Octagons

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[with Till Bargheer and Frank Coronado]

arxiv:1904.00965.

Positions (cross-ratios)

't Hooft coupling

Number of colours

In N=4 SYM, we would like to compute

$$\langle O(y_1, x_1) \ldots O(y_n, x_n) \rangle = F(z, \ldots | \lambda, N)$$

where $O \equiv \mathrm{T}r\,(y \cdot \phi(x))^2$ is the simplest, smallest, protected, single trace operator in the theory.

In AdS, this computes fully quantum graviton scattering.

(n=2 and 3 are protected by SUSY and known for a long time)
[Lee, Minwalla, Rangamani, Seiberg]
[Freedman, Mathur, Matusis, Rastelli]

This is too hard.

e.g. [Arutyunov, Dolan, Osborn, Sokatchev],
[Eden, Heslop, Korchemsky, Sokatchev]
[Chicherin, Drummond, Heslop, Sokatchev]

e.g. [D'Hoker, Freedman, Mathur, Matusis, Rastelli],
[Arutyunov, Frolov], [Rastelli, Zhou],[Gonçalves],
[Caron-Huot, Trinh].[Gonçalves,Pereira,Zhou]

We can do very *small coupling* or very *large coupling, typically in the planar limit,* sometimes for the first few terms in the large N expansion. Finite coupling seems a bit intractable for now. Hopefully in a future Strings meeting.

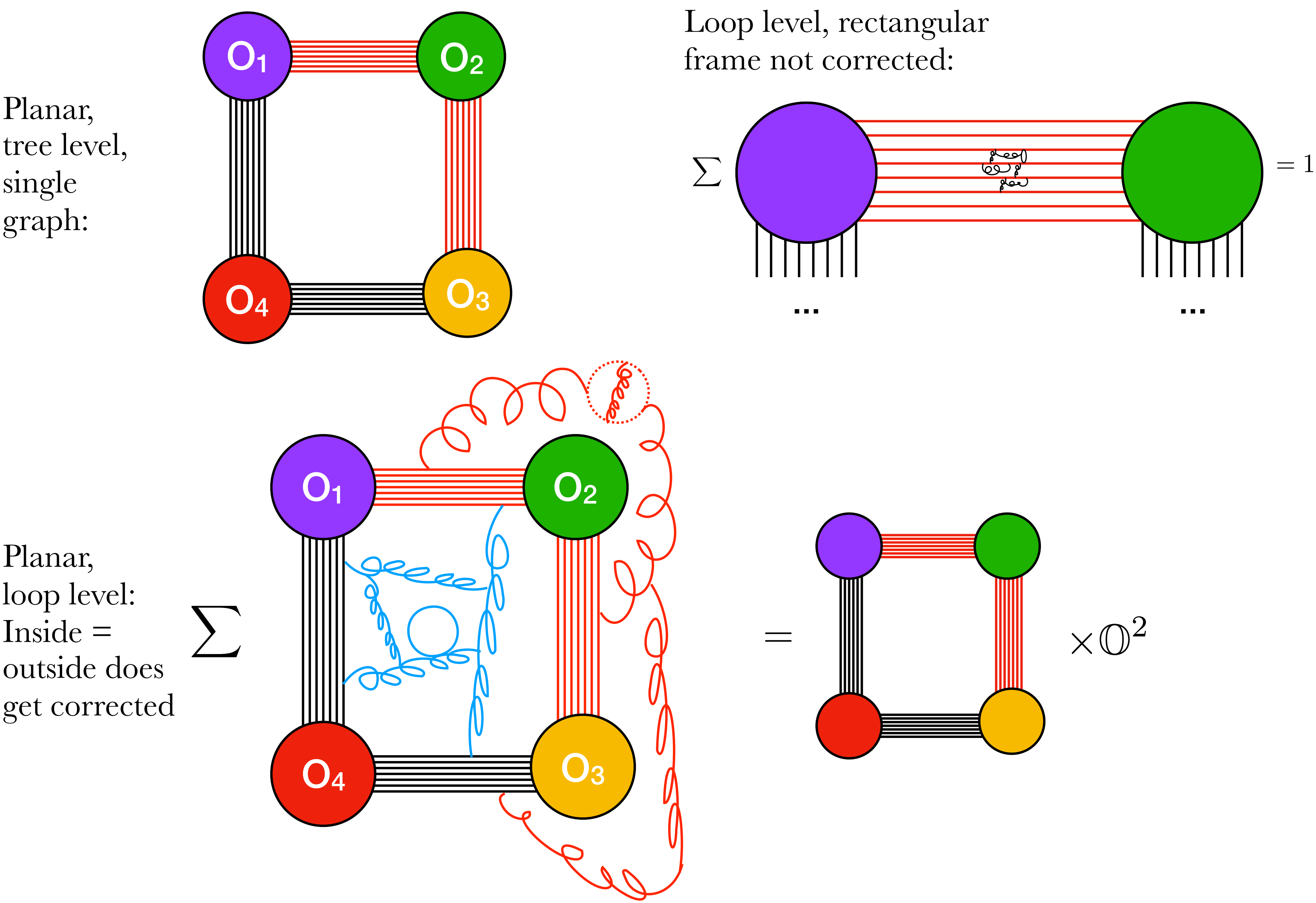
e.g. [Basso,Komatsu,PV],[Fleury, Komatsu],
[Bargheer,Caetano,Fleury, Komatsu,PV],
[Beem, Rastelli, van Rees]

e.g. [Alday,Bissi,Perlmutter], [Aprile,Drummond,Heslop,Paul],[Alday, Caron-Huot]

For now we try to start with large operators $O \to O_k \equiv \mathrm{Tr}\,(y \cdot \phi)^k$ where k is very large.

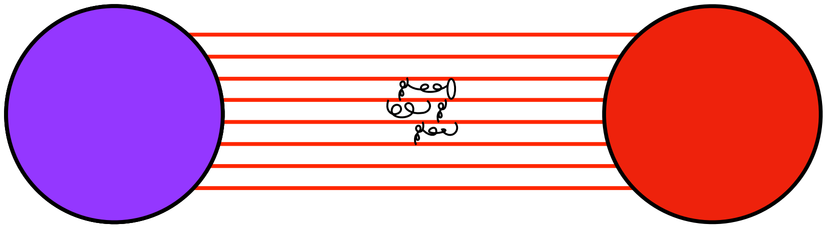
This is not too hard.

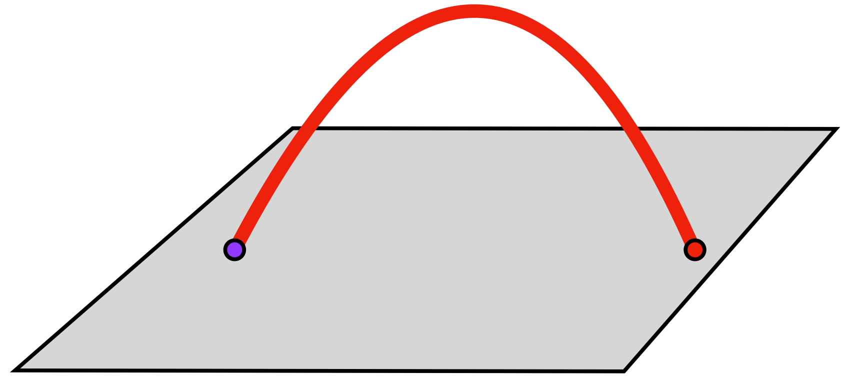
$$\mathcal{O}_2 = \mathrm{tr}(X^{2k})(z)\,, \quad \mathcal{O}_4 = \mathrm{tr}(Z^{2k})(\infty)\,, \quad \mathcal{O}_1 = \mathrm{tr}(\bar{Z}^k\,\bar{X}^k)(0) + \texttt{permutations}\,, \quad \mathcal{O}_3 = \mathrm{tr}(\bar{Z}^k\,\bar{X}^k)(1) + \texttt{permutations}\,,$$



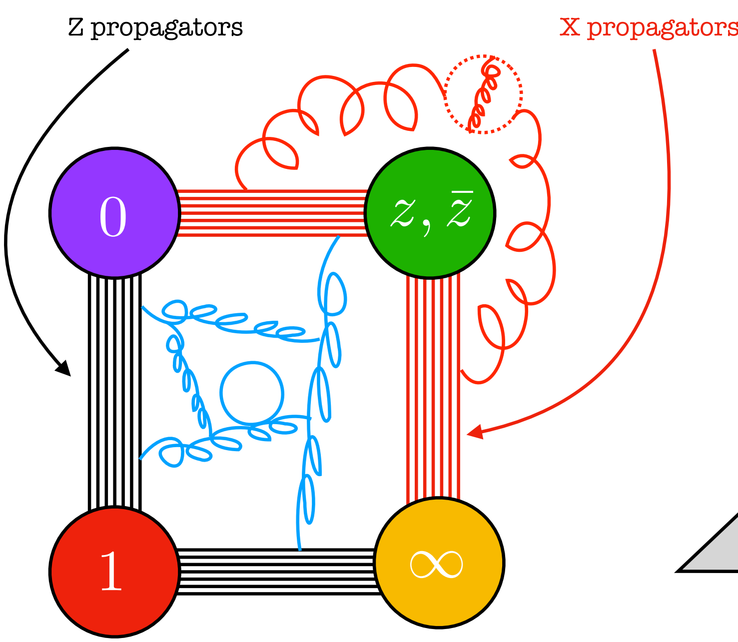
AdS pictures:

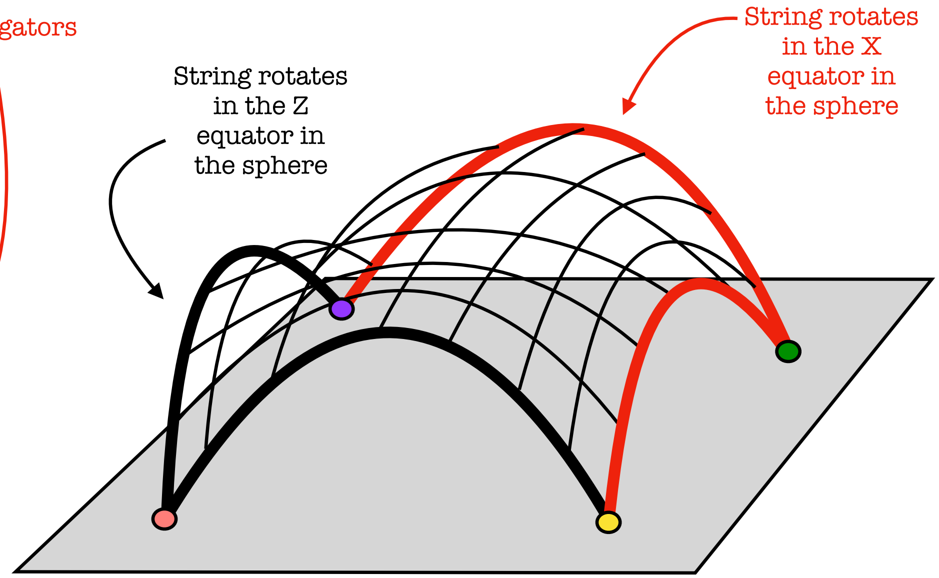
Two point function = classical geodesic:





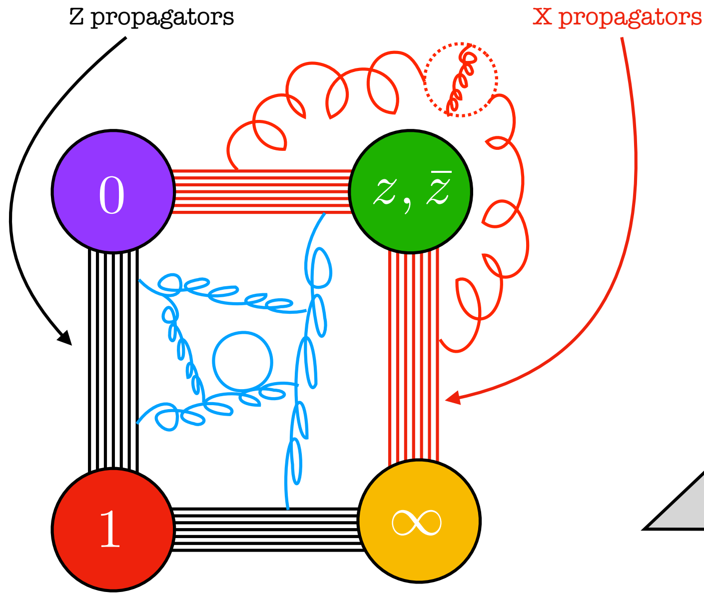
Four point function = quantum folded string:

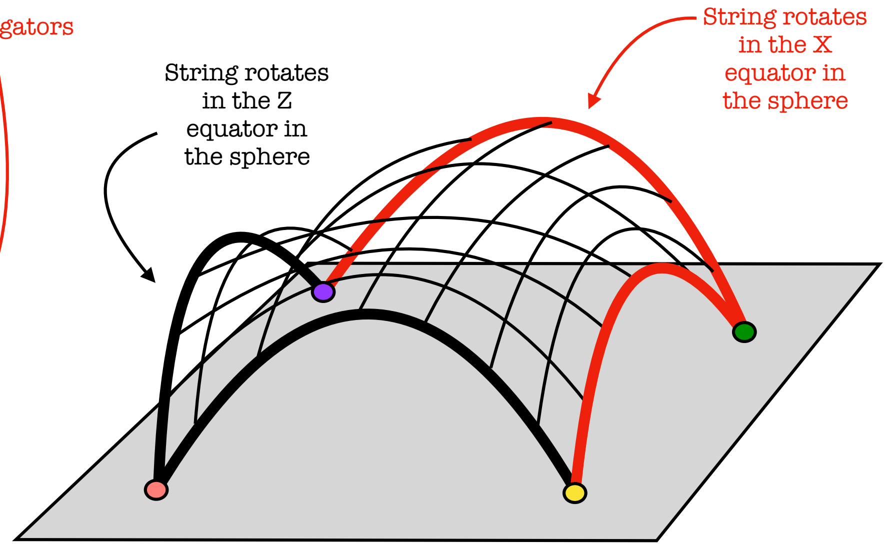




$$\mathbb{O}(z,\bar{z})^2 = \mathbb{O} \times \mathbb{O} = \texttt{inside graphs} \times \texttt{outside graphs}$$

$$= \texttt{top of folded string} \times \texttt{bottom of folded string}$$





$$\mathbb{O}(z,\bar{z})^2 = \mathbb{O} \times \mathbb{O} = \texttt{inside graphs} \times \texttt{outside graphs}$$
$$= \texttt{top of folded string} \times \texttt{bottom of folded string}$$

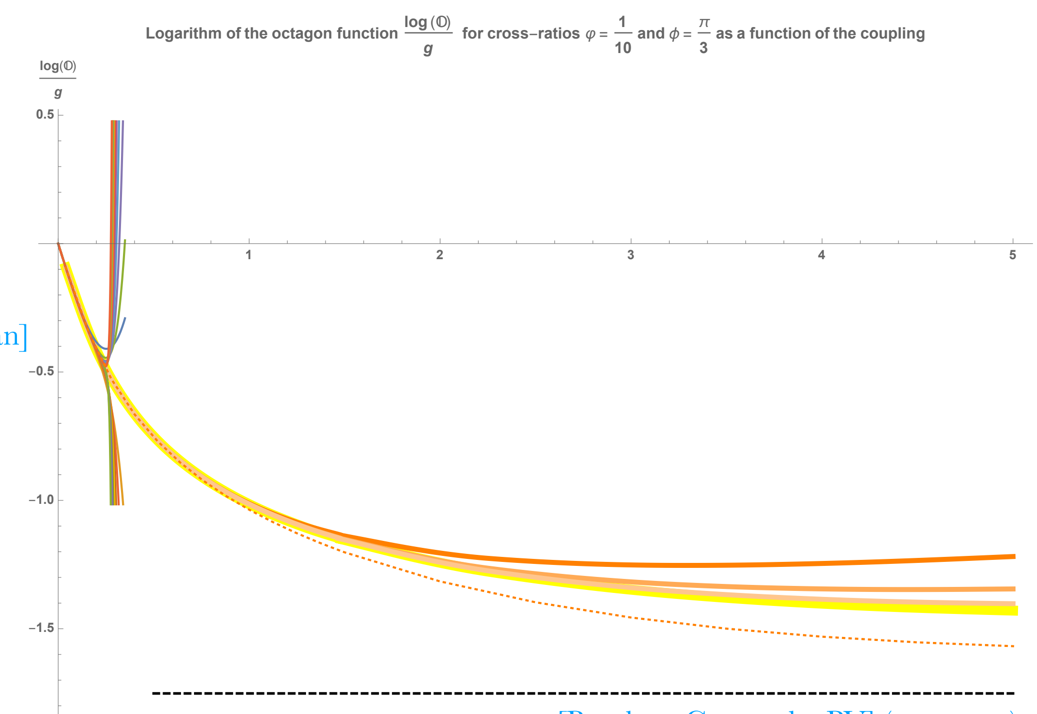
$\mathbb{O}$ is the octagon function. It was computed from *integrability* and even *bootstrapped*.
It is thus known.

[Coronado]

[Coronado], [Kostov,Petkova,Serban]

We don't need its explicit form here.
All we need is for the octagon to exist.

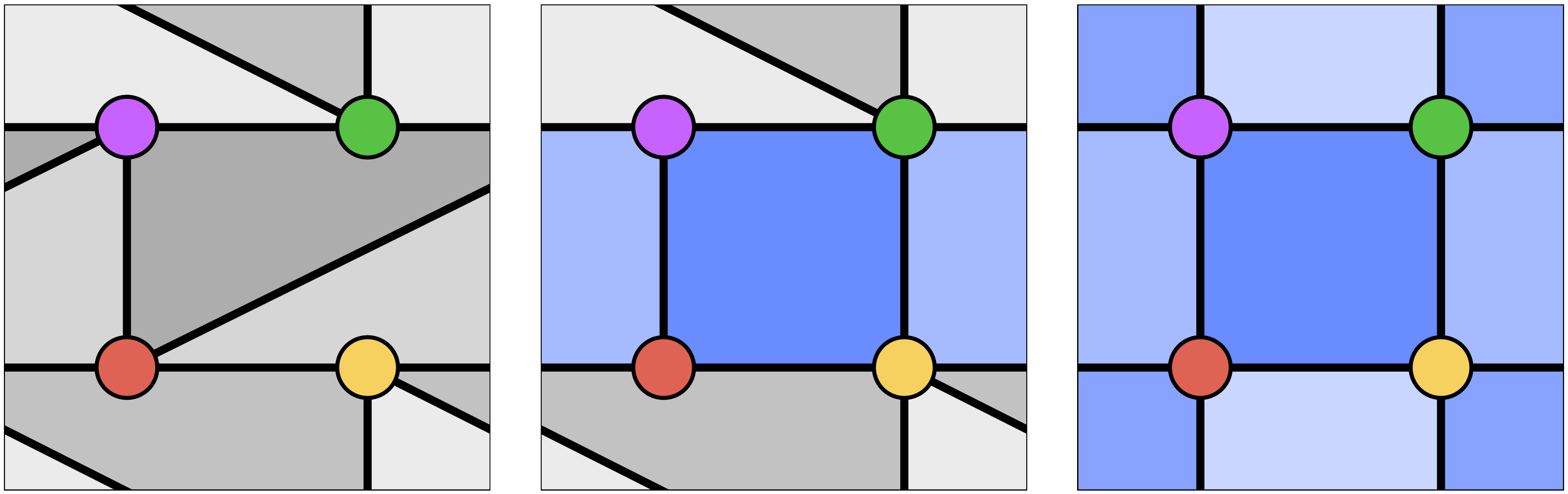
(For other theories such octagon would be different but what comes next would still apply)



[Bargheer,Coronado, PV] (to appear)

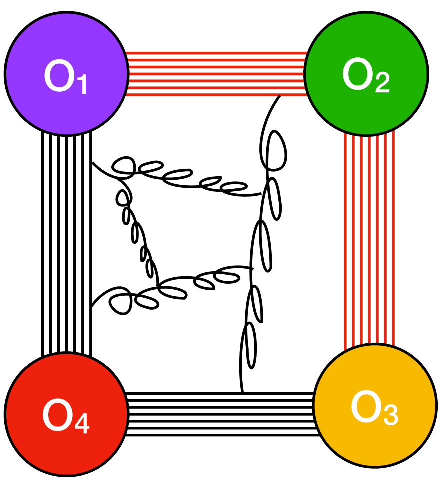
In the planar limit we had a single rectangular frame which we then fill in (and out).

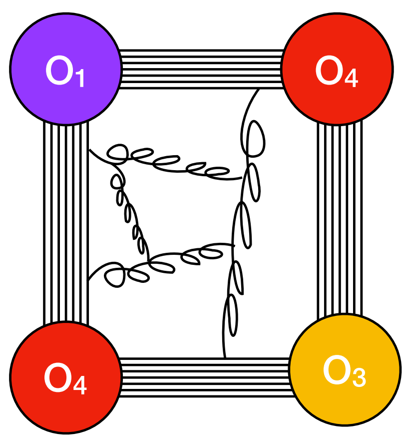
What is the analogue of the rectangular frame at genus one and higher?
There are now more than one option, e.g.:

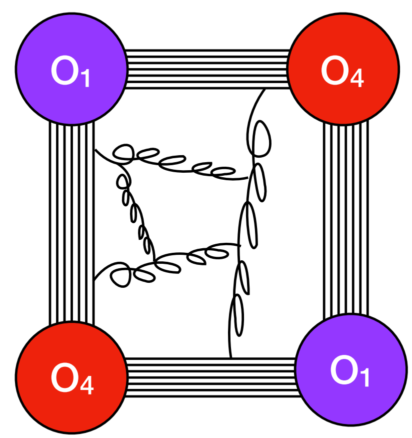


$\mathbb{O}^0$ $\mathbb{O}^2$ $\mathbb{O}^4$

And we will now have one non-BPS octagon and a few BPS octagons:

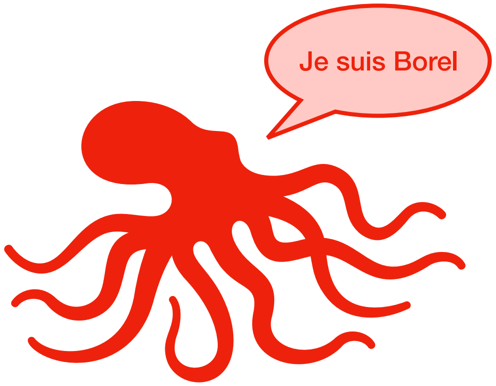
$\sum$ 

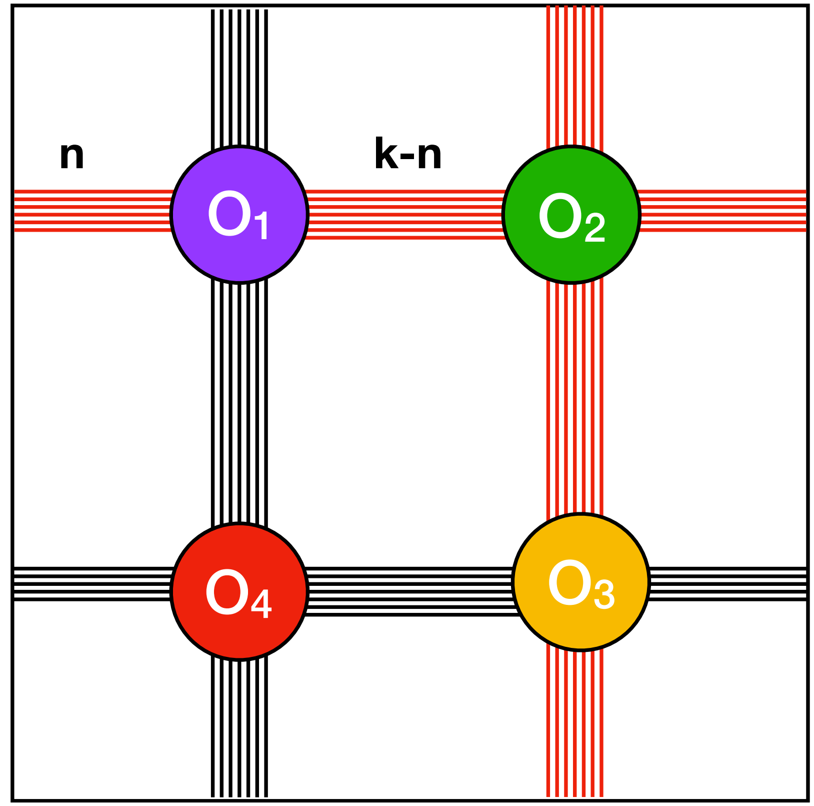
$= \mathbb{O}(z, \bar{z}|\lambda)$, $\sum$ 

$= \sum$ 

$= 1$

2 important points:





$$\sum_n = \underbrace{\sum_{n<n_0} + \sum_{k-n<n_0}}_{O(1)} + \underbrace{\sum_{n,k-n\geq n_0}}_{O(k)}$$

see also [Kristjansen,Plefka,Semenoff,Staudacher], [Constable, Freedman, Headrick, Minwalla, Motl], [Bargheer,Caetano,Fleury,Komatsu,PV]

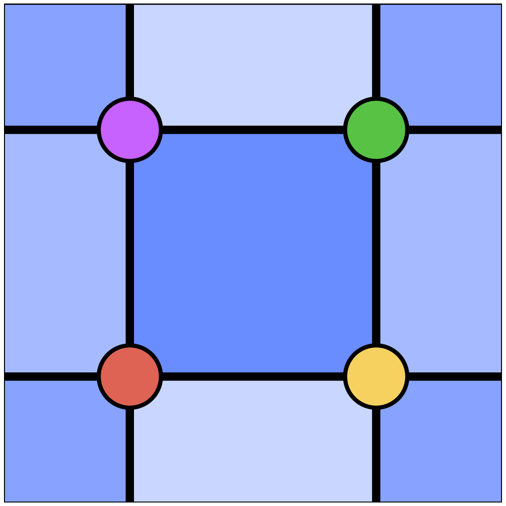
1. Octopus principle. Configurations where some available propagator bundles are not massively occupied are strongly suppressed in the large charge limit. All skeleton graphs are thus maximal graphs, where no further propagator bundle can be added without increasing the genus at hand.

$$\sum_{\substack{k_1,\dots,k_n \\ k_1+\dots+k_n=k}} = \frac{k^{n-1}}{(n-1)!} + \mathcal{O}(k^{n-2}),$$

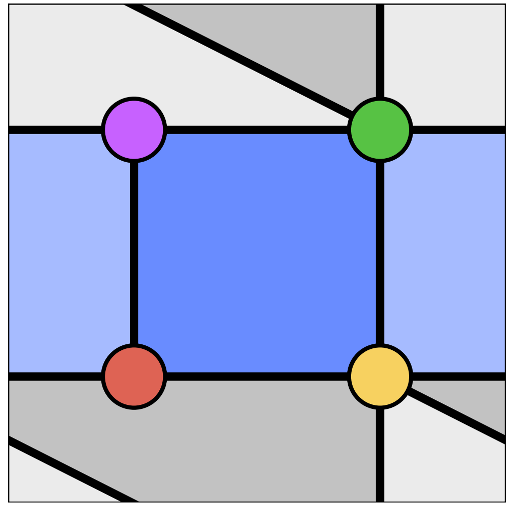
Combinatorial factor for distributing lines in the skeleton graph. Should add it to our counting.

2. Maximal graphs are all *quadrangulations* since 13 and 24 do not connect.

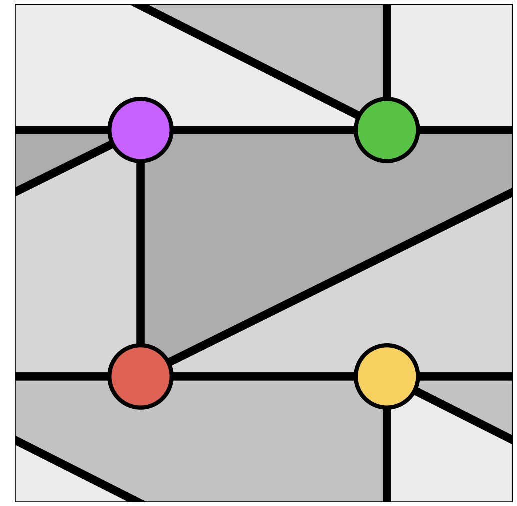
More AdS pictures:



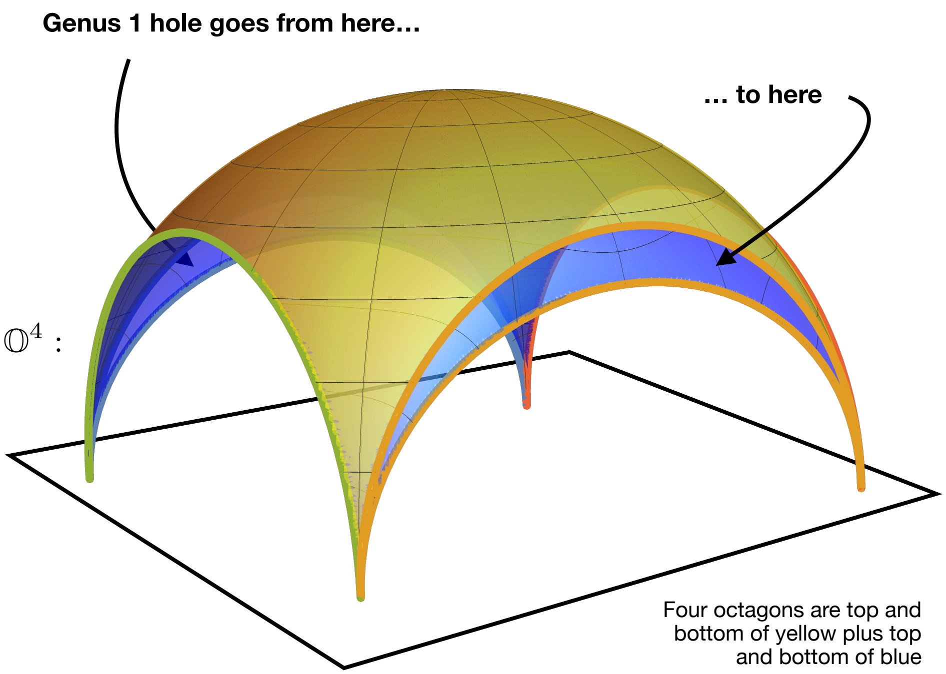
$\mathbb{O}^4$

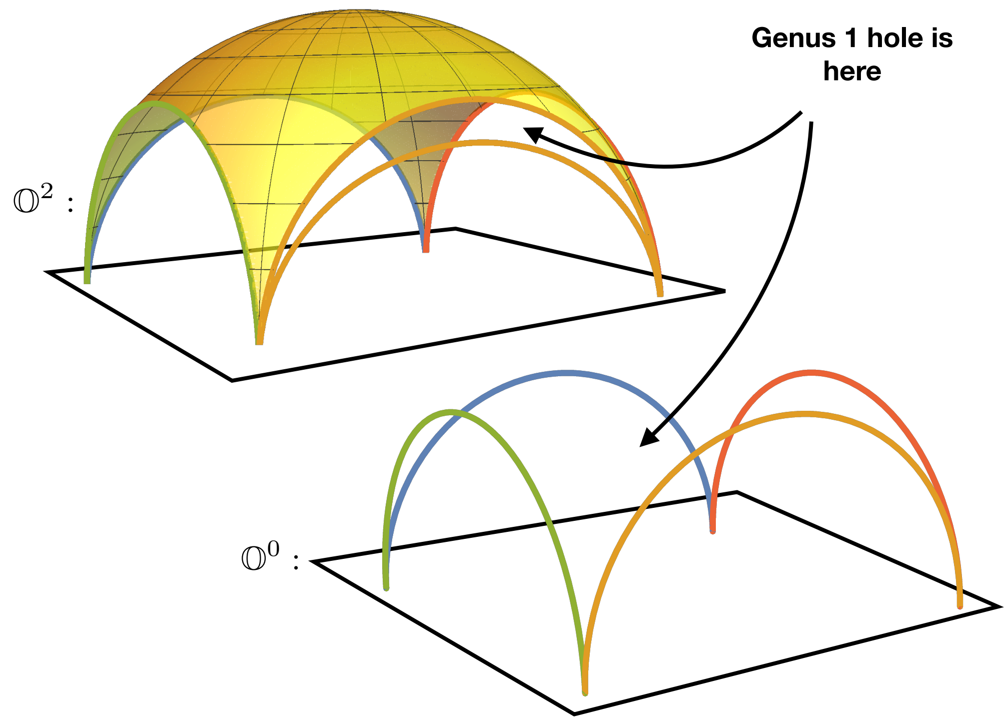


$\mathbb{O}^2$



$\mathbb{O}^0$

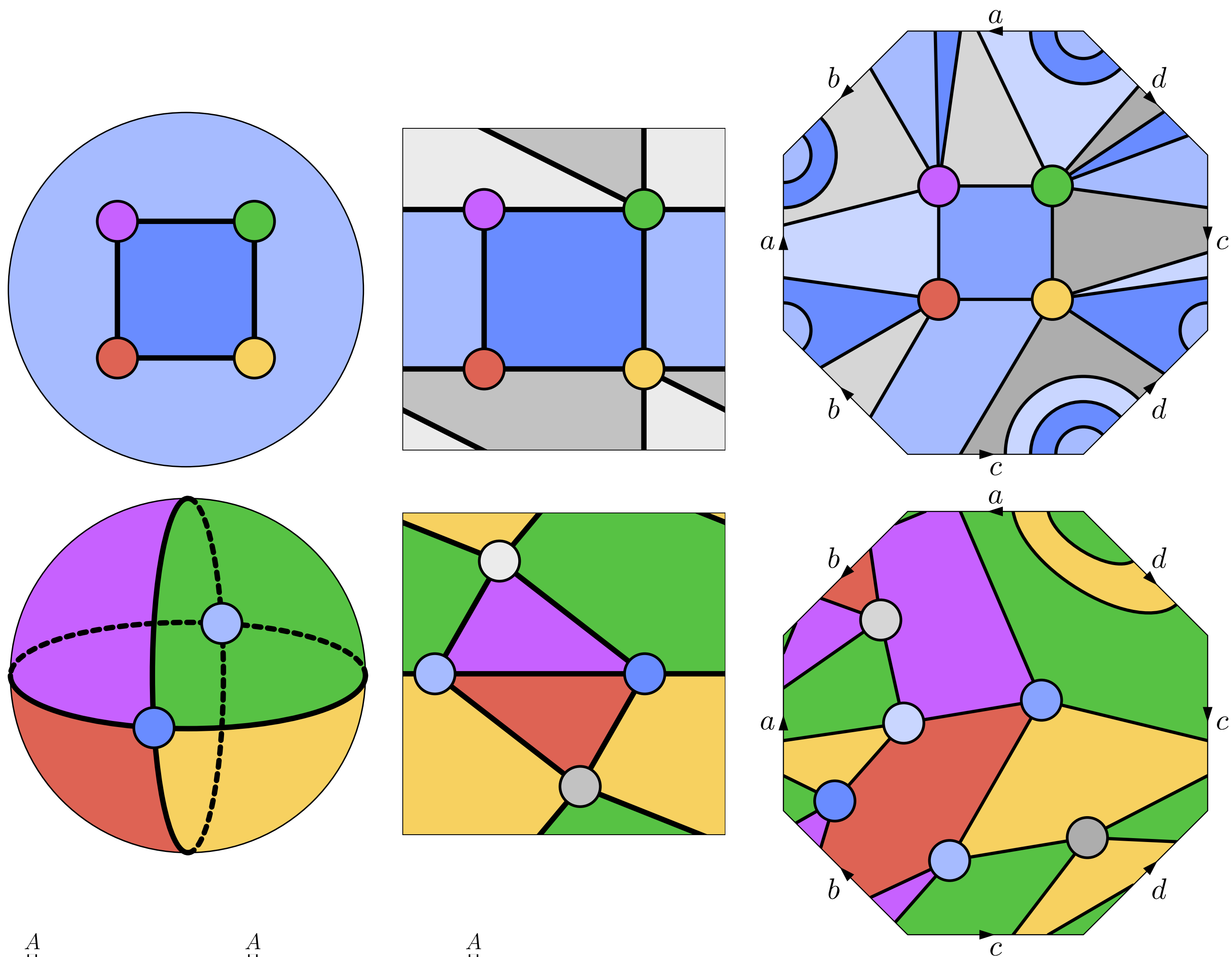




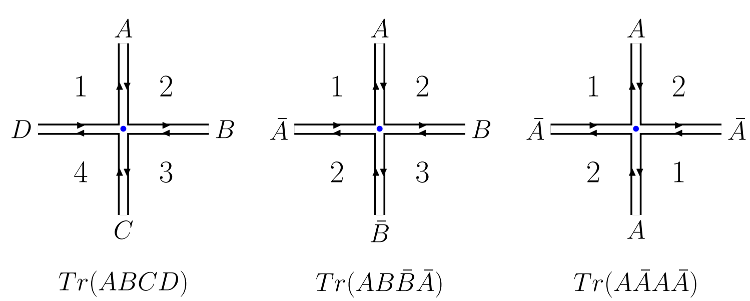
Quadrangulations = Quartic Matrix Model

All faces are squares. There are four vertices of unfixed valency = the four operators in our correlation function.

There are 2g+2 *quartic* vertices and four *faces* only in a ***dual*** graph picture. Four faces = the four operators in our correlation function.



4 matrices for the 4 connections between operators i and i+1, e.g.:



Amusingly, four faces = minimal number of faces = small N limit of sorts four the matrix model

All in all, the partition function of our matrix model is

$$Z \equiv \int [\mathcal{D}A][\mathcal{D}B][\mathcal{D}C][\mathcal{D}D]\exp\Big(-S_{\text{kin}}[A,B,C,D]+S_{\text{int}}[A,B,C,D]\Big)\,,$$

with the kinetic action term

$$S_{\text{kin}} = \operatorname{tr}\left[\frac{A\bar{A}}{k_1}+\frac{B\bar{B}}{k_2}+\frac{C\bar{C}}{k_3}+\frac{D\bar{D}}{k_4}\right]$$

and the interaction term

$$\begin{aligned} S_{\text{int}} &= \mathbb{O}\operatorname{tr}(ABCD)+\mathbb{O}\operatorname{tr}(\bar{D}\bar{C}\bar{B}\bar{A}) \\ &\quad+\operatorname{tr}\left[\frac{(A\bar{A})^2+(B\bar{B})^2+(C\bar{C})^2+(D\bar{D})^2}{2}+AB\bar{B}\bar{A}+BC\bar{C}\bar{B}+CD\bar{D}\bar{C}+DA\bar{A}\bar{D}\right]\,. \end{aligned}$$

We are interested in four faces where all operators are connected at least once:

$$Z = \cdots + N^4 k_1k_2k_3k_4\left(\mathcal{Z}\equiv\sum_{g=0}^{\infty}\tilde{P}_{4g|g+1}(k_1,k_2,k_3,k_4|\mathbb{O}^2)\right)+\ldots$$

Then our four point function is simply $\mathcal{Z}$ *Borel resummed* due to the octopus principle:

$$P_{4g|g+1}(k_1,k_2,k_3,k_4|\mathbb{O}^2)=\tilde{P}_{4g|g+1}(k_1,k_2,k_3,k_4|\mathbb{O}^2)\Big|_{k_1^{n_1}\ldots k_4^{n_4}\to\frac{k_1^{n_1}\ldots k_4^{n_4}}{n_1!\ldots n_4!}}\,.$$

We could also solve this matrix model in some limits. At the end of the day, this is what we know:

$$\frac{\langle \mathrm{tr}(\bar{\boldsymbol{Z}}^k \bar{\boldsymbol{X}}^k)(0)\,\mathrm{tr}(\boldsymbol{X}^{2k})(z)\,\mathrm{tr}(\bar{\boldsymbol{X}}^k \bar{\boldsymbol{Z}}^k)(1)\,\mathrm{tr}(\boldsymbol{Z}^{2k})(\infty)\rangle}{\text{same at } \lambda = 0 \text{ and genus} = 0} \equiv \mathcal{A}(\zeta|\mathbb{O})$$

in the double-scaling limit where N_{c} and k are both very large with $\zeta = k/\sqrt{N_c}$ held fixed.

Up to genus 5:

$$\begin{aligned}\mathcal{A} = \mathbb{O}^2 &+ \zeta^4 \left(1 + \tfrac{9}{2}\mathbb{O}^2 + \tfrac{1}{2}\mathbb{O}^4\right)\\ &+ \zeta^8 \left(\tfrac{3}{2} + \tfrac{607}{80}\mathbb{O}^2 + \tfrac{97}{36}\mathbb{O}^4 + \tfrac{1}{16}\mathbb{O}^6\right)\\ &+ \zeta^{12} \left(\tfrac{81}{80} + \tfrac{7321}{1120}\mathbb{O}^2 + \tfrac{953}{216}\mathbb{O}^4 + \tfrac{5689}{12960}\mathbb{O}^6 + \tfrac{5}{1296}\mathbb{O}^8\right)\\ &+ \zeta^{16} \left(\tfrac{459}{1120} + \tfrac{75553}{22400}\mathbb{O}^2 + \tfrac{44971}{12600}\mathbb{O}^4 + \tfrac{5587171}{7257600}\mathbb{O}^6 + \tfrac{2903}{86400}\mathbb{O}^8 + \tfrac{31}{207360}\mathbb{O}^{10}\right)\\ &+ \mathcal{O}(\zeta^{20})\,.\end{aligned}$$

For very small octagon (very non-perturbative regime): $\mathcal{A} = \left(\dfrac{\sinh(\frac{3}{2}\zeta^2)}{\frac{3}{2}\zeta}\right)^4$

For very large octagon (also very non-perturbative regime):

$$\mathcal{A} = \mathbb{O}^2 \int_0^1 dt \int_0^1 ds \left[\frac{ts}{t+s-1}\, I_0\!\left(2\sqrt{t\,s}\,\zeta^2\mathbb{O}\right) - \frac{(1-t)(1-s)}{t+s-1}\, I_0\!\left(2\sqrt{(1-t)(1-s)}\,\zeta^2\mathbb{O}\right)\right]$$

For $\mathbb{O} \simeq 1$ which is the trivial zero coupling regime, we could not solve the matrix model. Ups.

Note that although A was computed by a matrix model it is perfectly convergent. This is because of the octopus principle factors which end up (more than) Borel re-summing the MM partition function. I have a slide about this if you are curious :)

Would be very nice to fully solve this matrix model.

It is a *four* complex matrix model and we want to extract its *four* face contribution.

It can be simplified into a matrix model with *two* Hermitian matrices and *two* complex matrices where we need to extract its *two* face contribution:

$$\langle F\rangle \equiv \int [\mathcal{D}\mathbb{M}_1][\mathcal{D}\mathbb{M}_2][\mathcal{D}\mathbb{X}][\mathcal{D}\mathbb{Y}]F \exp\left[-\frac{1}{2}\operatorname{tr}\mathbb{M}_1^2 - \frac{1}{2}\operatorname{tr}\mathbb{M}_2^2 - \operatorname{tr}\begin{pmatrix}\mathbb{X} & \mathbb{Y}\end{pmatrix}\begin{pmatrix}1 & \mathbb{O}\\ \mathbb{O} & 1\end{pmatrix}^{-1}\begin{pmatrix}\bar{\mathbb{X}}\\ \bar{\mathbb{Y}}\end{pmatrix}\right].$$

Then we have the rather compact expression

$$\mathcal{Z} = \frac{\left\langle \operatorname{tr}\log\left(\mathbb{I} - \frac{k_2}{\mathbb{I}-k_2\mathbb{M}_2}\bar{\mathbb{X}}\frac{k_1}{\mathbb{I}-k_1\mathbb{M}_1}\mathbb{X}\right)\operatorname{tr}\log\left(\mathbb{I} - \frac{k_3}{\mathbb{I}-k_3\mathbb{M}_2}\bar{\mathbb{Y}}\frac{k_4}{\mathbb{I}-k_4\mathbb{M}_1}\mathbb{Y}\right)\right\rangle_{\text{two faces}}}{k_1k_2k_3k_4}$$

More bravely, would be great to find a full matrix model representation for any correlation function in N=4 SYM at any genus and any coupling, even for the smallest 20' operators introduced in the beginning.

The building blocks should now be cubic vertices since the octagons (i.e. squares) ought to be replaced by hexagons (i.e. triangles) and we will need some further degrees of freedom to capture the mirror particles which glue these hexagons together.

This sounds hard. We should at least start with the large cyclic operator setup, reduce the size of the operators slowly and see how far we could go.

Conclusion

- Coronado's octagon function is a powerful building block.

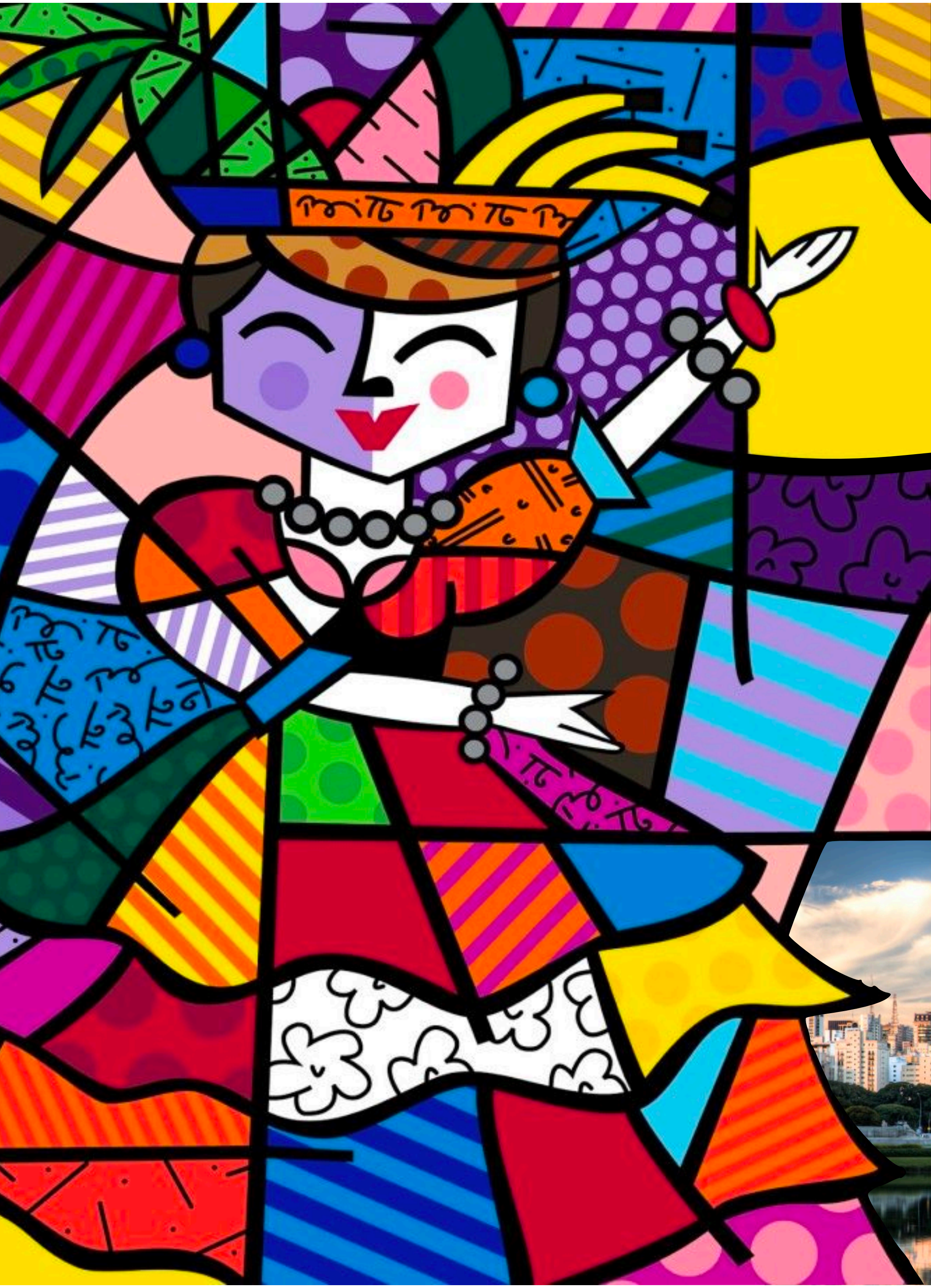
- There is no conceptual obstacle for applying integrability beyond the planar limit: it is the same world-sheet after all.

- Correlation functions of large operators are within reach both at planar and non-planar level. We can probably already probe some interesting bulk locality and other nice kinematical limits.

- Small operators/light strings are harder. They were harder even for the spectrum problem; they were eventually tamed there so we should be optimistic here as well.

  Just last week there was a beautiful paper by Basso and Zhong where three point functions of two large BPS operators and a small non-BPS operator was considered at strong coupling. The technology developed therein could be very useful for higher point functions as well.

- In the end, all these recent (skeleton graphs) + (integrability) ideas can be seen as a realization of an old program by Gopakumar, Razamat,… based on the beautiful relations between ribbon graphs and Riemann surface moduli [Striebel] (upgraded by integrability which was absent at the time).



STRINGS 2021

At the **ICTP-SAIFR** in
Sao Paulo, Brazil:
21 to 26 June
(Monday to Saturday)

Sunday and Monday:
ICTP-SAIFR's
10 year anniversary

Satellite Activities:
String Math 2021
(IMPA, Rio de Janeiro)
String workshop
(IIP, Natal)
String school
(Buenos Aires)

Much more for sure!

Extra Slides

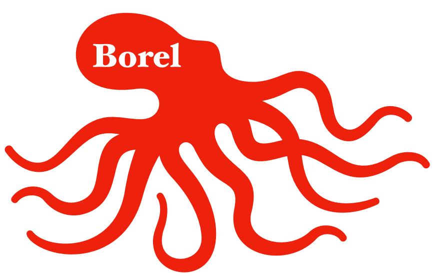
Borel and Large Octagon

For large octagon

$$\mathcal{A} = \frac{1}{\zeta_1\zeta_2\zeta_3\zeta_4} \sum_{g=0}^{\infty} \frac{(\zeta_1\zeta_2\zeta_3\zeta_4\mathbb{O}^2)^{g+1}}{(g+1)^2\, g!^4} \sum_{m=0}^{g} m!^2 (g-m)!^2$$

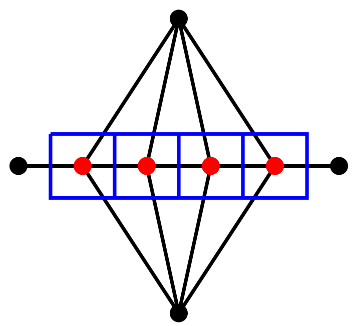
$$\sum_{m=0}^{g} m!^2 (g-m)!^2 \simeq 2(g!)^2 \simeq \sqrt{4\pi g}\, 4^{-g}\, (2g)! \qquad \text{for} \qquad g \gg 1$$

As expected: [Shenker] [Polchinski]



Since multi-strings = multi-folded strings plus multi geodesics, the growth of moduli space is strongly tamed by the large charge limit

- The octagon can be found at all loops as the solution of a bootstrap exercise consisting of 3 remarkable analytic properties:
  1. A basis of (products of) single-valued polylogarithms. Such as the ones in the Ladder Feynman integrals:

$L_4 =$  (1)

  2. A OPE channel dominated by double-trace operators, where the exponent of $log(1-z)$ truncates at all loops.

$$\boxed{\lim_{z,\bar{z}\to 1} \mathbb{O}(z,\bar{z}) = \mathrm{a}(z,\bar{z},\lambda) + \mathrm{b}(z,\bar{z},\lambda)\,\log\left((1-z)(1-\bar{z})\right)} \qquad (2)$$

  3. Null square limit $x_{12}^2, x_{13}^2, x_{24}^2, x_{34}^2 \to 0$. Now the truncation is in the logarithm of $\mathbb{O}$.

$$\boxed{\lim_{z\to 0,\, \bar{z}\to\infty} \log \mathbb{O}(z,\bar{z}) \;=\; \tilde{\Gamma}(\lambda)\log^2(z/\bar{z})} \qquad (3)$$