Ambitwistor Strings beyond Tree-level
Worldsheet Models of QFTs

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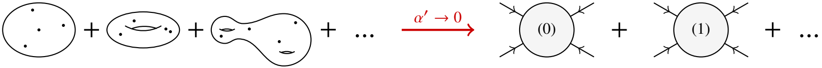
YG, L. Mason, R. Monteiro, P. Tourkine

arxiv:170x.xxxx with R. Monteiro

Motivation: worldsheet models

String Theory



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- map: $\Sigma \to M$

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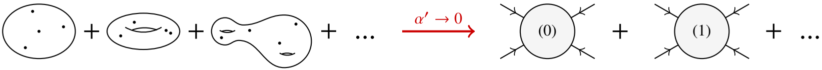

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Worldsheet models of QFT

[Witten, Berkovits, RSV; Hodges, Cachazo-YG, Skinner-Mason]

- map: $\Sigma \to \mathbb{T} \cong \mathbb{CP}^{3|4}$
- $D = 4$, maximal supersymmetry

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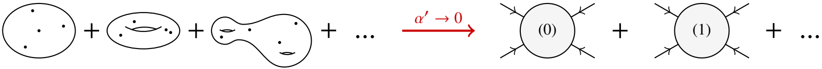

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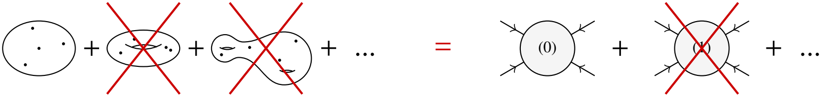

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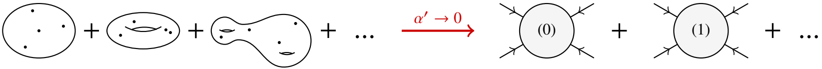

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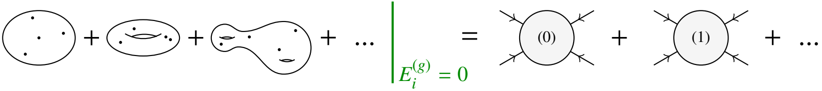
[CHY, Skinner-Mason, Adamo-Casali-Skinner, CGMMR]



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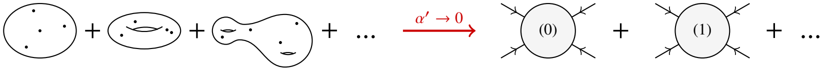

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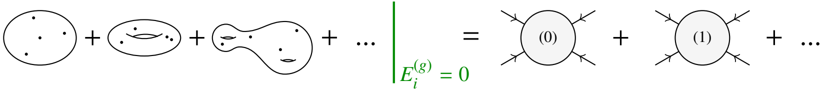

- integration fully localised on the Scattering Equations $\{E_i^{(g)}\}$
- map: $\Sigma \to \mathbb{A}$ = {phase space of complex null rays}
- upshot: generic for massless QFTs

Ambitwistor Strings:

- two-dimensional chiral CFTs
- auxiliary target space:
  complexified phase-space of null geodesics

Starting Point: Tree-level S-matrix

CHY formulae [Cachazo-He-Yuan]

$$\mathscr{M}_{n,0} = \int_{\mathfrak{M}_{0,n}} \frac{\mathrm{d}\sigma^n}{\mathrm{vol}\, G} \prod_i \bar{\delta}\left(\sum_{j\neq i} \frac{k_i \cdot k_j}{\sigma_i - \sigma_j}\right) \mathcal{I}_n$$

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- Integration over $\mathfrak{M}_{0,n}$ localised onto the *scattering equations* E_i
- Integrand $\mathcal{I}_n(\epsilon, k, \sigma) \in \bigotimes K_i^2$ encodes kinematic data

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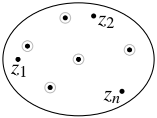
- Integration over $\mathfrak{M}_{0,n}$ localised onto the *scattering equations* E_i
- Integrand $\mathcal{I}_n(\epsilon, k, \sigma) \in \bigotimes K_i^2$ encodes kinematic data
- progress on evaluation without solving the SE
  [Cachazo-Gomez, Baadsgaard et al, Sogard-Zhang, ...]
- family of massless models:
  - Gravity: $\mathcal{I}_n = \mathrm{Pf}'(M)\,\mathrm{Pf}'(\tilde{M})$
  - Yang-Mills theory: $\mathcal{I}_n = C_n\, \mathrm{Pf}'(M)$
  - Bi-adjoint scalar: $\mathcal{I}_n = C_n\, \tilde{C}_n$

Scattering Equations

[Fairlie-Roberts, Gross-Mende][Cachazo-He-Yuan, Mason-Skinner]

Construction: for n null momenta k_i, define
$P_\mu(\sigma, \sigma_i) \in \Omega^0(\Sigma, K_\Sigma)$

$$P_\mu(\sigma) = \sum_{i=1}^{n} \frac{k_{i\mu}}{\sigma - \sigma_i} d\sigma \,.$$



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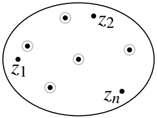

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Scattering Equations at tree-level

Enforce $P^2 = 0$ on Σ:

$$E_i = \text{Res}_{\sigma_i} P^2(\sigma) = k_i \cdot P(\sigma_i) = \sum_{j \neq i} \frac{k_i \cdot k_j}{\sigma_i - \sigma_j} = 0 \, .$$

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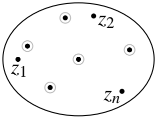

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- $\sum \sigma_i^a E_i = 0$ for $a = 0, 1, 2$
  $\rightarrow$ $\mathrm{SL}(2, \mathbb{C})$ invariant
  $\rightarrow$ $(n-3)$ independent equations = $\dim(\mathfrak{M}_{0,n})$
  $\rightarrow$ $(n-3)!$ solutions
- factorisation properties [Dolan-Goddard, YG-Mason-Monteiro-Tourkine, ...]

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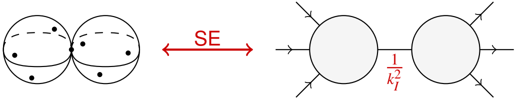

boundary of $\mathfrak{M}_{0,n}$ factorisation channel

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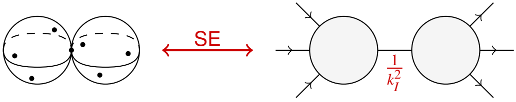

boundary of $\mathfrak{M}_{0,n}$ factorisation channel

- With $\sigma_i = \sigma_I + \varepsilon x_i$ for $i \in I$, the pole is given by

$$\sum_{i\in I} x_i E_i^{(I)} = \sum_{i,j\in I} x_i \frac{k_i \cdot k_j}{x_i - x_j} = \frac{1}{2} \sum_{i,j\in I} k_i \cdot k_j = k_I^2\,.$$

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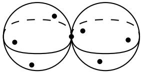

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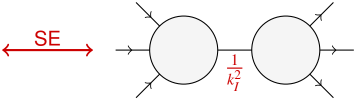

factorisation channel

- Upshot: amplitudes take the form

$$\mathcal{M} = \sum_{\sigma_j | E_i(\sigma_j)=0} \frac{\mathcal{I}(\sigma_i, k_i, \epsilon_i)}{J(\sigma_i, k_i)}\,.$$

The Ambitwistor String

Moduli integral: CHY $\sim$ correlator of CFT on Σ.

$$S = \frac{1}{2\pi}\int P\cdot\bar\partial X + \frac{1}{2}\sum_r \Psi_r\cdot\bar\partial\Psi_r - \frac{e}{2}P^2 - \chi_r P\cdot\Psi_r\,,$$

where $P_\mu \in \Omega^0(\Sigma, K_\Sigma)$, $\Psi_r^\mu \in \Pi\Omega^0(\Sigma, K_\Sigma^{1/2})$.

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- Geometrically:
  - gauge fields e and χ_r impose the constraints $P^2 = P\cdot\Psi_r = 0$ target space: Ambitwistor space $\mathbb{A}$
  - gauge freedom: $\delta X^\mu = \alpha P^\mu$, $\delta P_\mu = 0$, $\delta e = \bar\partial\alpha$ ...

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- BRST quantisation: $Q = \oint cT + \tilde{c} P^2 + \gamma_r P \cdot \Psi_r$
  Vertex operators: $V = c\tilde{c}\delta^2(\gamma)\, \epsilon_\mu \epsilon_\nu \Psi_1^\mu \Psi_2^\nu e^{ik\cdot X}$
  - $Q^2 = 0$ for $d = 10$
  - $[Q, V] = 0 \quad \Rightarrow \quad k^2 = \epsilon \cdot k = 0$
    $\Rightarrow$ spectrum: type II sugra

Localisation and the Scattering Equations

Action: $S = \frac{1}{2\pi}\int P\cdot\bar{\partial}X + \frac{1}{2}\sum_r \Psi_r\cdot\bar{\partial}\Psi_r - \frac{e}{2}P^2 - \chi_r P\cdot\Psi_r\,,$

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- Moduli of gauge field e forces $P^2=0$;
  scattering equations ⇔ map to $\mathbb{A}$

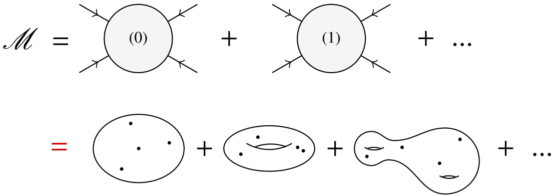
$$\mathrm{Res}_{\sigma_i}P^2(\sigma) = k_i\cdot P(\sigma_i) = 0\,.$$

- Correlator = CHY

Loop Integrands from the Riemann Sphere

Upshot – Loops

Worldsheet models of QFT



$$E_i^{(g)} = 0$$

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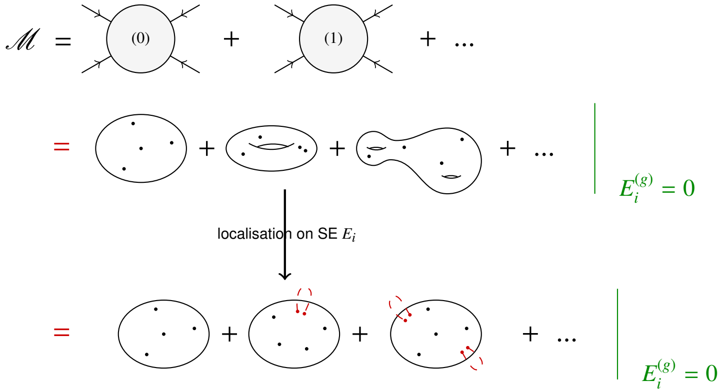
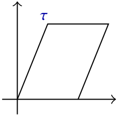


Scattering Equations on the Torus

Genus 1

- moduli space of torus Σ_τ:



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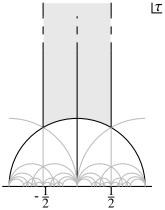
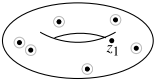
$$\tau \sim \tau + 1 \sim -1/\tau$$



- Solve $\bar{\partial} P = 2\pi i \sum_i k_i \bar{\delta}(z - z_i) dz$:

$$P_\mu = \left(2\pi i \, \ell_\mu + \sum_i k_{i\mu} \frac{\theta_1'(z - z_i)}{\theta_1(z - z_i)} \right) dz \, ,$$



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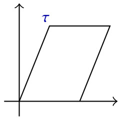

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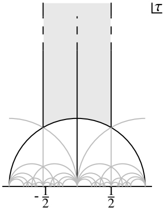

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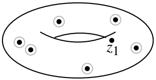

Scattering Equations on the torus: $P^2(z|\tau) = 0$

$$\mathrm{Res}_{z_i} P^2(z) := 2k_i \cdot P(z_i) = 0 \,,$$
$$P^2(z_0) = 0 \,.$$

The 1-loop Integrand

One-loop integrand of type II supergravity

$$\mathcal{M}^{(1)}_{\text{SG}} = \int d^{10}\ell \, d\tau \, \underbrace{\bar{\delta}(P^2(z_0)) \prod_{i=2}^{n} \bar{\delta}(k_i \cdot P(z_i))}_{\text{Scattering Equations}} \underbrace{\left(\sum_{\text{spin struct.}} Z^{(1)}(z_i) Z^{(2)}(z_i) \right)}_{\equiv \mathcal{I}_q, \text{ fermion correlator}}$$

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- modular invariance: $\tau \sim \tau + 1 \sim -1/\tau$
- Scattering Equations: $P^2(z|\tau) = 0$
- checks:
  - factorisation
  - correct tensor structure at $n = 4$: $t_8 t_8 R^4$

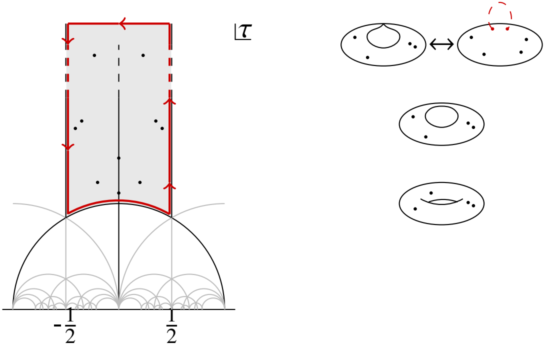
Why rational function?

From the Torus to the Riemann Sphere

Contour argument

- localisation on scattering equations
  $\Rightarrow$ contour integral argument in fundamental domain
- modular invariance: sides and unit circle cancel

$\Rightarrow$ localisation on $q \equiv e^{2i\pi\tau} = 0 \Leftrightarrow \tau = i\infty$



Contour argument in the fundamental domain

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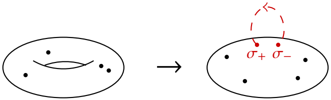

Residue theorem: elliptic curve $\longrightarrow$ nodal Riemann spere at $q = 0$.

From the Torus to the Riemann Sphere

More explicitly: residue theorem

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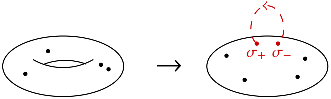

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$$\mathscr{M}^{(1)}_{SG} = \frac{1}{2\pi i} \int d^{10}\ell \, \frac{dq}{q} \, \bar{\partial}\left(\frac{1}{P^2(z_0)}\right) \prod_{i=2}^{n} \bar{\delta}(k_i \cdot P(z_i)) \, \mathcal{I}_q$$

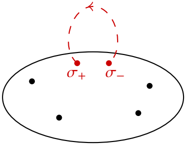
$$= - \int d^{10}\ell \, dq \, \frac{1}{P^2(z_0)} \, \bar{\delta}(q) \prod_{i=2}^{n} \bar{\delta}(k_i \cdot P(z_i)) \, \mathcal{I}_0$$

$$= - \int \frac{d^{10}\ell}{\ell^2} \prod_{i=2}^{n} \bar{\delta}(k_i \cdot P(z_i)) \, \mathcal{I}_0 \Big|_{q=0} .$$

One-loop off-shell scattering equations

On the nodal Riemann Sphere:

$$P = \left(\frac{\ell}{\sigma - \sigma_{\ell^+}} - \frac{\ell}{\sigma - \sigma_{\ell^-}} + \sum_{i=1}^{n} \frac{k_i}{\sigma - \sigma_i} \right) d\sigma \,.$$



Define $S = P^2 - \left(\frac{\ell}{\sigma - \sigma_{\ell^+}} - \frac{\ell}{\sigma - \sigma_{\ell^-}}\right)^2 d\sigma^2$.

One-loop off-shell scattering equations

$$\mathrm{Res}_{\sigma_i} S = k_i \cdot P(\sigma_i) = \frac{k_i \cdot \ell}{\sigma_i - \sigma_{\ell^+}} - \frac{k_i \cdot \ell}{\sigma_i - \sigma_{\ell^-}} + \sum_{j \neq i} \frac{k_i \cdot k_j}{\sigma_i - \sigma_j} = 0 \,,$$

$$\mathrm{Res}_{\sigma_{\ell^-}} S = - \sum_j \frac{\ell \cdot k_j}{\sigma_{\ell^-} - \sigma_j} = 0 \,,$$

$$\mathrm{Res}_{\sigma_{\ell^+}} S = \sum_j \frac{\ell \cdot k_j}{\sigma_{\ell^+} - \sigma_j} = 0 \,.$$

The integrand

One-loop integrand on the nodal Riemann sphere

$$\mathscr{M}^{(1)} = -\int \frac{d^d\ell}{\ell^2} \frac{d^{n+2}\sigma}{\text{vol}\,(G)} \underbrace{\prod_{i,\ell^\pm} \bar{\delta}\left(\text{Res}_{\sigma_i} S\right)}_{\text{off-shell scattering equations}} \mathcal{I}$$

- type II sugra: $\mathcal{I} = \mathcal{I}|_{q\to 0}$, limit of $g = 1$ correlator
- manifestly rational
- upshot: widely applicable for supersymmetric and non-supersymmetric theories

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Puzzle: Only depends on $1/\ell^2$, remainder $\ell \cdot k_i$, $\ell \cdot \epsilon_i$, ...

Solution: Shifted integrands

- partial fractions: $\frac{1}{\prod_i D_i} = \sum_i \frac{1}{D_i \prod_{j\neq i}(D_j - D_i)}$
- shift: $D_i \to \ell^2$
- formalised: Q-cuts [Baadsgaard et al]

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Example:

$$\frac{1}{\ell^2(\ell+K)^2} = \frac{1}{\ell^2(2\ell \cdot K + K^2)} + \frac{1}{(\ell+K)^2(-2\ell \cdot K - K^2)}$$
$$\to \frac{1}{\ell^2}\left(\frac{1}{2\ell \cdot K + K^2} + \frac{1}{-2\ell \cdot K + K^2}\right)$$

Integrands – Double Copy again

One-loop integrand on the nodal Riemann sphere

$$\mathscr{M}^{(1)} = -\int \frac{d^d\ell}{\ell^2}\,\frac{d^{n+2}\sigma}{\mathrm{vol}\,(G)} \prod_{i,\ell^\pm} \bar{\delta}(\mathrm{Res}_{\sigma_i} S)\;\mathcal{I}$$

Supersymmetric:

- $\mathcal{I}_{\text{sugra}} = \dfrac{1}{(\sigma_{\ell^+\,\ell^-})^4}\,\mathcal{I}_0\,\widetilde{\mathcal{I}}_0$
- $\mathcal{I}_{\text{sYM}} = \dfrac{1}{(\sigma_{\ell^+\,\ell^-})^4}\,\mathcal{I}_0\,\mathcal{I}^{PT}$

Building blocks

- Parke-Taylor: $\mathcal{I}^{PT} = \sum_{i=1}^{n} \frac{\sigma_{\ell^+\,\ell^-}\, tr(T^{a_1}T^{a_2}\ldots T^{a_n})}{\sigma_{\ell^+\,i}\sigma_{i+1\,i}\sigma_{i+2\,i+1}\ldots\sigma_{i+n\,\ell^-}}$ + non-cycl.
- Pfaffian: $\mathcal{I}_0 = \sum_r \mathrm{Pf}\,'(M^r_{\text{NS}}) - \frac{c_d}{\sigma^2_{\ell^+\,\ell^-}}\mathrm{Pf}\,(M_2)$

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Non-supersymmetric

- $\mathcal{I}_{\text{YM}} = (\sum_r \mathrm{Pf}\,'(M^r_{\text{NS}}))\ \mathcal{I}^{PT}$
- $\mathcal{I}_{\text{grav}} =$
  $(\sum_r \mathrm{Pf}\,'(M^r_{\text{NS}}))^2 - \alpha\,(\mathrm{Pf}\,(M_3)\big|_{q^0})^2\,.$

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Checks: numerical, factorisation

BCJ numerators

Manifest Colour-Kinematics duality:

- Tree-level: ✓ [Fu-Du-Huang-Feng]
- Loop-level: susy ✓ [He-Schlotterer-Zhang]

  non-susy?

Algorithm:

1. Pick an 'ordered gauge' $(+ \prec 1 \prec \ldots \prec n \prec -)$
2. For a disjoint split into ρ_1, ρ_2, find all ordered splittings of ρ_2
3. Calculate

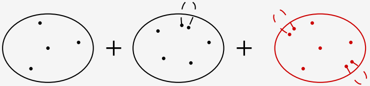
$$n = \sum_{\rho_1} (-1)^{n_\rho} W_{(+\rho_1-)} \left(\sum_{\text{OS}(\rho_2)} \prod_{\text{subsets}} \tilde{W} \right)$$

with $W_{(+\rho_1-)} = \mathrm{tr}(F_{\rho_1} \ldots F_{\rho_{n_\rho}})$

$\tilde{W} = \epsilon_{p_{n_p}} \cdot F_{p_{n_p-1}} \cdot \ldots \cdot F_{p_1} \cdot Z_{p_1}$ for OS $\supset \{p_1 \ldots p_{n_p}\}$

Proof: Expansion of Pfaffian into PT factors

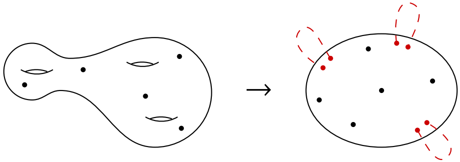
Question: Beyond one loop?

$\mathscr{M} =$  $+ \ \ldots$

$E_i(\sigma_j) = 0$

Heuristics: Higher genus

Riemann surface Σ_g $\xrightarrow[\text{contract } g \text{ } a\text{-cycles}]{\text{residue theorems}}$ nodal RS



- $2g - 3$ moduli $\Leftrightarrow$ $2g$ new marked points mod $\mathrm{SL}(2, \mathbb{C})$
- 1-form P_μ

$$P = \sum_{r=1}^{g} \ell_r \omega_r^{(g)} + \sum_i k_i \frac{d\sigma}{\sigma - \sigma_i} \, ,$$

$\omega_r^{(g)} = \frac{(\sigma_{r^+} - \sigma_{r^-})\, d\sigma}{(\sigma - \sigma_{r^+})(\sigma - \sigma_{r^+})}$: basis of g global holomorphic 1-forms

The Scattering Equations

The multiloop off-shell scattering equations are

$$\mathrm{Res}_{\sigma_A} S = 0\,, \quad A = 1, \ldots, n + 2g$$

$$S(\sigma) := P^2 - \sum_{r=1}^{g} \ell_r^2 \omega_r^2 + \sum_{r<s} a_{rs}(\ell_r^2 + \ell_s^2)\omega_r \omega_s\,.$$

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At two-loops:

- Factorisation: $a_{12} = 1$

Two-loop integrand

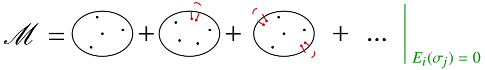
$$\widehat{\mathscr{M}}^{(2)} = \frac{1}{\ell_1^2 \ell_2^2} \int_{\mathfrak{M}_{0,n+4}} \frac{d^{n+4}\sigma}{\mathrm{Vol}\, G} \prod_{A=1}^{n+4} \bar{\delta}\big(\mathrm{Res}_A S(\sigma_A)\big)\, \mathcal{I}\,,$$

- integrands known for $n = 4$ sYM and type II sugra
- Similar complexity as tree amplitudes with $n+4$ particles!

Conclusion and Outlook

Summary:

- massless QFTs from chiral worldsheet theories
- moduli integrals localised on scattering equations
- loop integrands on nodal Riemann spheres



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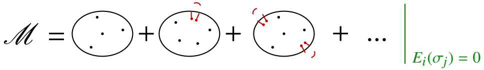

Future directions:

- Short-term: explore one and two loops
- Medium-term: model on nodal Riemann sphere

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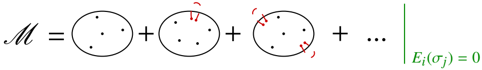

Future directions:

- Short-term: explore one and two loops
- Medium-term: model on nodal Riemann sphere
- Long-term: connection with *string theory*