Bulk Causality from the Conformal Bootstrap

Simon Caron-Huot

McGill University

talk at Strings 2017, Tel Aviv

based on 1703.00278

What is the basic mechanism of AdS/CFT?

[cf. Takayanagi's talk]

Conjecture:

Any CFT with:

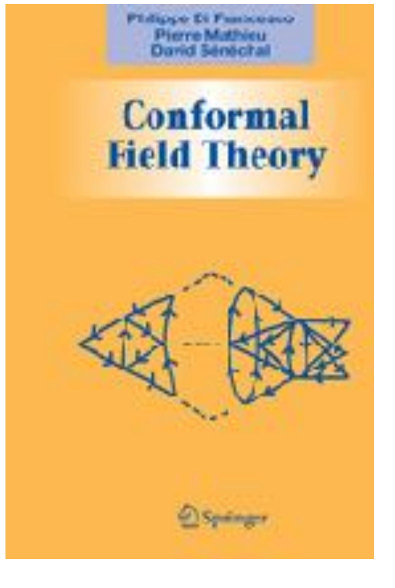
1. Large-N expansion
2. Large gap in operator dimensions

has a bulk dual that is local to lengths $\ell_{\rm AdS}/\Delta_{\rm gap}$.

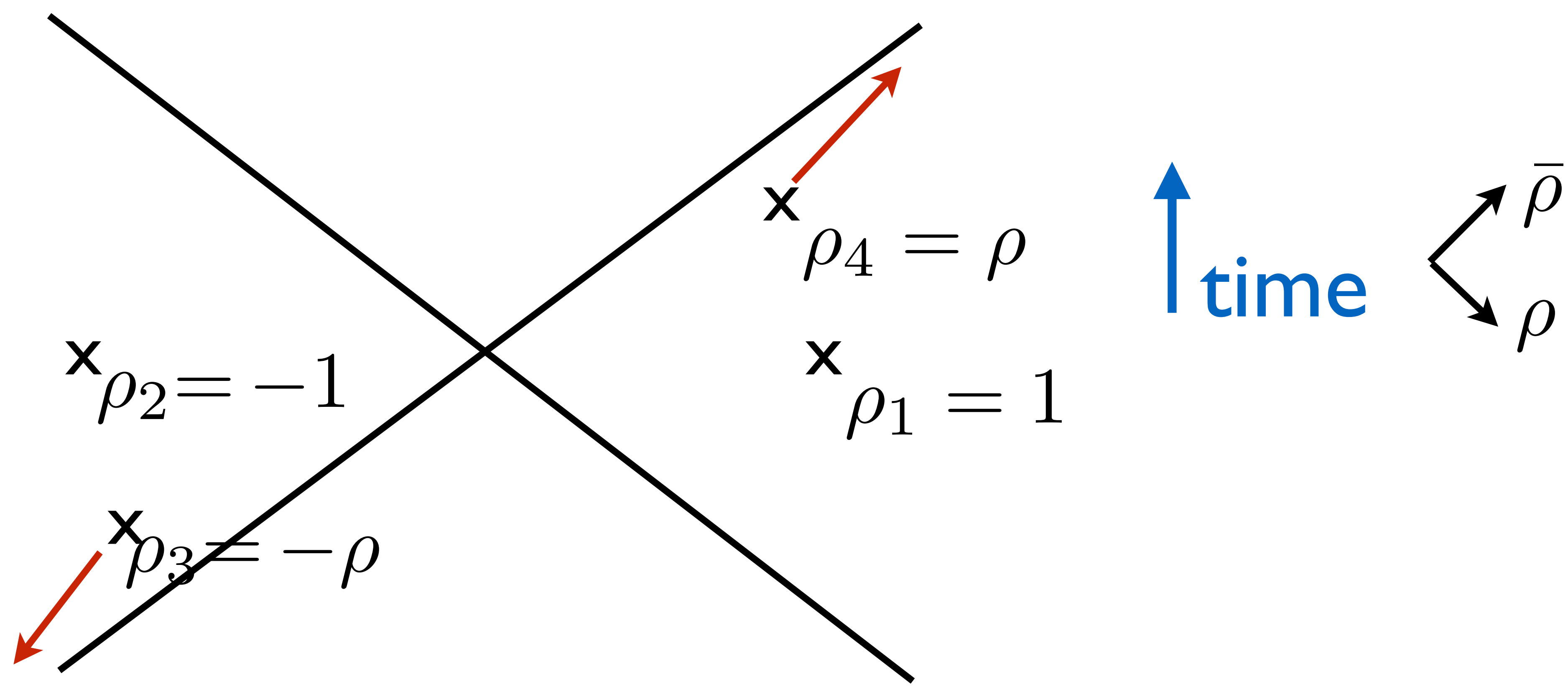
[Heemskerk, Penedones, Polchinski & Sully '09]

CFT 'bootstrap' problem:

Find consistency conditions which ensure this!



We'll study Lorentzian 4-point correlator in CFT$_d$



-stay within two Rindler wedges

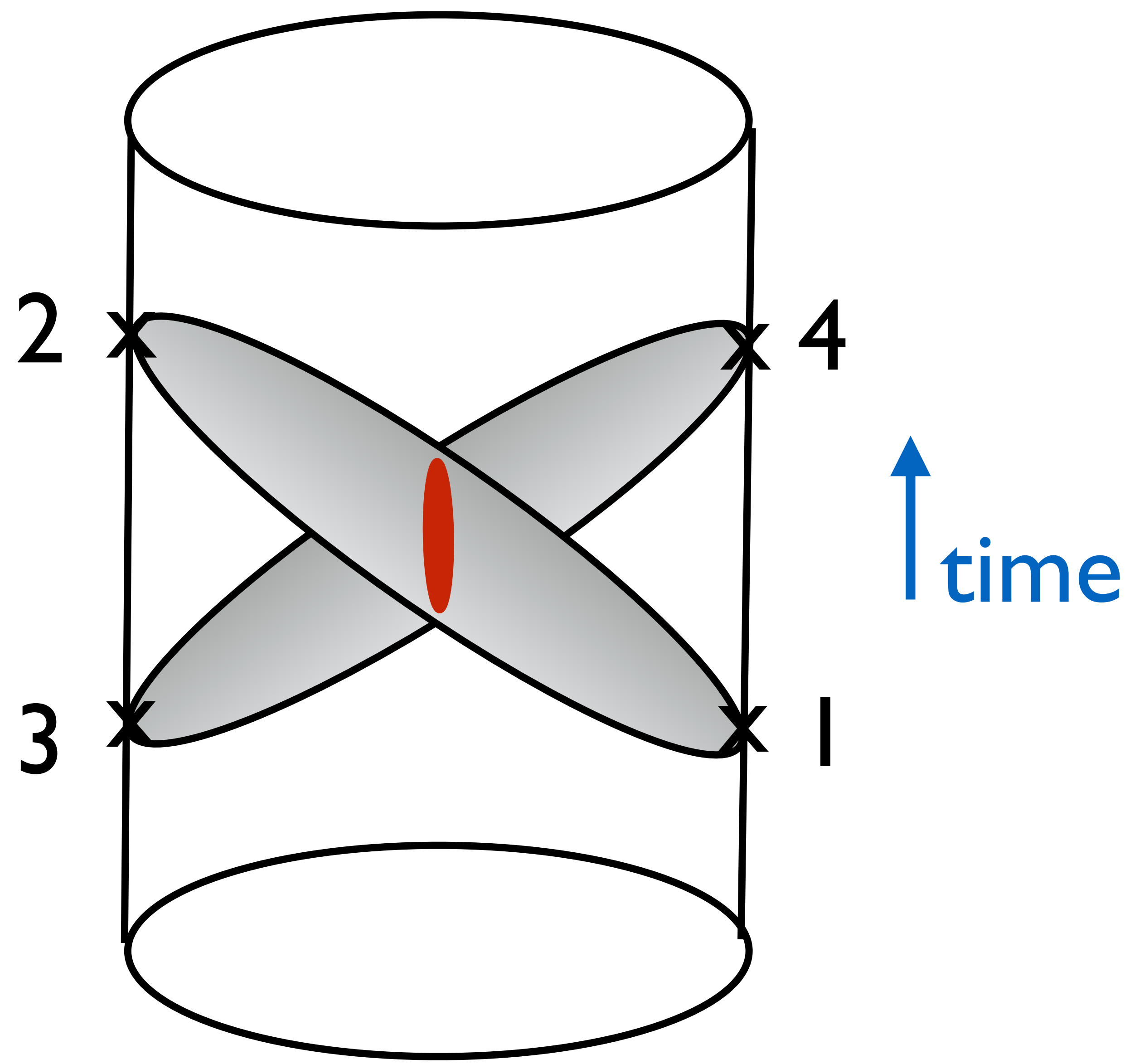
-get constraints from Regge limit: $\bar{\rho} \to \infty, \rho\bar{\rho}$ fixed

[Hogervorst&Rychkov '13]

Regge limit:

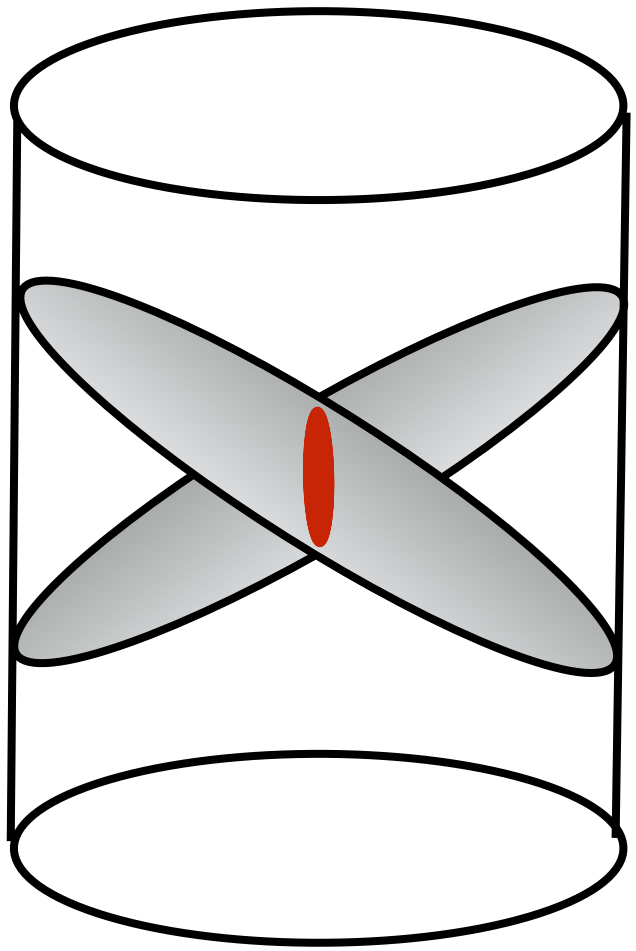
-localizes in time (in two null directions)

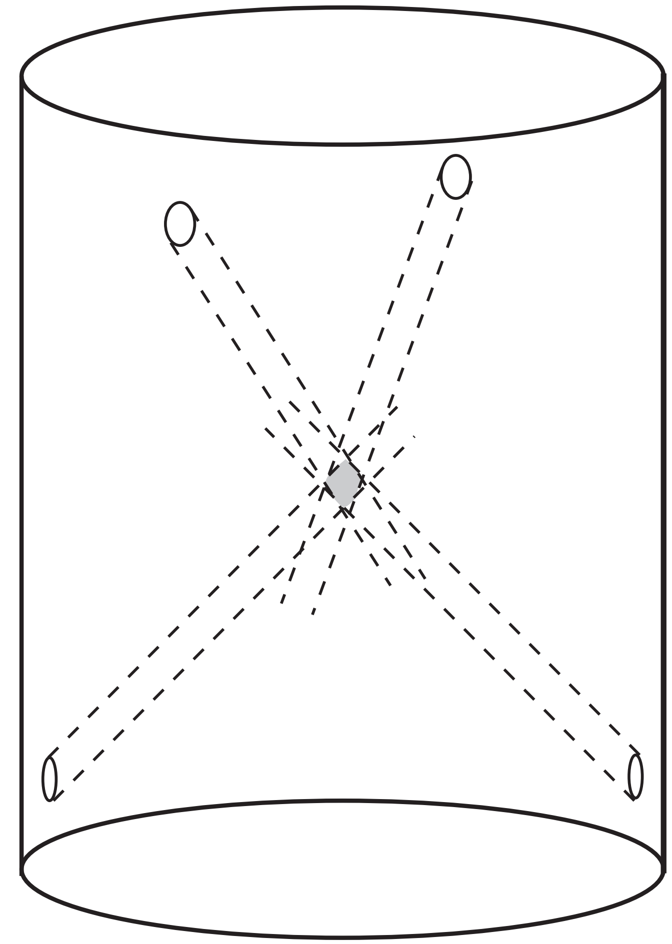
-spreads transversely over AdS_{d-2}



[Cornalba, Costa, Penedones ’06-…]

Contrast with 'focusing things into the bulk'





[fig: Heemskerk,Penedones, Polchinski& Sully]

we won't do the latter since it can't be done from within Rindler wedges (unless $\Delta >> 1$)

Locality in time alone will take us surprisingly far…

Logic:

Locality in time (aka causality) $\Leftrightarrow$ analyticity in complex energies $\Leftrightarrow$ dispersion relations

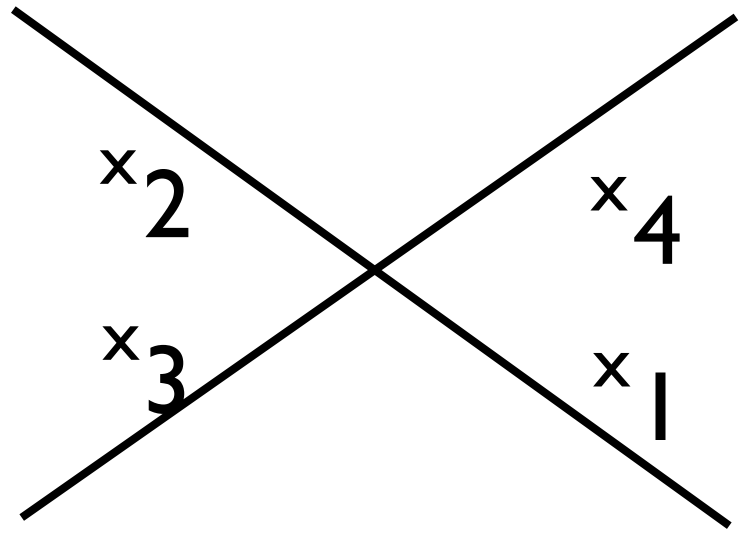
For gapped S-matrices in flat space:

$$\mathcal{M}(s,t) = \int \frac{dt'}{\pi(t-t')} \mathrm{Im}\,\mathcal{M}(s,t') \; + (t \leftrightarrow u)$$

Claim: this generalizes to any unitary CFT:

$$G(\rho', \bar{\rho}') = \int d\rho d\bar{\rho}\, K(\rho', \bar{\rho}'; \rho, \bar{\rho})\, \mathrm{dDisc}\,[G(\rho, \bar{\rho})] + (t \leftrightarrow u)$$

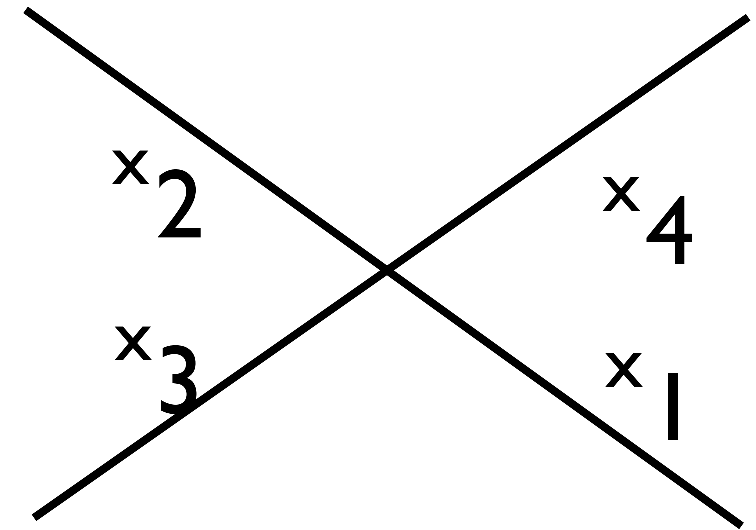
$$\mathrm{dDisc}\, G \equiv \tfrac{1}{2}\langle 0|[\phi_2,\phi_3][\phi_1,\phi_4]|0\rangle$$



Properties:

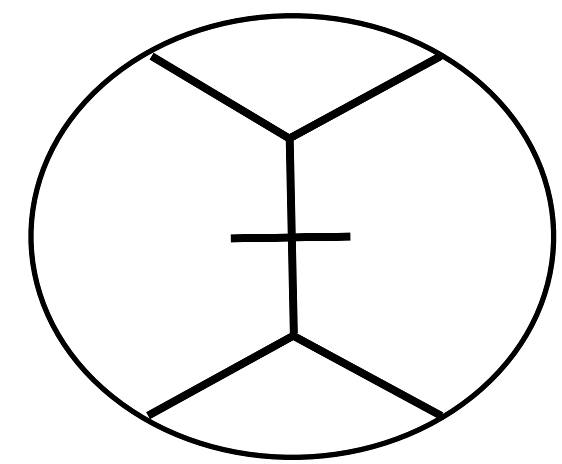
- Positive & bounded
- Saturated by single-traces at large-N

$$\text{dDisc}\, G \equiv \frac{1}{2}\langle 0|[\phi_2, \phi_3][\phi_1, \phi_4]|0\rangle$$



Properties:

- Positive & bounded
- Saturated by single-traces at large-N



Intuition: view correlator as scattering amplitude

$$\langle 0|T\phi_1 \cdots \phi_4|0\rangle \equiv S = G_E + i\mathcal{M}$$

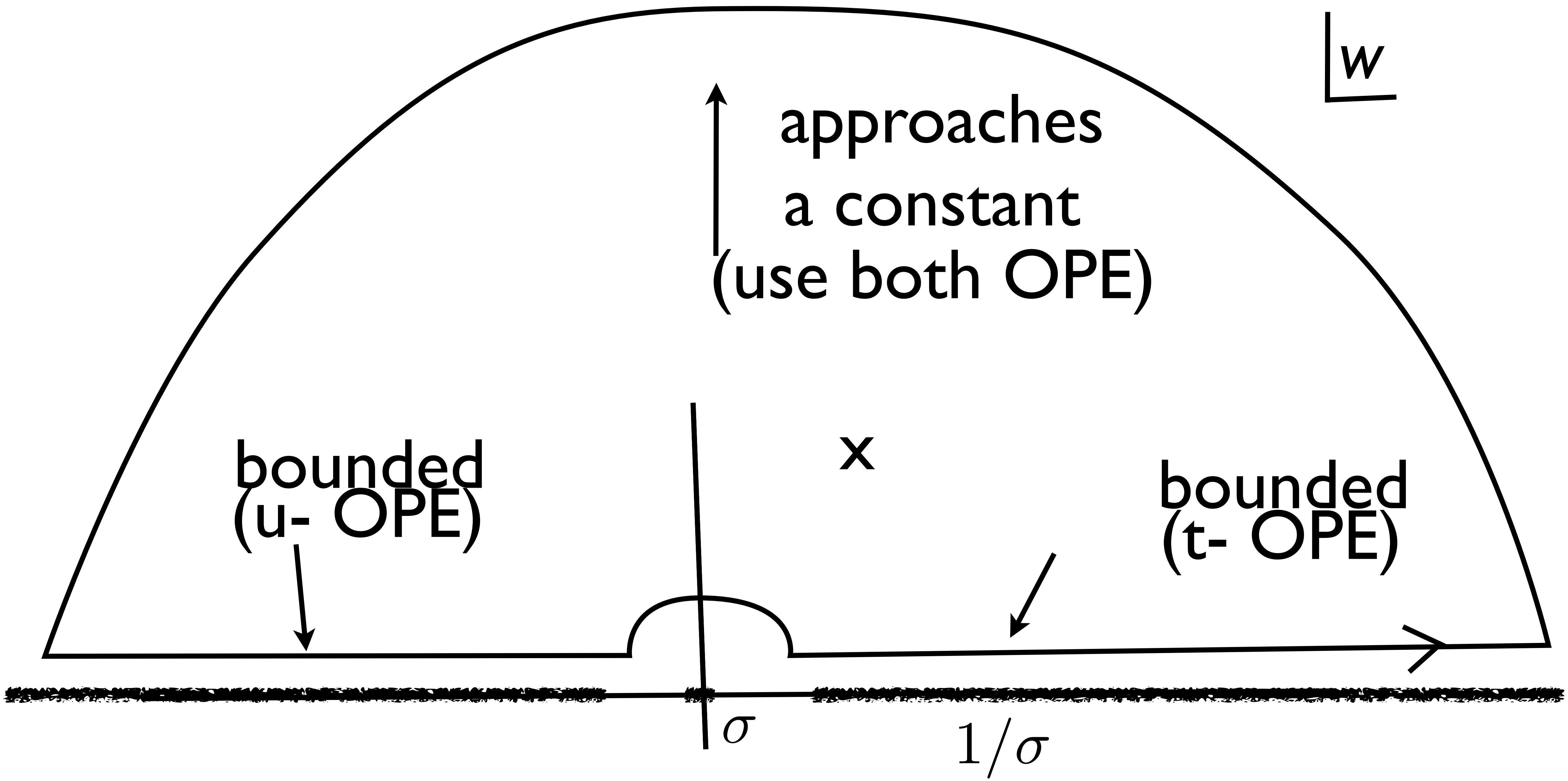
$$\langle 0|\bar{T}\phi_1 \cdots \phi_4|0\rangle \equiv S^* = G_E - i\mathcal{M}^*$$

$$\langle 0|\phi_2\phi_3\phi_1\phi_4|0\rangle \equiv G_E$$

$\Rightarrow$ dDisc G is natural CFT version of Im *M*!!

dispersion relation for *M* at fixed-$\rho\bar{\rho}$:

$$\rho = \sigma w$$
$$\bar{\rho} = \sigma / w$$



$$\mathcal{M}(w) = C + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{dw' \, \mathrm{dDisc}\, G(\sigma, w')}{w - w'} + (\text{small cut})$$

[Hartman,Kundu&Tajdini ’16]

Implications. [Ignore the sign-indefinite small cut in Regge limit]

1. Imaginary part is positive in upper-half-plane

$$\mathrm{Im}\,\mathcal{M}(x+iy) = \int \frac{y\,dx'\,\mathrm{Im}\mathcal{M}(x')}{(x'-x)^2+y^2} > 0$$

⇒ANEC: $\mathcal{M}(w) \approx w\langle \int_{-\infty}^{\infty} dx^+ T_{++}\rangle_{34} \Rightarrow \langle \int dx^+ T_{++}\rangle > 0$

[Hartman,Kundu&Tajdini ’16]

Implications. [Ignore the sign-indefinite small cut in Regge limit]

1. Imaginary part is positive in upper-half-plane

$$\mathrm{Im}\,\mathcal{M}(x+iy) = \int \frac{y\,dx'\,\mathrm{Im}\mathcal{M}(x')}{(x'-x)^2+y^2} > 0$$

⇒ANEC: $\mathcal{M}(w) \approx w\langle \int_{-\infty}^{\infty} dx^+ T_{++}\rangle_{34} \Rightarrow \langle \int dx^+ T_{++}\rangle > 0$

[Hartman,Kundu&Tajdini '16]

2. Rate of growth locally bounded in imaginary direction:

$$(y\partial_y - 1)\Big[\log \mathrm{Im}\mathcal{M}(x+iy)\Big] = -2\frac{\int \frac{dx'\,y^2\,\mathrm{Im}\mathcal{M}(x')}{((x'-x)^2+y^2)^2}}{\int \frac{dx'\,\mathrm{Im}\mathcal{M}(x')}{(x'-x)^2+y^2}} \leq 0$$

⇒chaos bound $\lambda < 2\pi T$ (in Rindler time $w = e^{t/(2\pi T)}$)

[Maldacena,Shenker&Stanford '15]

- Near perfect analogy with ingredients in the proof of the Froissart-Matrin bound*:

$$\mathrm{Im}\,\mathcal{M}(s,t) > 0 \quad \forall \quad 0 \leq t < 4m^2,\ 4m^2 < s < \infty$$

$$\lim_{|s|\to\infty} |\mathcal{M}(s,t)| < C|s|^{1+\epsilon} \quad \epsilon < 1$$

[Martin '63; Jin&Martin '64;
see de Rham, Melville, Tolley &Zhou '17]

- CFT has effective mass gap since 'AdS is like a box' (and OPE makes proofs much easier!)

*impact parameter Im M(s,b) might also provide a good analogy

- Near perfect analogy with ingredients in the proof of the Froissart-Matrin bound:

$$\mathrm{Im}\,\mathcal{M}(s,t) > 0 \quad \forall \quad 0 \leq t < 4m^2, \; 4m^2 < s < \infty$$

$$\lim_{|s|\to\infty} |\mathcal{M}(s,t)| < C|s|^{1+\epsilon} \quad \epsilon < 1$$

[Martin '63; Jin&Martin '64; de Rham, Melville, Tolley &Zhou '17]

- CFT has effective mass gap since 'AdS is like a box' (and OPE makes proofs much easier)

What does this imply for OPE coefficients?

partial waves: $$a_j(s) = \int_{-\pi}^{\pi} d\theta \cos(j\theta) \mathcal{M}(s, t(\cos\theta))$$

+

disp. relation: $$\mathcal{M}(s,t) = \int \frac{dt'}{\pi(t-t')} \operatorname{Im} \mathcal{M}(s,t') + (t \leftrightarrow u)$$

=

analyticity in spin $$a_j(s) = \int_{\eta_0}^{\infty} d\eta \, e^{-j\eta} \mathcal{M}(s, t(\cosh(\eta)) + (-1)^j (t \leftrightarrow u)$$

(for Re j>j_0)

[Froissart-Gribov]

What does this imply for OPE coefficients?

partial waves: $$a_j(s) = \int_{-\pi}^{\pi} d\theta \, \cos(j\theta) \mathcal{M}(s, t(\cos\theta))$$

+

disp. relation: $$\mathcal{M}(s,t) = \int \frac{dt'}{\pi(t-t')} \mathrm{Im}\, \mathcal{M}(s,t') + (t \leftrightarrow u)$$

=

analyticity in spin $$a_j(s) = \int_{\eta_0}^{\infty} d\eta \, e^{-j\eta} \mathcal{M}(s, t(\cosh(\eta))) + (-1)^j (t \leftrightarrow u)$$

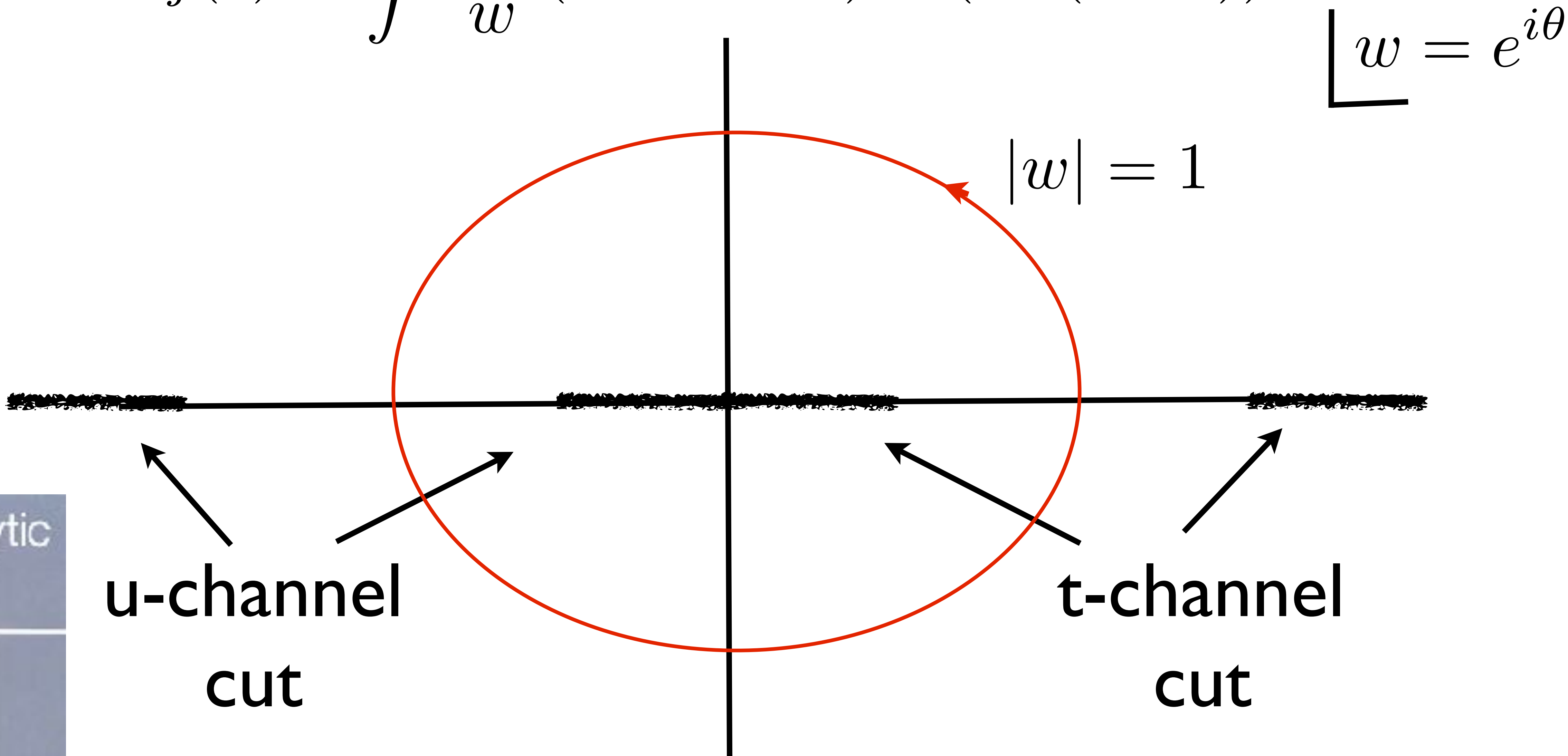
(for Re $j>j_0$)

in CFT, we (currently) lack the second step, away from Regge limit!

[Froissart-Gribov]

‘tactical retreat’: alternative contour integral derivation

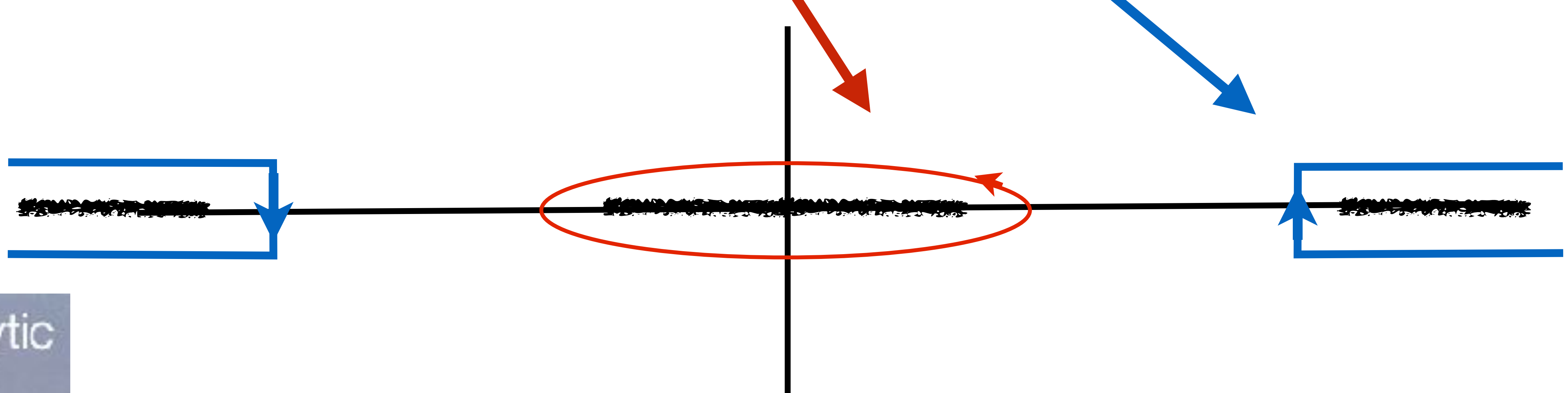
$$a_j(s) = \oint \frac{dw}{w}(w^j + w^{-j})\mathcal{M}(s, t(\cos\theta))$$



Alternative contour integral derivation ('tactical retreat')

$$a_j(s) = \oint \frac{dw}{w}(w^j + w^{-j})\mathcal{M}(s, t(\cos\theta))$$

$w = e^{i\theta}$



⇒get integrals over Im *M*

CFT steps are the same: Euclidean OPE:

$$G(z,\bar{z}) = \sum_{j,\Delta} f_{j,\Delta}^2 G_{j,\Delta}(z,\bar{z})$$

- Actually, we first have to make Δ continuous:

$$G(z,\bar{z}) = \delta_{12}\delta_{34} + \sum_{j=0}^{\infty} \int_{d/2-i\infty}^{d/2+i\infty} \frac{d\Delta}{2\pi i}\, c(j,\Delta)\, F_{j,\Delta}(z,\bar{z}).$$

✓

$$F_{j,\Delta} = g_{j,\Delta} + g_{j,d-\Delta}$$

[Costa,Goncalves&Penedones'12]

[see also: Mazac'16; Hogervorst&van Rees '17, Gadde '17]

= single-valued, needed for self-adjointness of Casimir

[Simmons-Duffin '12]

- Mellin-like contour, encodes OPE through poles:

$$c(j,\Delta') \approx \frac{f^2_{OO\to j,\Delta}}{\Delta - \Delta'}$$

invert OPE using orthogonality for principal series Δ=d/2+iν

$$c(j,\Delta) = N(j,\Delta)\int_0^1 d(\rho\bar{\rho}) \oint \frac{dw}{w} \mu(\rho,\bar{\rho})\, G(\rho,\bar{\rho})\, F_{j,\Delta}(\rho,\bar{\rho})$$

$$\rho = \sigma w, \bar{\rho} = \sigma/w$$

Tricky part is to find the analog of

$$2\cos(j\theta) = w^j + w^{-j}$$

so we can split the 'block+shadow':

$$F_{j,\Delta}(z,\bar{z}) = F^{(+)}_{j,\Delta} + F^{(-)}_{j,\Delta}$$

$\sim w^j$ $(w \to 0)$

$\sim w^{-j}$ $(w \to \infty)$

- tricky because there are 8 basic solutions to conformal Casimirs diff eqs.: (quadratic and quartic)

$$g^{\text{pure}}_{j,\Delta}(z,\bar{z}) \sim z^{\frac{\Delta-j}{2}}\,\bar{z}^{\frac{\Delta+j}{2}} \qquad (0 \ll z \ll \bar{z} \ll 1)$$

- Solutions related by symmetries:

$$j \longleftrightarrow 2-d-j, \qquad \Delta \longleftrightarrow d-\Delta, \qquad \Delta \longleftrightarrow 1-j\,.$$

- Only 2 are nice (convergent) in Regge limit:

$$g^{\text{pure}}_{\Delta+1-d,j+d-1}, \quad g^{\text{pure}}_{1-\Delta,j+d-1} \sim (z\bar{z})^{j/2}$$

- So we have 4 parameters and 8 constraints

- tricky because there are 8 basic solutions to conformal Casimirs diff eqs.: (quadratic and quartic)

$$g^{\text{pure}}_{j,\Delta}(z,\bar{z}) \sim z^{\frac{\Delta-j}{2}}\,\bar{z}^{\frac{\Delta+j}{2}} \qquad (0 \ll z \ll \bar{z} \ll 1)$$

- Solutions related by symmetries:

$$j \longleftrightarrow 2-d-j, \qquad \Delta \longleftrightarrow d-\Delta, \qquad \Delta \longleftrightarrow 1-j\,.$$

- Only 2 are nice (convergent) in Regge limit:

$$g^{\text{pure}}_{\Delta+1-d,j+d-1}, \quad g^{\text{pure}}_{1-\Delta,j+d-1} \sim (z\bar{z})^{j/2}$$

- So we have 4 parameters and 8 constraints

one solution (!!!!)

Result: CFT Froissart-Gribov formula

$$c(J,\Delta) = \int_{\diamond} [\text{Inverse block}] \times [\text{dDisc}\, G]$$

s-channel
OPE coefficients

block with
j and Δ
exchanged

convergent
t-channel sum

converges for j>1

(proved using unitarity&OPE
convergence in cross-channels)

A (boring) test: 2D Ising

$$G(\rho,\bar{\rho}) = \left|\frac{1}{(1-\rho^2)^{1/4}}\right|^2 + \left|\frac{\sqrt{\rho}}{(1-\rho^2)^{1/4}}\right|^2$$

- Double discontinuity:

$$\frac{1-\frac{1}{\sqrt{2}}(\sqrt{\rho}+\sqrt{\bar{\rho}})+\sqrt{\rho\bar{\rho}}}{(1-\rho^2)^{1/4}(1-\bar{\rho}^2)^{1/4}} > 0$$

- Factorized integral against 2d (global) blocks

$$c_{j,\Delta} = f_0(j{+}\Delta)f_0(j{+}2{-}\Delta) - \tfrac{1}{2}f_{1/4}(j+\Delta)f_0(j+2-\Delta) + \ldots$$

$$f_p(\alpha) = 2^{a-3+2p}\frac{\Gamma(\frac{7}{4})\Gamma(p+\frac{\alpha-2}{4})}{\Gamma(p+\frac{\alpha+5}{4})}\,{}_3F_2\big(\tfrac{1}{2},\tfrac{\alpha}{2},p+\tfrac{\alpha-2}{4};\tfrac{a+1}{2},p+\tfrac{\alpha+5}{4};1\big)\,. \qquad \text{(B.6)}$$

- Residues at all poles do match global OPE!*

✓

$$C_{j,\Delta} = -K_{j,\Delta}\mathrm{Res}_{\Delta'=\Delta}c(j,\Delta')$$

$$C_{0,1} = \frac{1}{4}, \quad C_{2,2} = \frac{1}{64}, \quad C_{4,4} = \frac{9}{40960}, \quad C_{0,4} = \frac{1}{4096}$$

$$C_{4,5} = \frac{1}{65536}, \quad C_{6,6} = \frac{35}{3670016} \quad C_{2,6} = \frac{9}{2621440}, \quad C_{6,7} = \frac{1}{1310720}, \ldots$$

*never trust Mathematica's Residue on 3F2's……..

Application: Large spin bootstrap

Usual story: double-light-cone limit $(z, \bar{z}) \to (0, 1)$

non-analytic behaviour in $(1 - \bar{z})$ needs large spin:

$$\sum_j \frac{1}{j^\alpha} F_j(\bar{z}) = (1 - \bar{z})^{\alpha/2} + \text{regular}$$

large spin in s-channel $\leftrightarrow$ low twist in t-channel

$\Rightarrow$ Solve OPE in asymptotic series in 1/j

[Komargodski&Zhiboedov,
Fitzpatrick,Kaplan,Poland&Simmons-Duffin,
Alday&Bissi&…,
Kaviraj,Sen,Sinha&…,
Alday,Bissi,Perlmutter&Aharony,…]

What about inversion formula?

$$c(j,\Delta) \sim \int_0^1 dz d\bar{z}\, z^{j-\Delta} \bar{z}^{j+\Delta} F_{j+\Delta}(\bar{z}) \mathrm{dDisc} G(z,\bar{z})$$

OPE data encoded in Δ-poles from G~$z^{\tau/2}$ as $z \to 0$:

$$c(j,\Delta) = \frac{1}{j-\Delta-\tau} \times \int_0^1 d\bar{z}\, \bar{z}^{j+\Delta} F_{j+\Delta}(\bar{z}) \mathrm{dDisc} G_\tau(\bar{z})$$

⇒large j+Δ pushes integral to (0,1) corner ✓

- Analytic result for t-channel power-law:

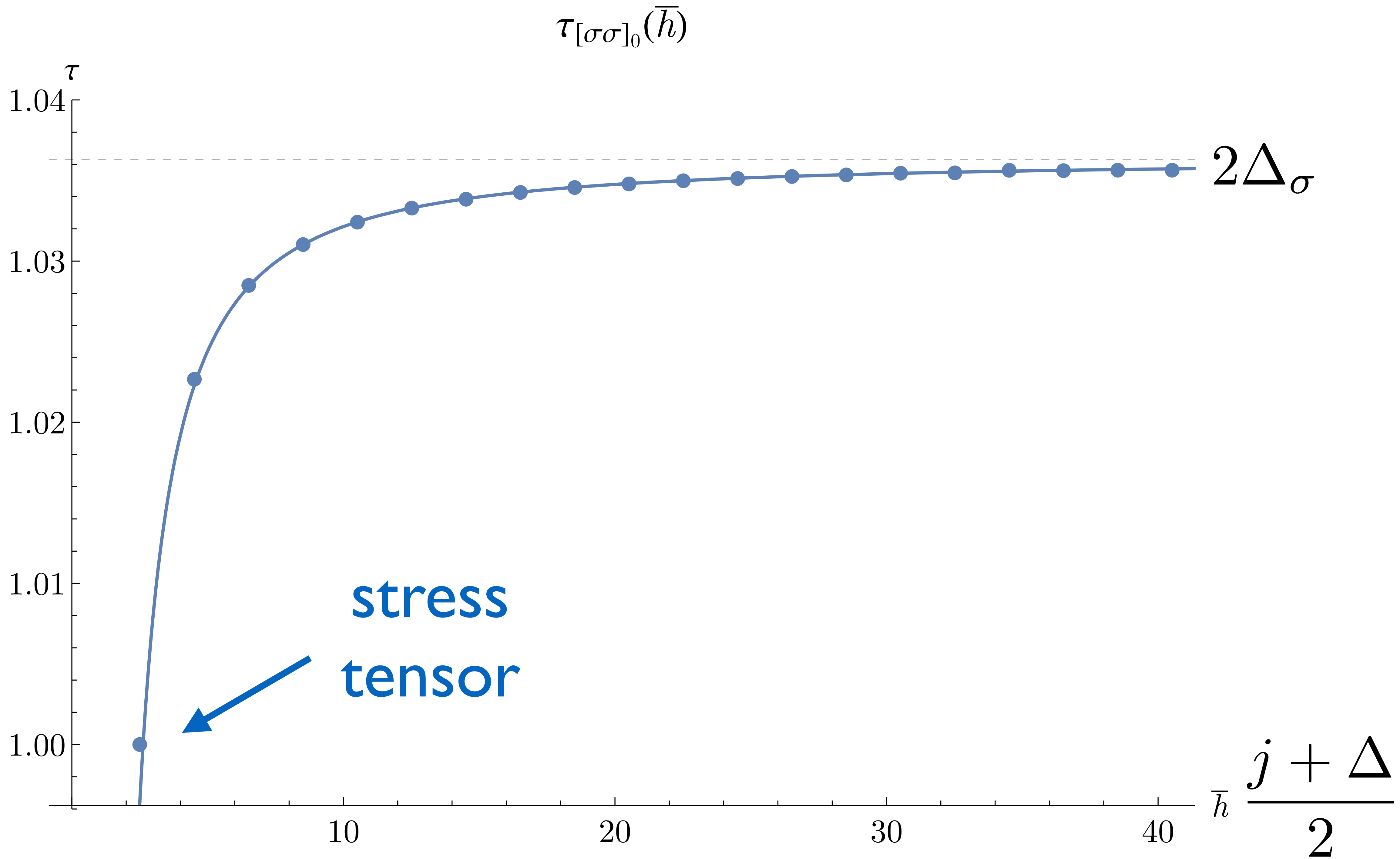
$$I^{a,b}_{\tau'}(\bar{h}) \equiv \int_0^1 \frac{d\bar{z}}{\bar{z}^2}(1-\bar{z})^{a+b}\kappa_{\bar{h}} k_{\bar{h}}(\bar{z})\,\mathrm{dDisc}\left[\left(\frac{1-\bar{z}}{\bar{z}}\right)^{\frac{\tau'}{2}-b}(\bar{z})^{-b}\right] \tag{4.7}$$

$$= \frac{1}{\Gamma(-\frac{\tau'}{2}-a)\Gamma(-\frac{\tau'}{2}+b)} \times \frac{\Gamma(\bar{h}-a)\Gamma(\bar{h}+b)}{\Gamma(2\bar{h}-1)} \times \frac{\Gamma(\bar{h}-\frac{\tau'}{2}-1)}{\Gamma(\bar{h}+\frac{\tau'}{2}+1)}.$$

$$\sim 1/\bar{h}^{\tau'} \qquad (\bar{h} = \tfrac{j+\Delta}{2})$$

- Earlier results reproduced by: 'expand cross-channel OPE in $\frac{1-\bar{z}}{\bar{z}}$ and integrate termwise using (4.7)' ✓

Asymptotic series in 3D Ising



[Plot from Simmons-Duffin '16; see Alday&Zhiboedov '15]

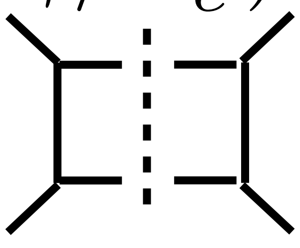
What's new:

- Replaced 1/J expansion by convergent sum (no need to expand in $(1-\bar{z})/\bar{z}$)
- Control over individual spins, not only averages over many spins ('no stick-out')
- Can try to bound errors?

Bulk Locality

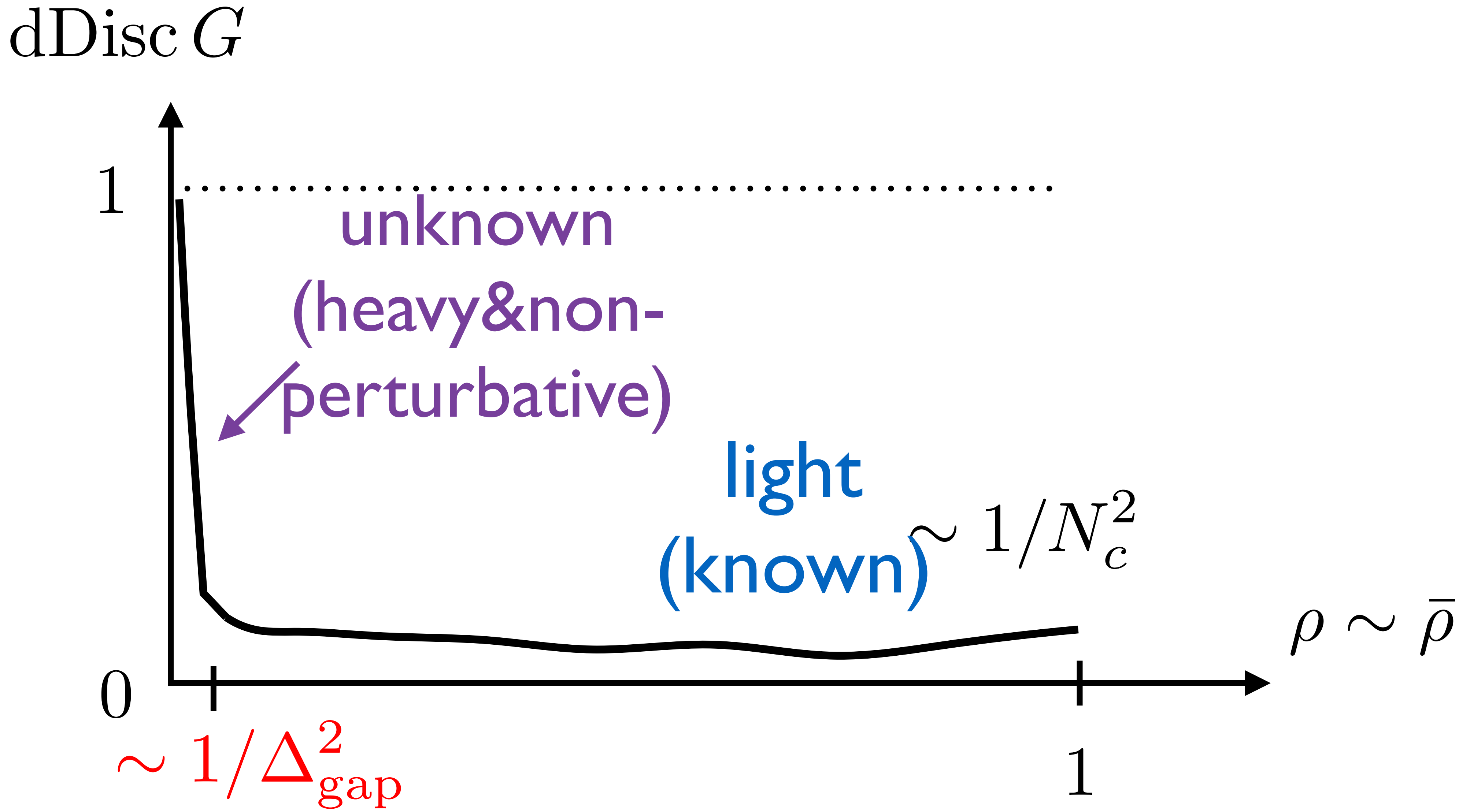
$$\mathrm{dDisc}\, G = \sum_{J',\Delta'} \sin^2\!\left(\tfrac{\pi}{2}(\Delta' - 2\Delta)\right) \left(\frac{1-\sqrt{\rho}}{1+\sqrt{\rho}}\right)^{\Delta'+J'} \left(\frac{1-\sqrt{\bar{\rho}}}{1+\sqrt{\bar{\rho}}}\right)^{\Delta'-J'}$$

- Double-traces killed at large N_c $(\Delta' = 2\Delta + 2n + \gamma/N_c^2)$



- Heavy operators killed unless $\rho, \bar{\rho} < \Delta_{\mathrm{gap}}^2$

Theories with classical AdS dual	=	CFTs where dDisc G is saturated by a handful of light primaries



$$c_{j,\Delta} = \int F_{j,\Delta}\, \mathrm{dDisc}\, G \quad = c_{j,\Delta}\Big|_{\text{light}} + c_{j,\Delta}\Big|_{\text{heavy}}$$

'minimal solution'

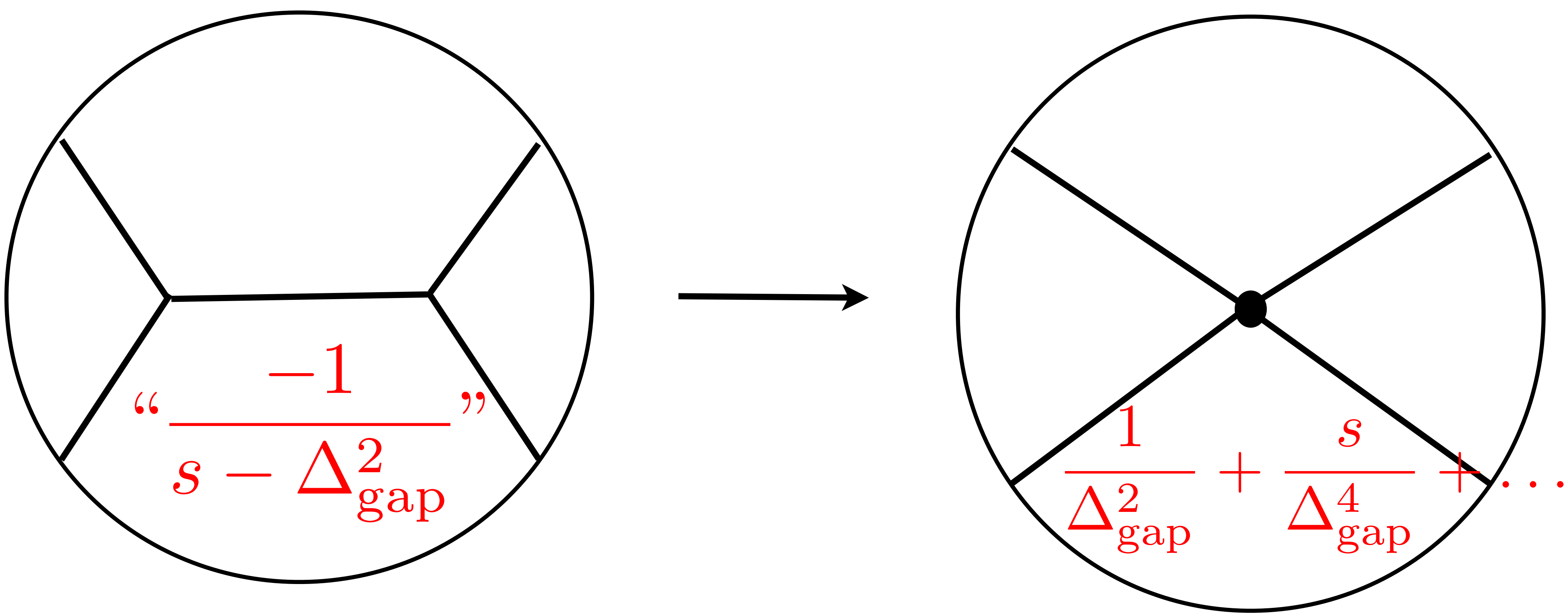
correction small for j>2

[see also: Alday, Bissi & Perlmutter; Li, Meltzer & Poland]

Since F$\sim (\rho\bar{\rho})^{J/2}$, heavy contribution decays with spin

Area set by stress-tensor two-point function

$$\left|c(j, \tfrac{d}{2} + i\nu)_{\text{heavy}}\right| \leq \frac{1}{c_T} \frac{\#}{(\Delta_{\text{gap}}^2)^{j-2}}$$



(Spin versus dimension)

- Consider an AdS interaction with flat-space limit:

$$stu$$

- This has spin two in the Regge limit in all channels:

$$stu = st(s+t) \sim s^2 \equiv s^j \qquad (s \to \infty, t \text{ fixed})$$

- Not constrained (Regge limit only localizes in time!), but due to crossing symmetry, any interaction with more derivatives will have a bounded coefficient
- For TT$\phi\phi$, only one unconstrained spin-2 contact interaction. For TTTT, none!

Summary

- Dispersion relation for OPE coefficients:

$$c(j,\Delta) \equiv \int_0^1 d\rho d\bar{\rho}\, g_{\Delta,j}\, \mathrm{dDisc}\, G$$

s-channel cross-channels

- Input: CFT versions of analyticity&positivity as used in Froissart-Martin's theorem, consequence here of the OPE in any unitary CFT_D.

- Output:
  -large-spin expansions with controllable errors
  -AdS/CFT correlators using only ~few light fields
  -bounds on higher-derivative terms → bulk locality