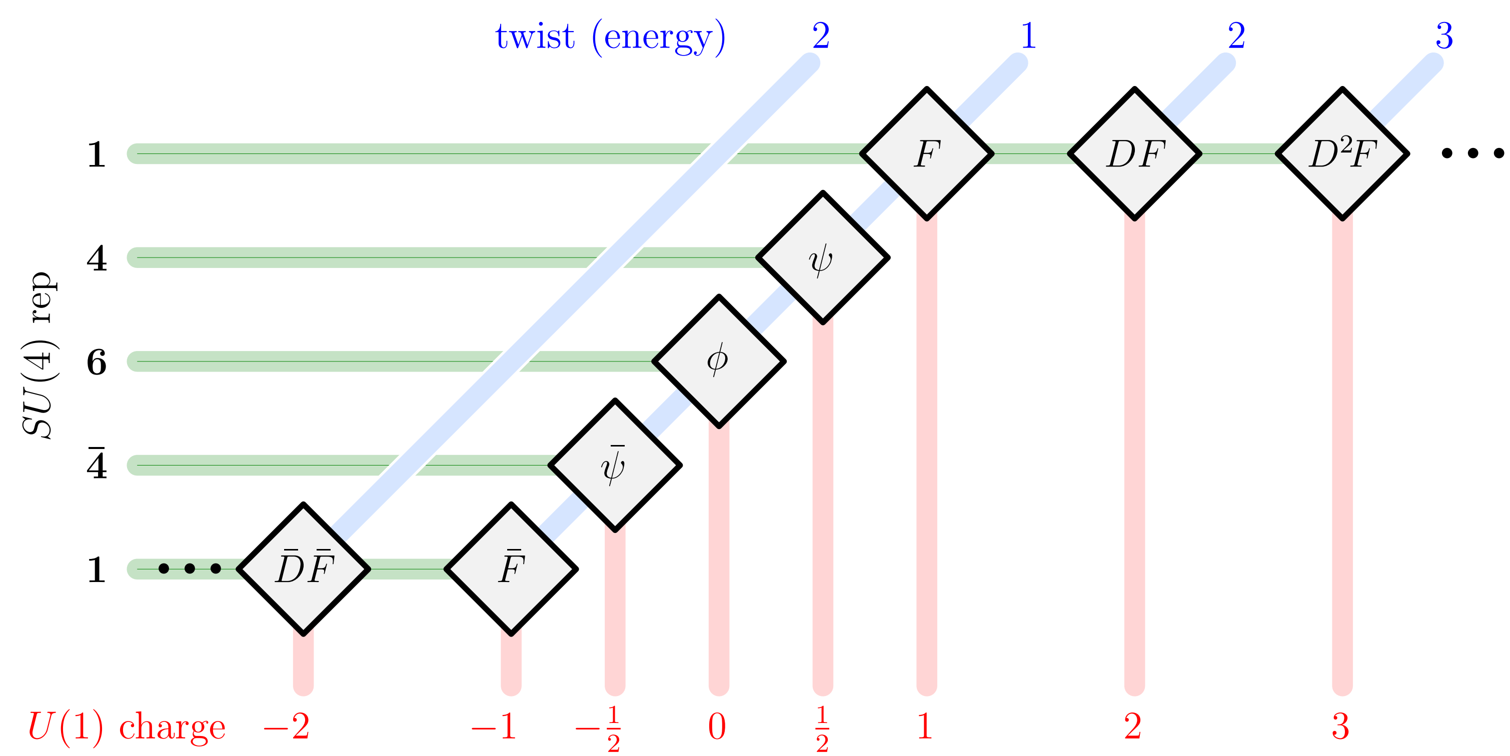
[Basso]

We see two excitations with mass $\sqrt{2}$ and one excitation with mass 2

The mass 2 excitation corresponding to the missing direction in AdS, is an *emergent* excitation which arises at strong coupling as a sort of bound-state made out of two *fermionic* excitations each of mass 1.

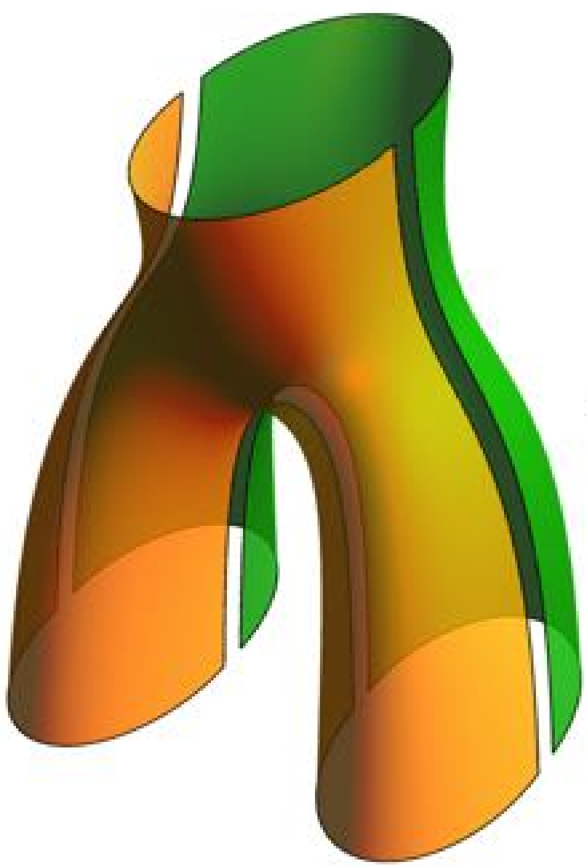
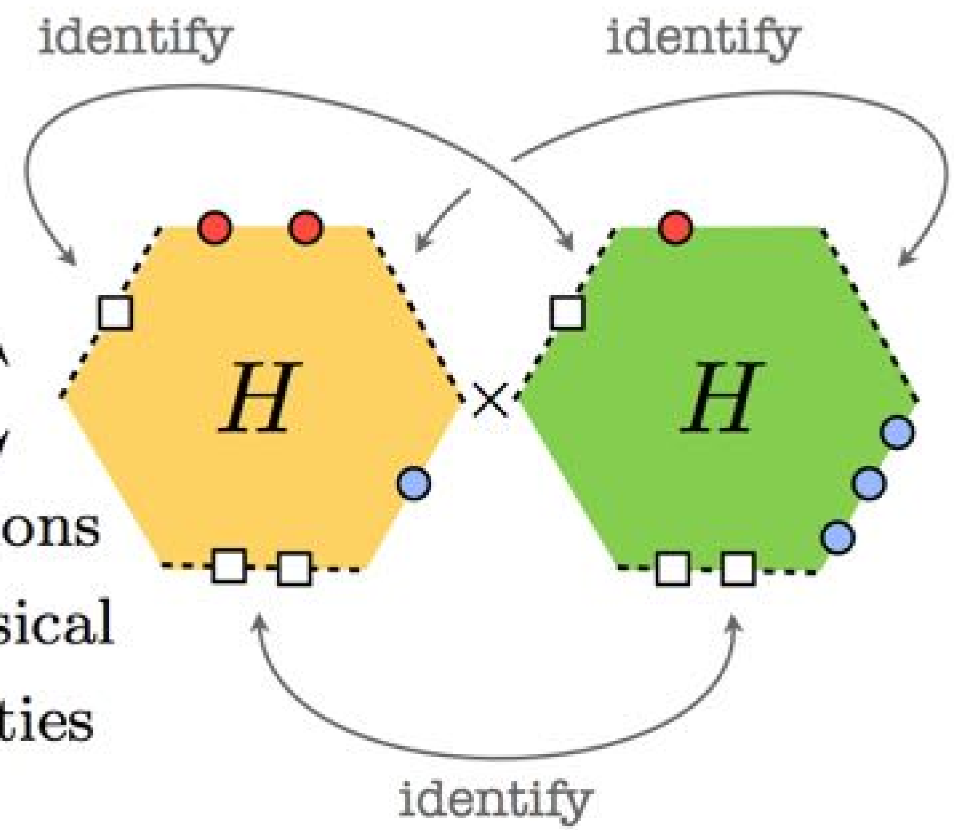
[Basso,Sever,PV]

Counterpart to the QCD flux tube axion excitation proposed by Flauger, Dubosky and Gorbenko? (next talk)



Tailoring 3pt Functions

[Basso, Komatsu, PV, 2015]

$C_{123} =$  $\sim \int$ (momentum of) mirror particles where we glue □ $\sum$ partitions of physical rapidities ○ 

The Hexagon twist operators can then once again be Bootstrapped using Integrability and the results can be then compared against direct perturbative computations:

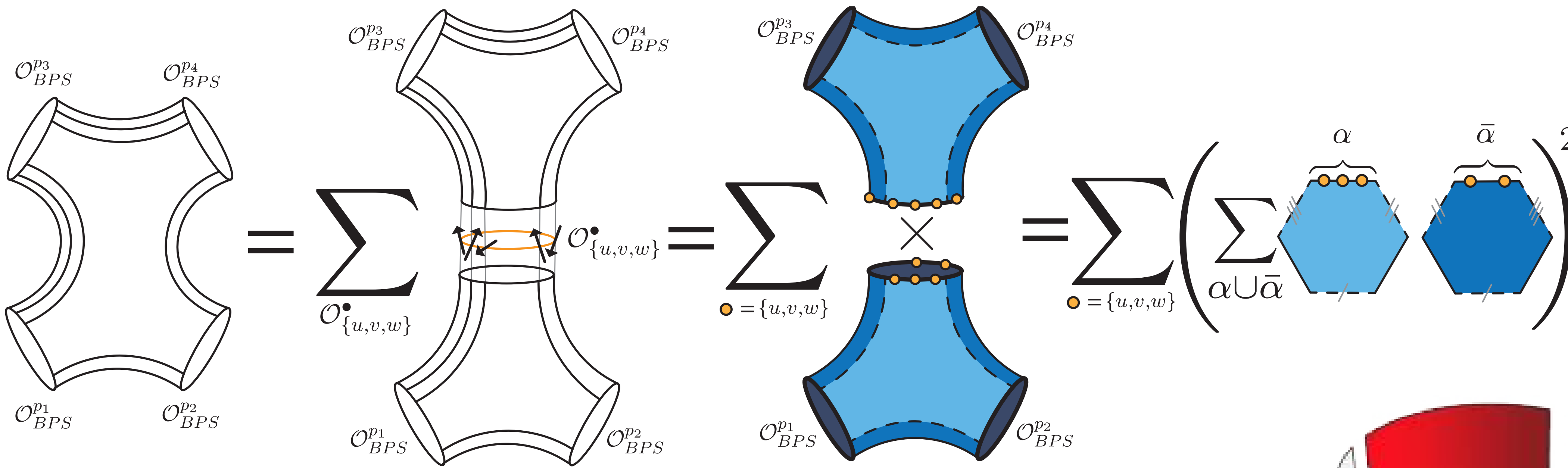
[Basso, Komatsu, PV, 2015] = [Dolan, Osborn 2001] up to 2 loops

[Basso, Gonçalves, Komatsu, PV, 2016] = [Eden 2012; Chicherin, Drummond, Heslop, Sokatchev] @ 3 loops

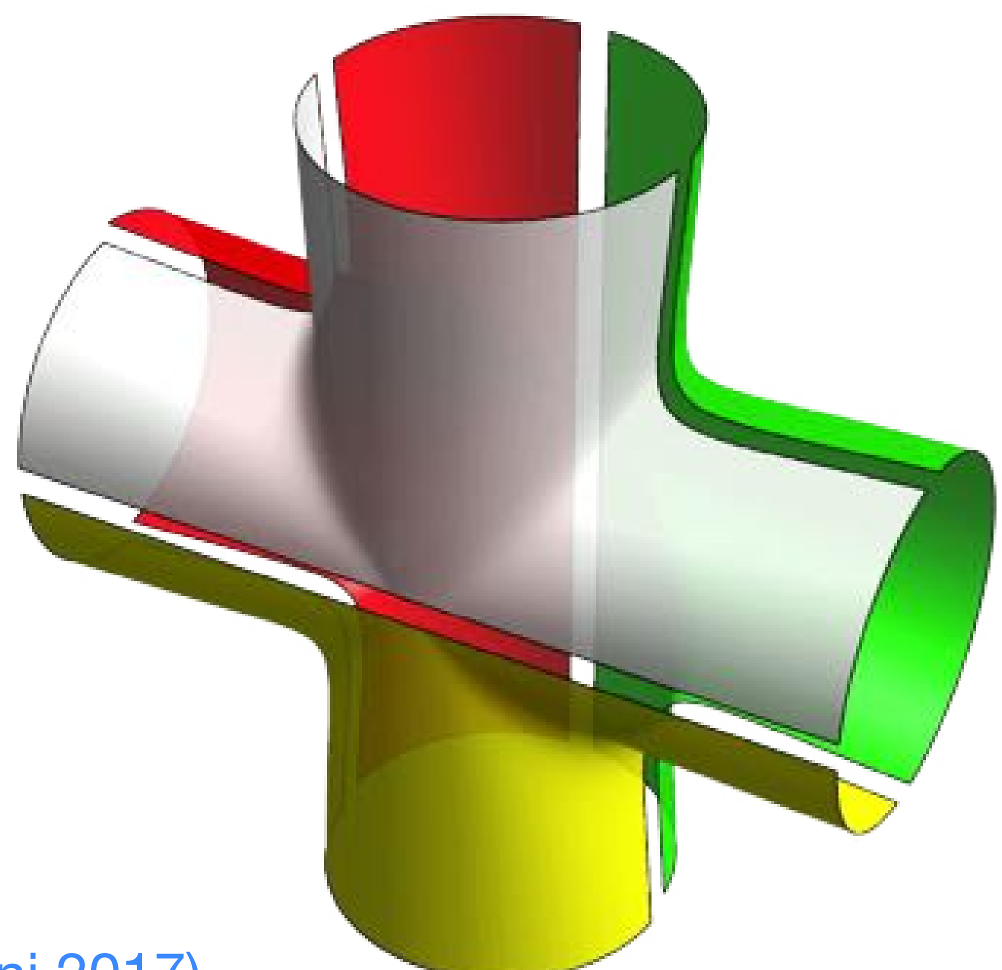
[Basso, Gonçalves, Komatsu, 2017] = [Gonçalves 2017; Eden, Paul 2016] @ 4 loops

The last one is more than just a check as it also fixes some ambiguities in the original prescription.

Four-Point Functions

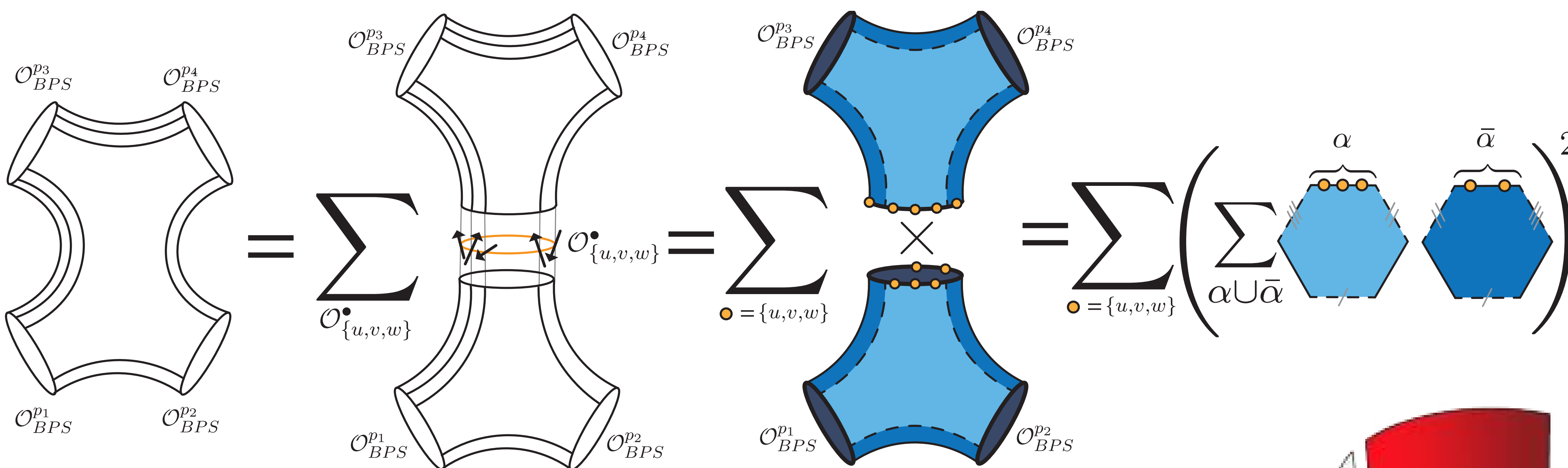


[Basso, Coronado, Komatsu, Tat Lam, PV, Zhong, 2017]



[Fleury, Komatsu 2017] (see also Eden,Stronfrini 2017)

Four-Point Functions

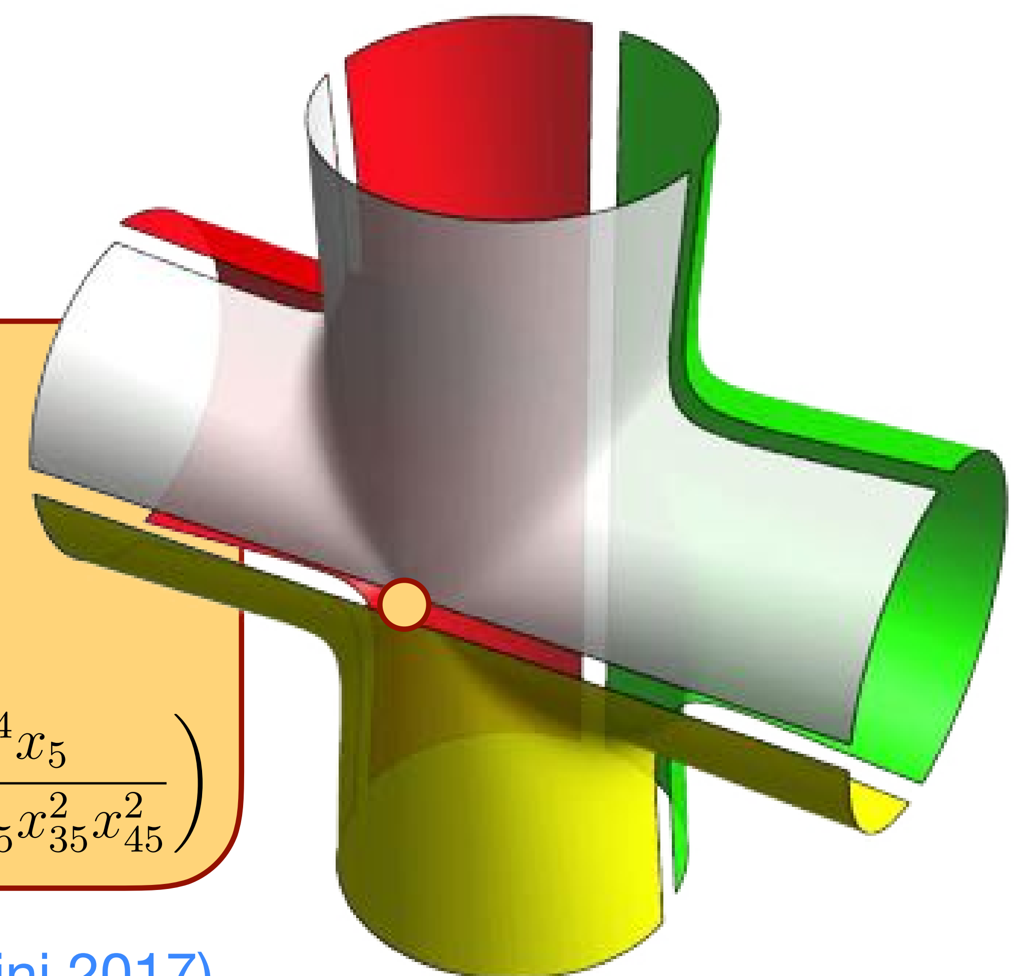


[Basso, Coronado, Komatsu, Tat Lam, PV, Zhong, 2017]

One loop example from Fleury and Komatsu:

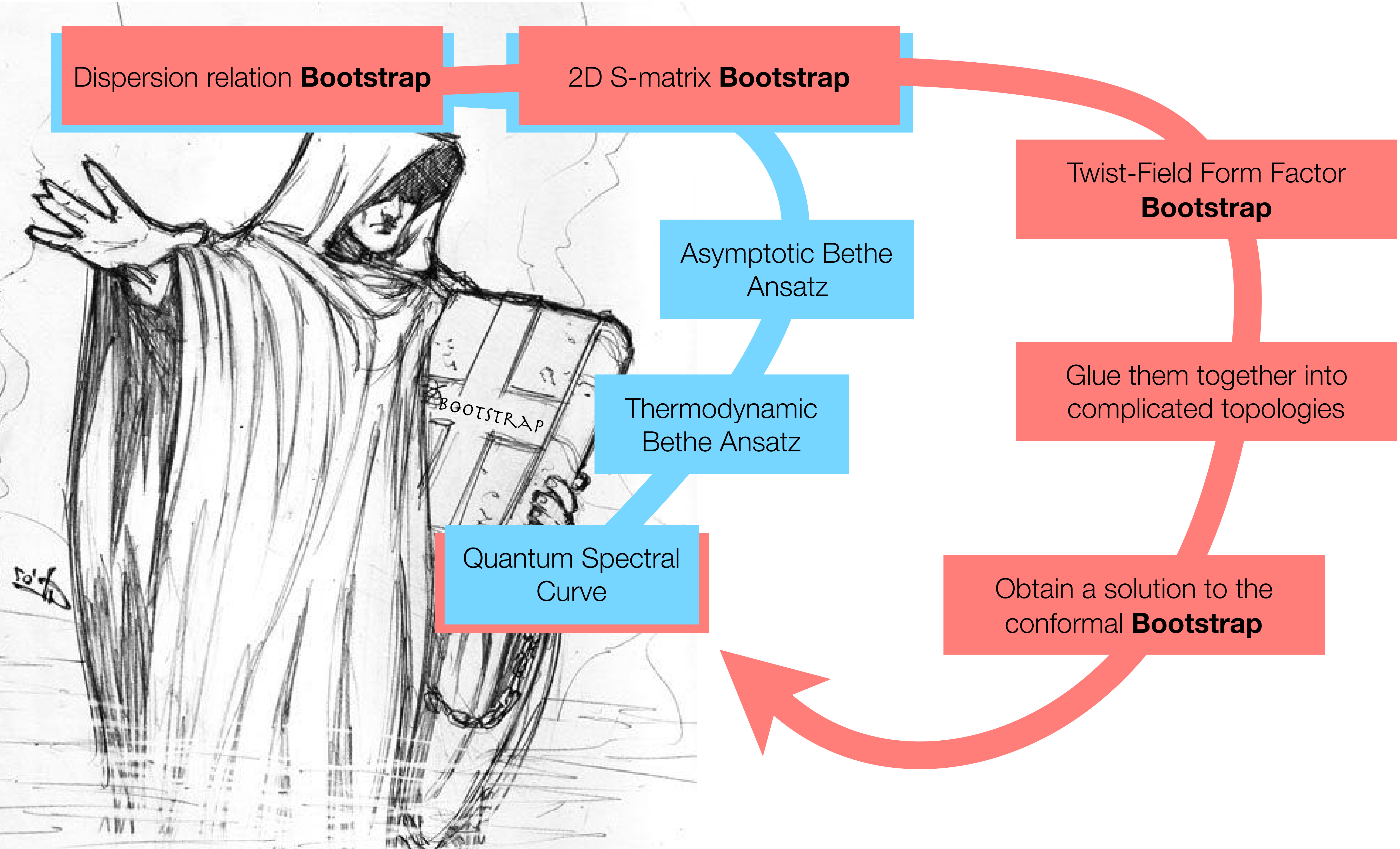
$$\sum_a \int dv \frac{\left(\frac{z}{\bar z}\right)^{a/2} - \left(\frac{\bar z}{z}\right)^{a/2}}{\left(\frac{z}{\bar z}\right)^{1/2} - \left(\frac{\bar z}{z}\right)^{1/2}} \times \frac{a}{v^2 + \frac{a^2}{4}} \times (z\bar z)^{-iv}$$

$$\propto \frac{2\mathrm{Li}_2(z) - 2\mathrm{Li}_2(\bar z) + \log z\bar z \log \frac{1-z}{1-\bar z}}{z - \bar z} \qquad \left(= \frac{x_{13}^2 x_{24}^2}{\pi^2} \int \frac{d^4 x_5}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2} \right)$$



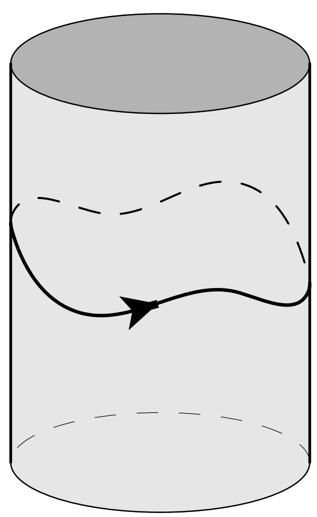
[Fleury, Komatsu 2017] (see also Eden,Stronfrini 2017)

An alternative different story full of *bootstraps*?

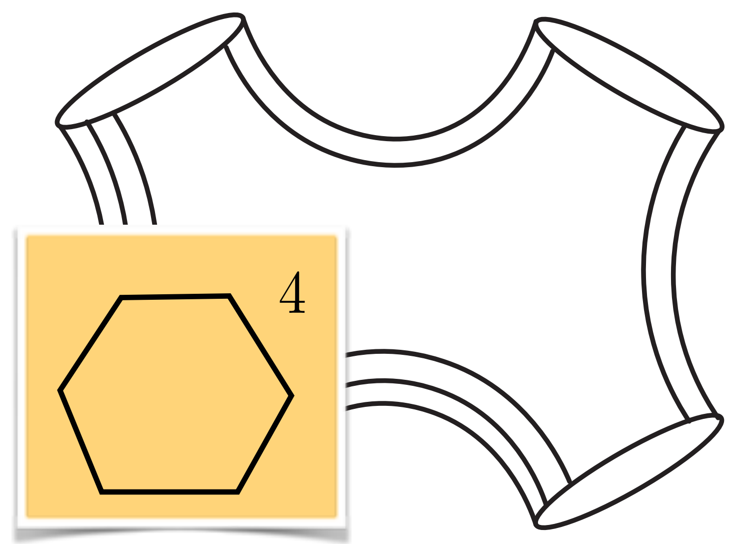


Unified Picture

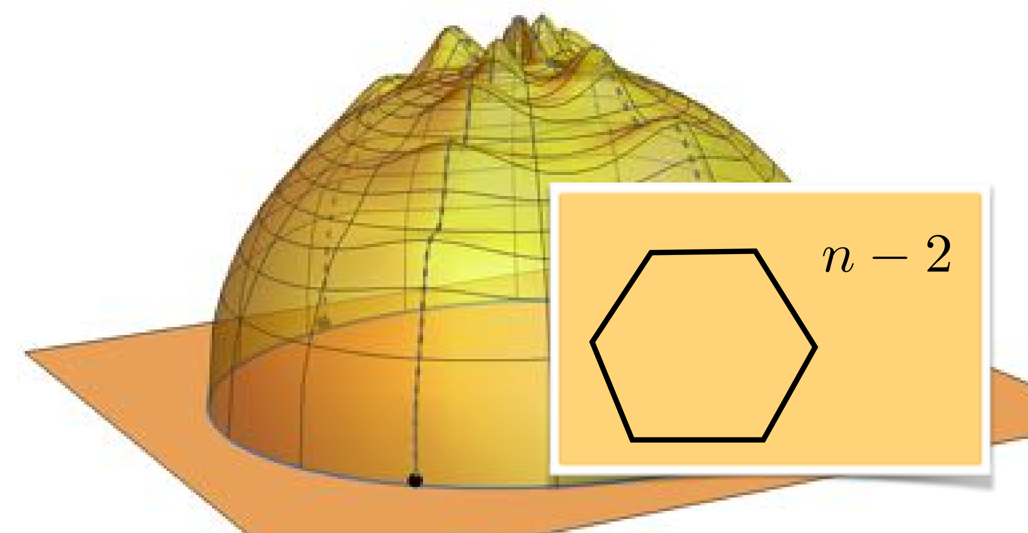
Cylinder



Sphere with Four Punctures

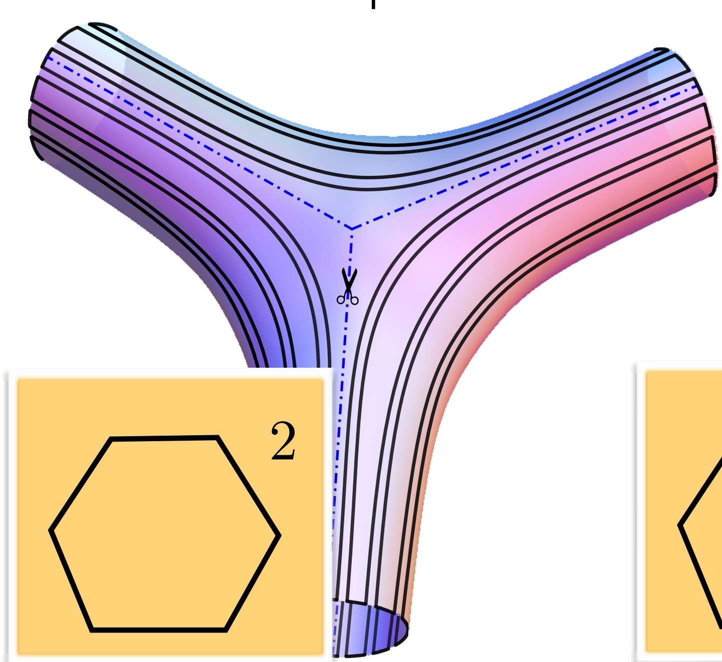


Disk with Circular Boundary
and n insertions

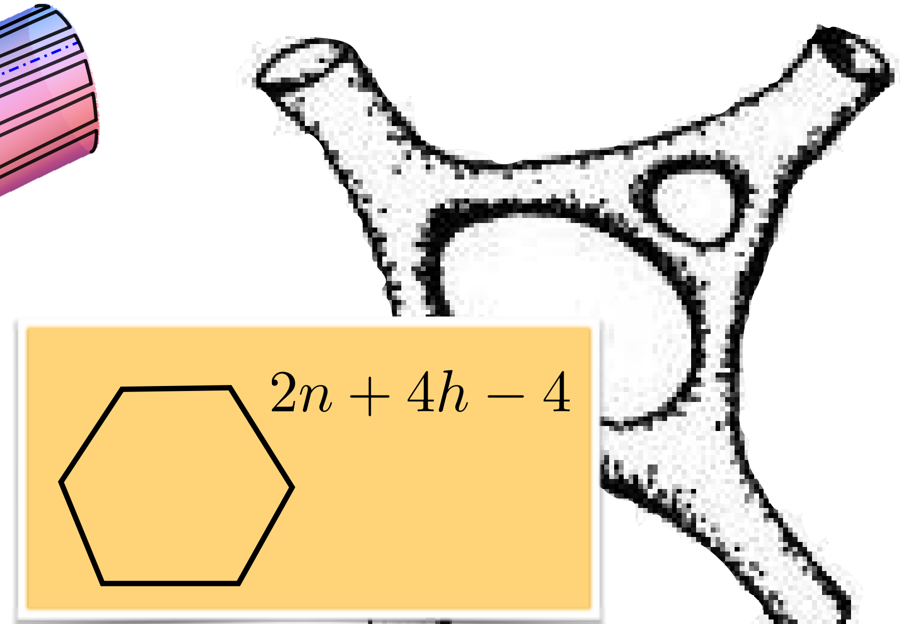


[Kim, Kiryu 2017]
[Kim, Kiryu, Komatsu, Nishimura to appear]

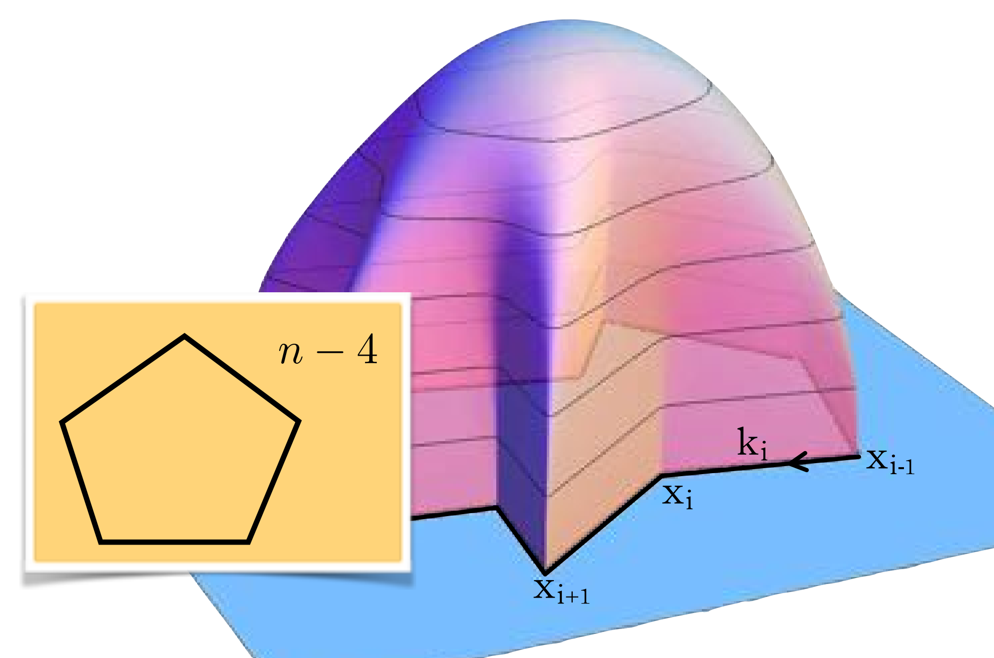
Pair of pants



Sphere with n Punctures
and h handles



Disk with null n-gon boundary



Open Problems

- Carry out some non-planar example in detail.
  Work in progress with Bargheer, Basso, Caetano, Komatsu, Fleury

- Connect Hexagons and Pentagons. (Large spin perhaps…)
  Inspiring first connection a few weeks ago by Basso and Dixon.

- Is there a master Quantum Curve for all quantities in N=4 SYM re-summing all these gluing sums and integrals?
  Very nice partial recent results by Bajnok, Janik. Are strong coupling Y-systems hints or red herrings? Partial resummations at strong couplings by Jiang, Komatsu, Kostov, Serban, see also Kazama, Komatsu, Nishimura

- Find interesting physical limits where the expressions simplify. Bulk Locality, Regge limit, Rastelli and Zhou's results, Heavy-Heavy-Light's…

- Relate the CFT/OPE cutting to the String theory/Hexagonalization cutting.
  Work in progress with Coronado and Komatsu

- General lessons for CFT's? General lessons for string theory? Can we *define* closed String theories as collections of hexagons obeying some set of consistency relations?