S-matrix Bootstrap revisited

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S-matrix Bootstrap I: QFT in AdS [arXiv:1607.06109]
S-matrix Bootstrap II: two-dimensional amplitudes [arXiv:1607.06110]
S-matrix Bootstrap III: higher dimensional amplitudes [to appear]

with M. Paulos, J. Toledo, B. Van Rees, P. Vieira

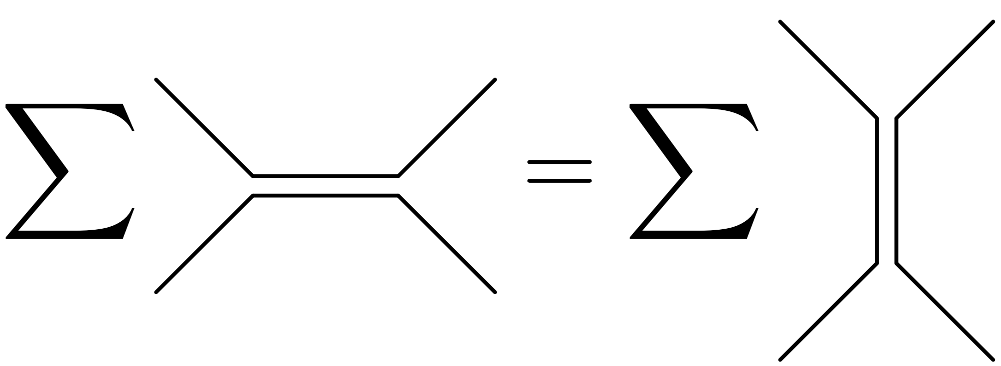
Strings 2017, Tel Aviv, 29th of June, 2017

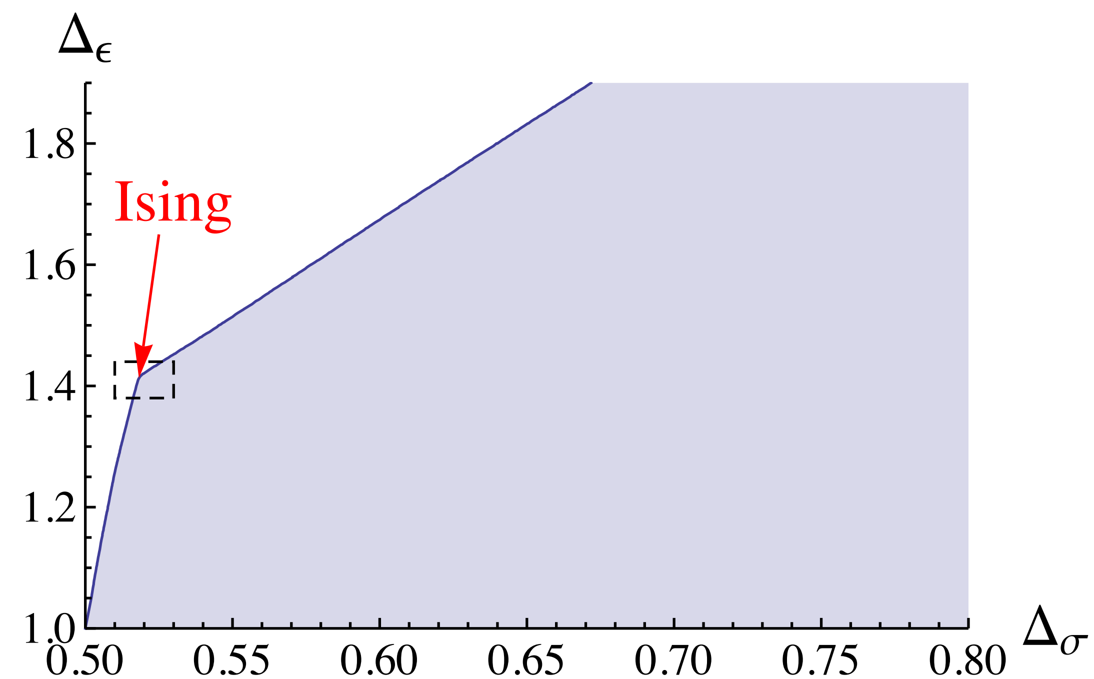
Motivation

Bootstrap Philosophy: bound the space of theories by imposing consistency conditions on physical observables.

Goal: extend recent success in CFT to massive QFT.

[Rattazzi, Rychkov, Tonni, Vichi '08] + many others





from [El-Showk et al '12]

Revisit the S-matrix Bootstrap program of the 60's and 70's.

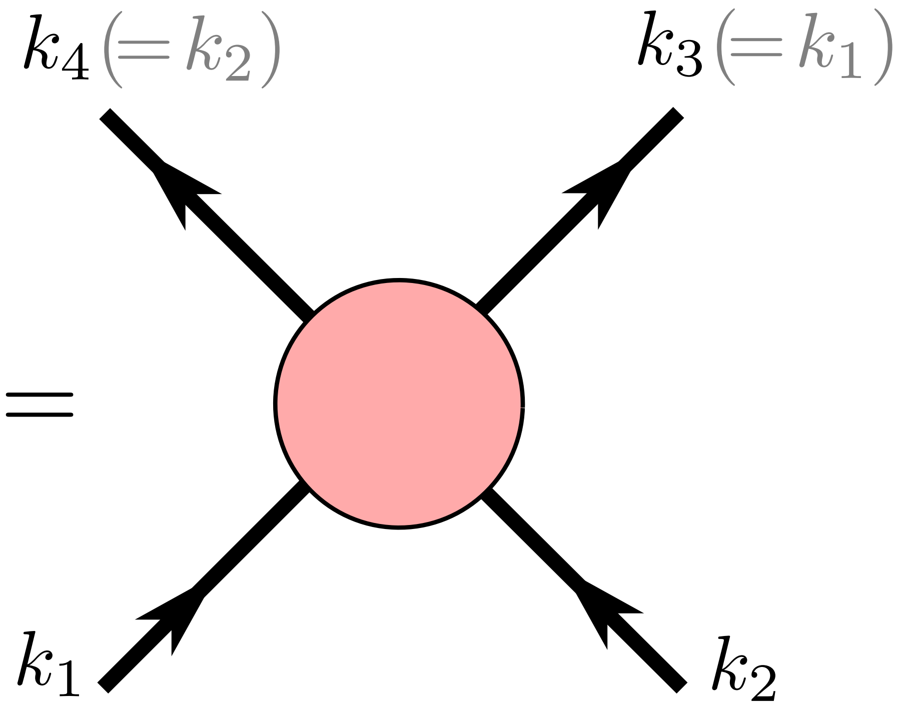
Outline

- S-matrix Bootstrap in D=2
- S-matrix Bootstrap in D>2
- S-matrix from Conformal Bootstrap
- Open questions

S-matrix Bootstrap in 2D QFT

2 to 2 Scattering Amplitude

$S(s) =$



$$k_i^2 = m^2$$

$$s \equiv (k_1 + k_2)^2$$

$$t \equiv (k_2 - k_3)^2 = 4m^2 - s$$

$$u \equiv (k_3 - k_1)^2 = 0$$

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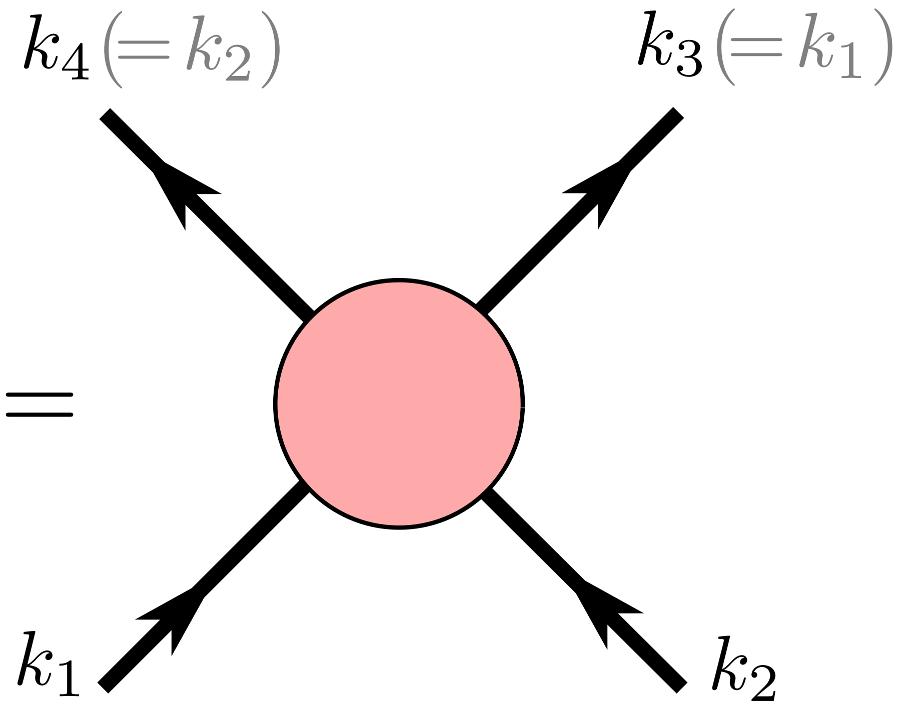

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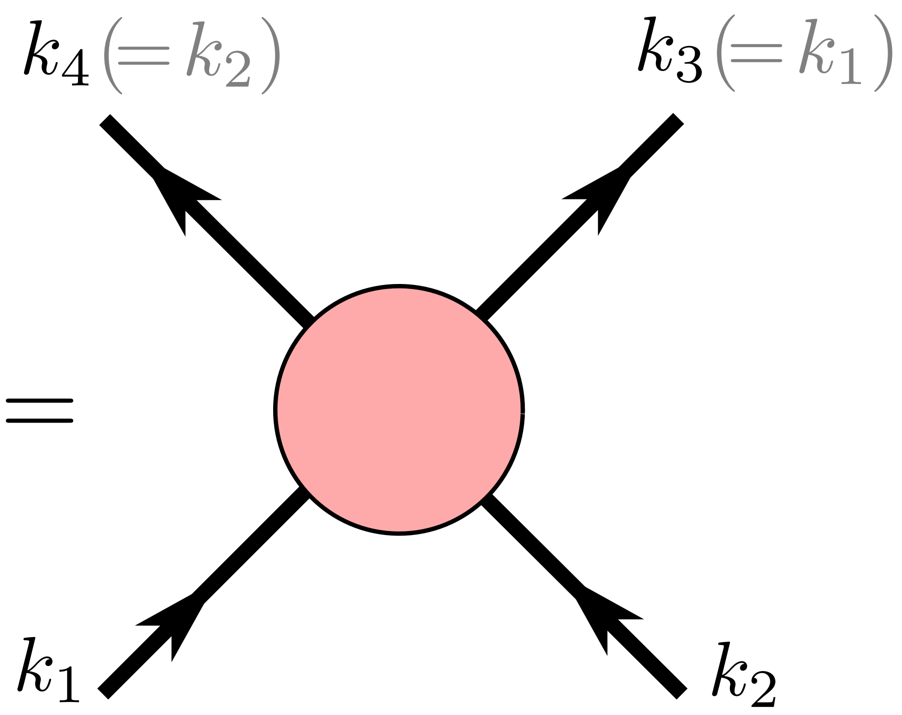

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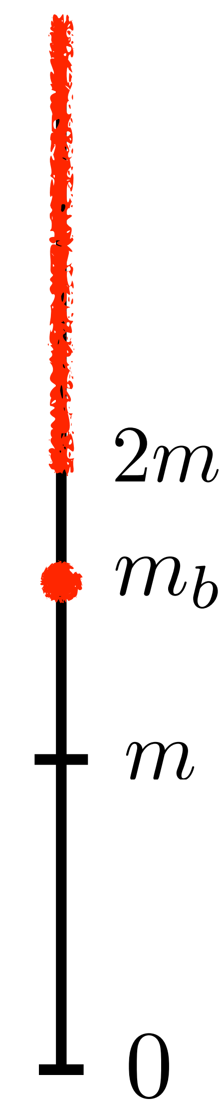

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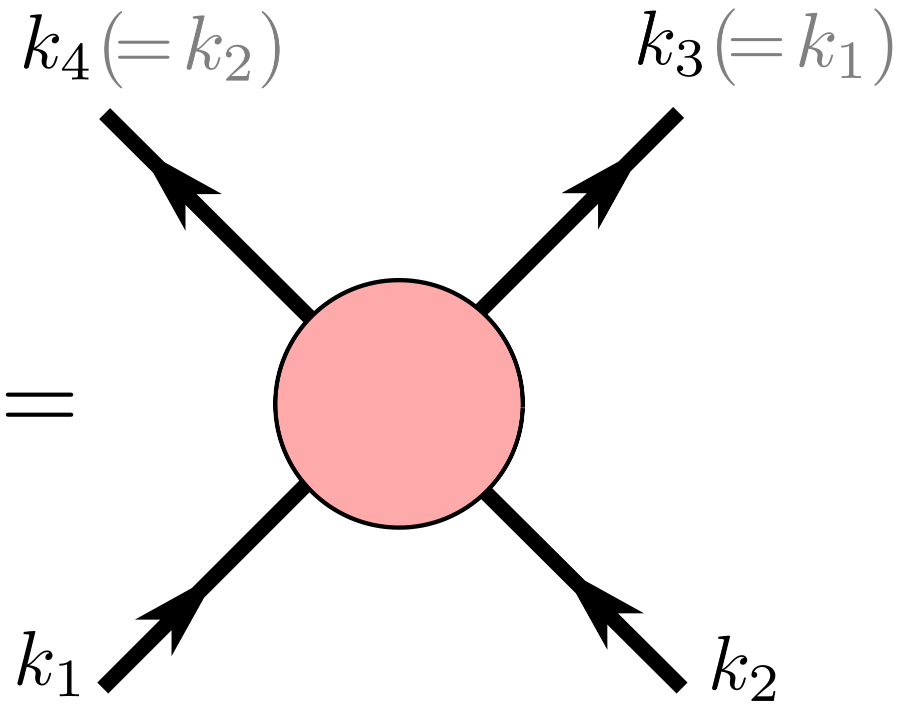

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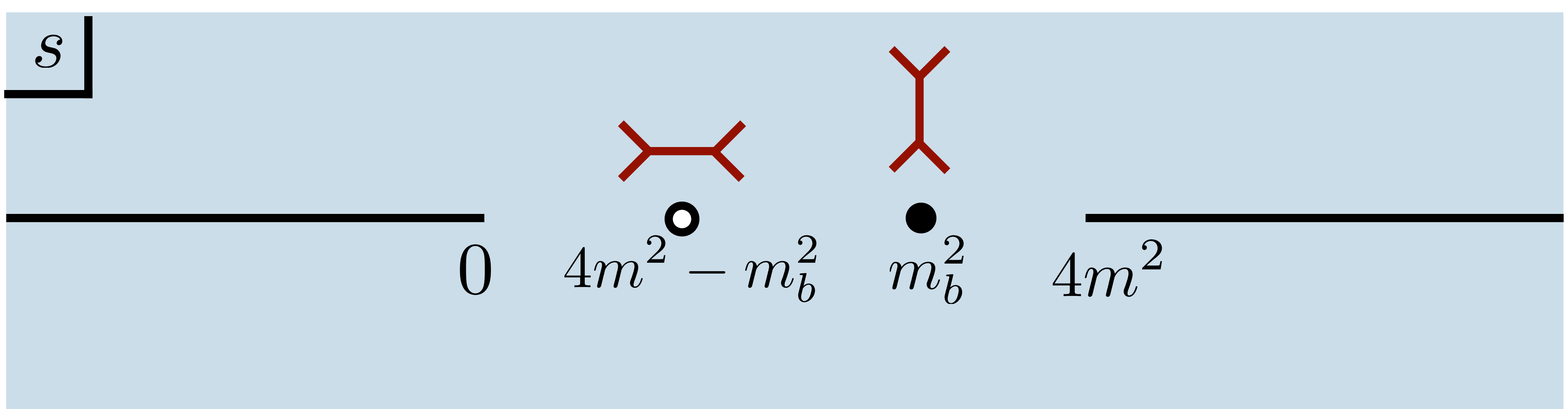

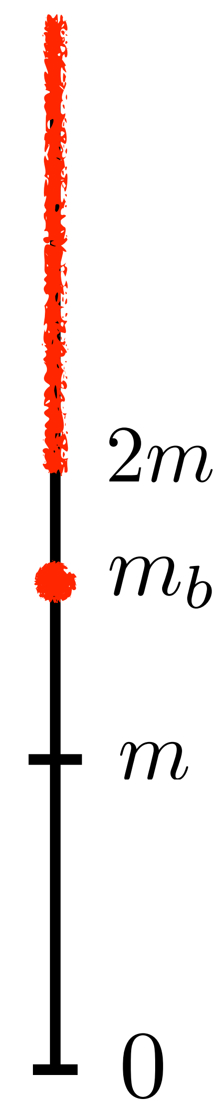

Constraints + question

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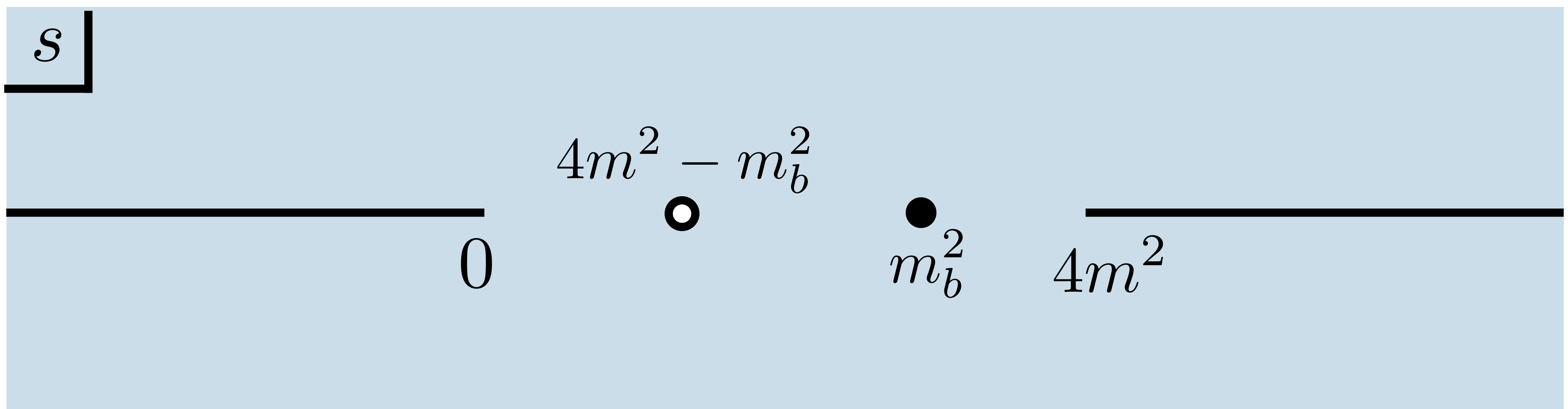

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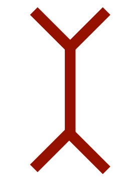
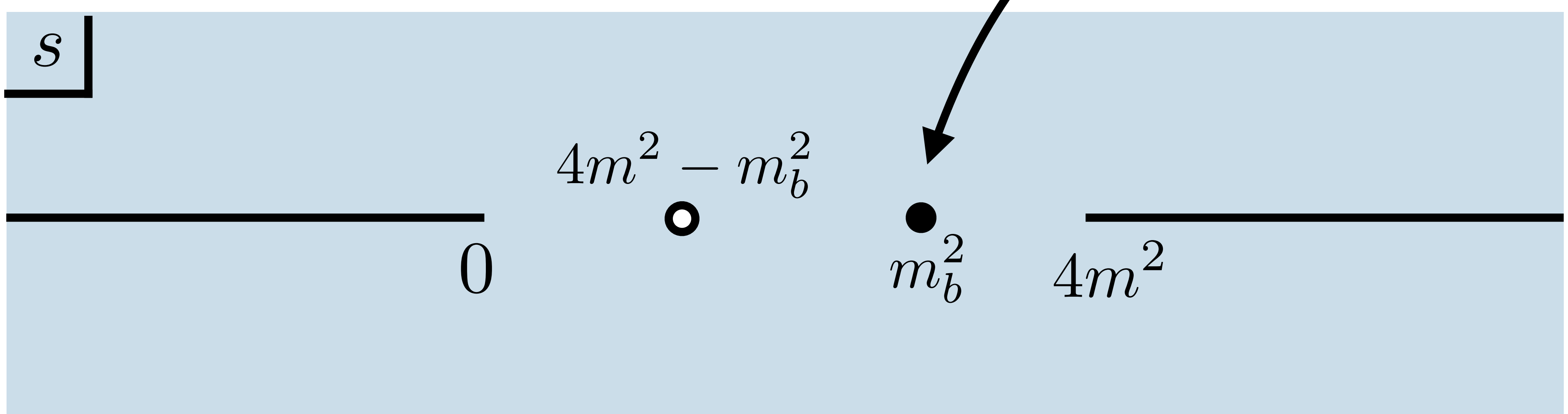
cubic coupling

$$S(s) \sim \frac{g_b^2}{s - m_b^2}$$





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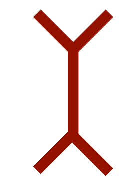

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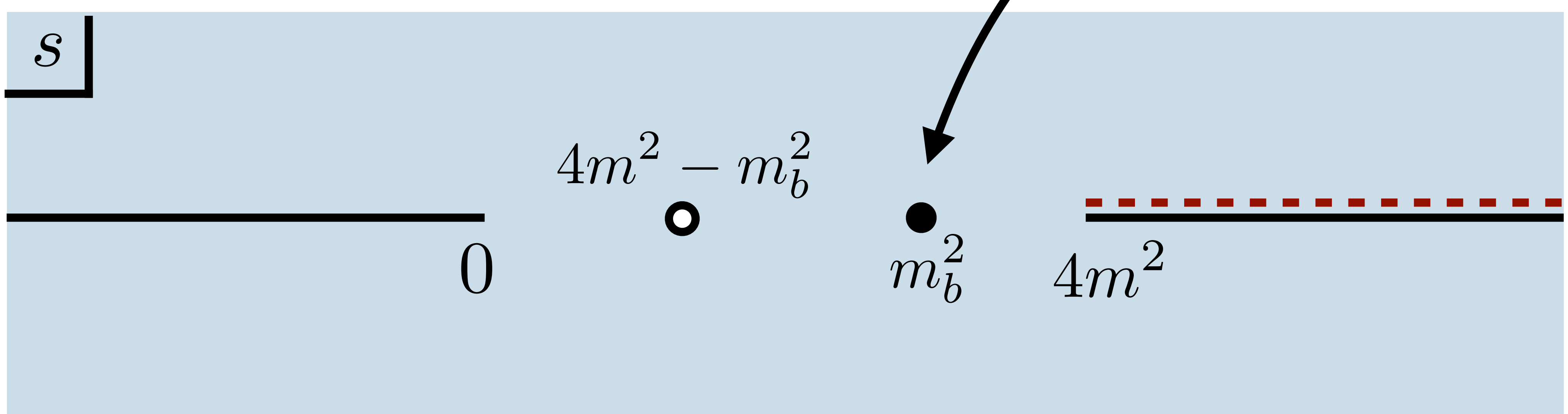

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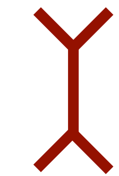

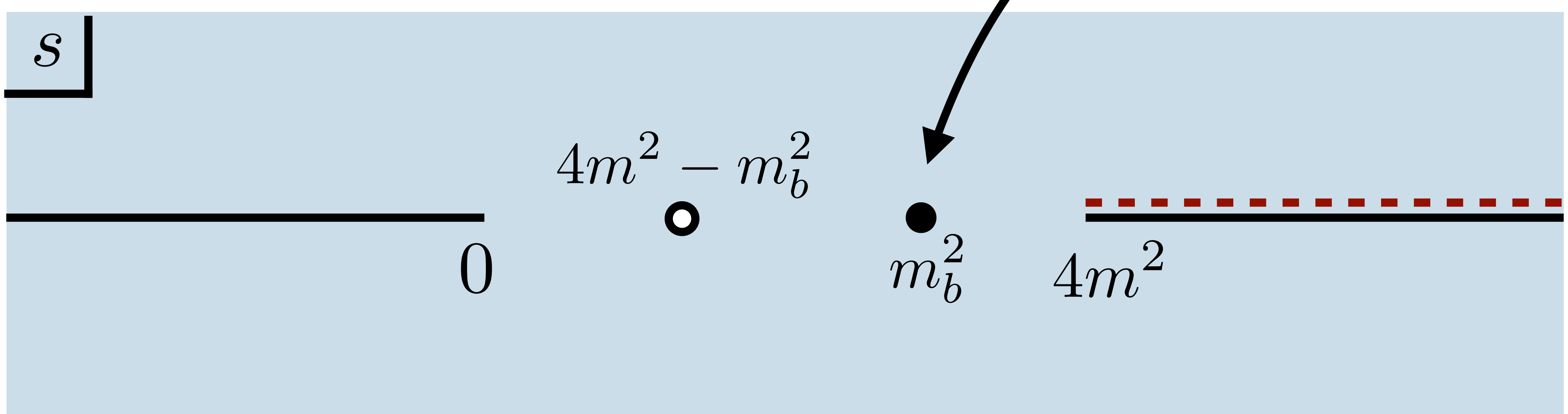

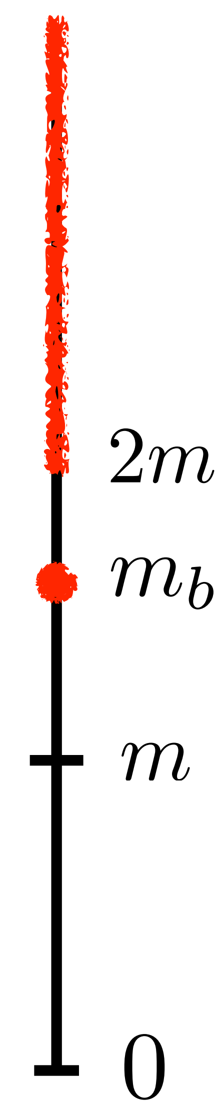

Unitarity: $|S(s)|^2 \leq 1\,, \qquad s > 4m^2\,.$

Question: for given spectrum, $\max\, g_b^2 = ?$

Analytic solution

$$S_{opt}(s) = \frac{\sqrt{s(4m^2-s)} + \sqrt{m_b^2(4m^2-m_b^2)}}{\sqrt{s(4m^2-s)} - \sqrt{m_b^2(4m^2-m_b^2)}} \equiv [m_b](s)$$

[Symanzik '61]
[Creutz '72]

CDD factor
[Castillejo, Dalitz, Dyson]

Pole at $s = m_b^2 > 2$

No particle production $|S_{opt}(s)|^2 = 1\,, \qquad s > 4m^2\,.$

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$h(s)$ analytic in the plane minus the cut

$|h(s)| \leq 1$ bounded at all boundaries

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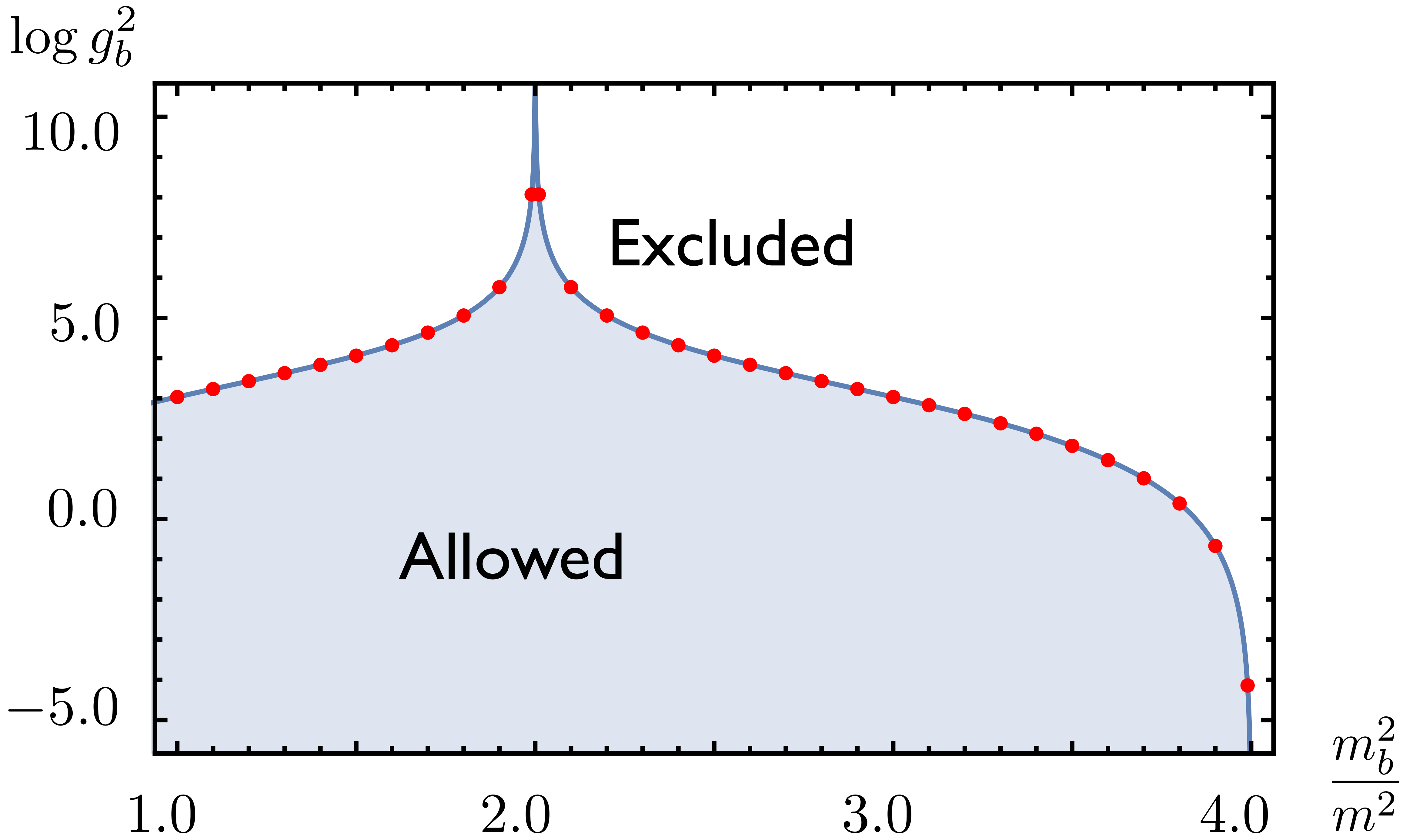
$\Downarrow$ maximum modulus principle

$$|h(m_b^2)| = \left| \frac{g_b^2}{\text{Res}_{s=m_b^2}[m_b](s)} \right| \leq 1$$

Maximum cubic coupling

$$S_{opt}(s) = \frac{\sqrt{s(4m^2 - s)} + \sqrt{m_b^2(4m^2 - m_b^2)}}{\sqrt{s(4m^2 - s)} - \sqrt{m_b^2(4m^2 - m_b^2)}}$$



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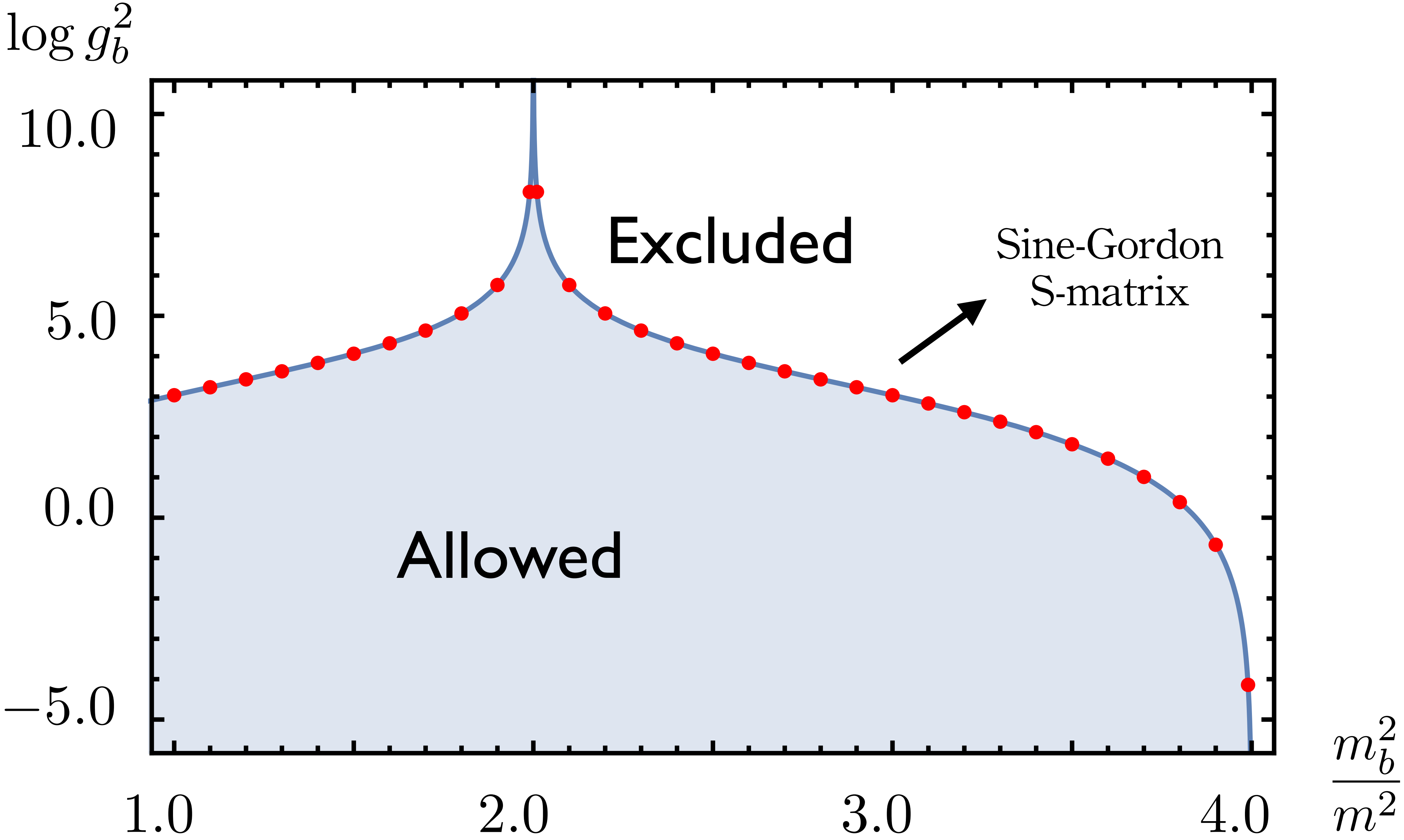
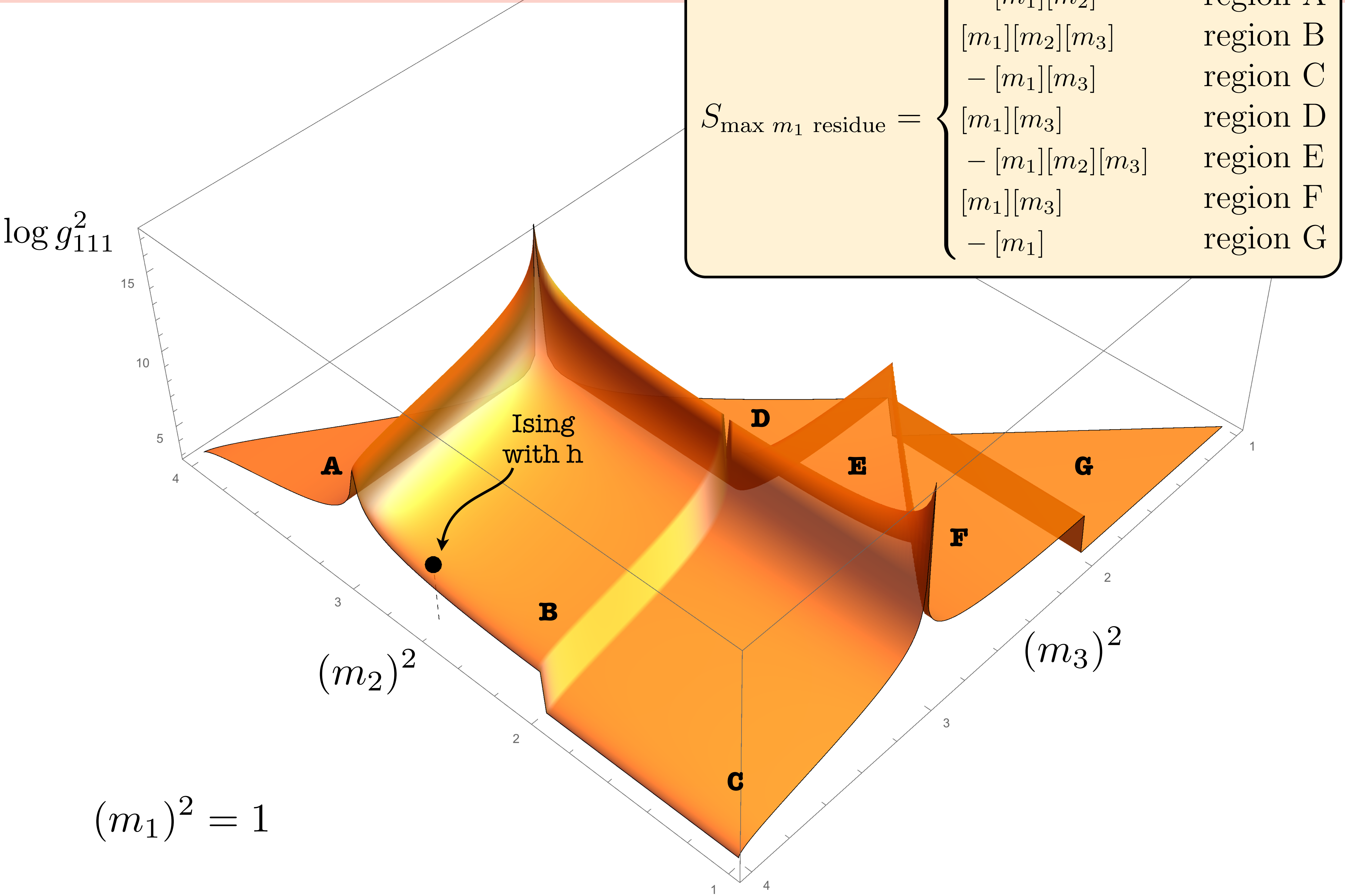
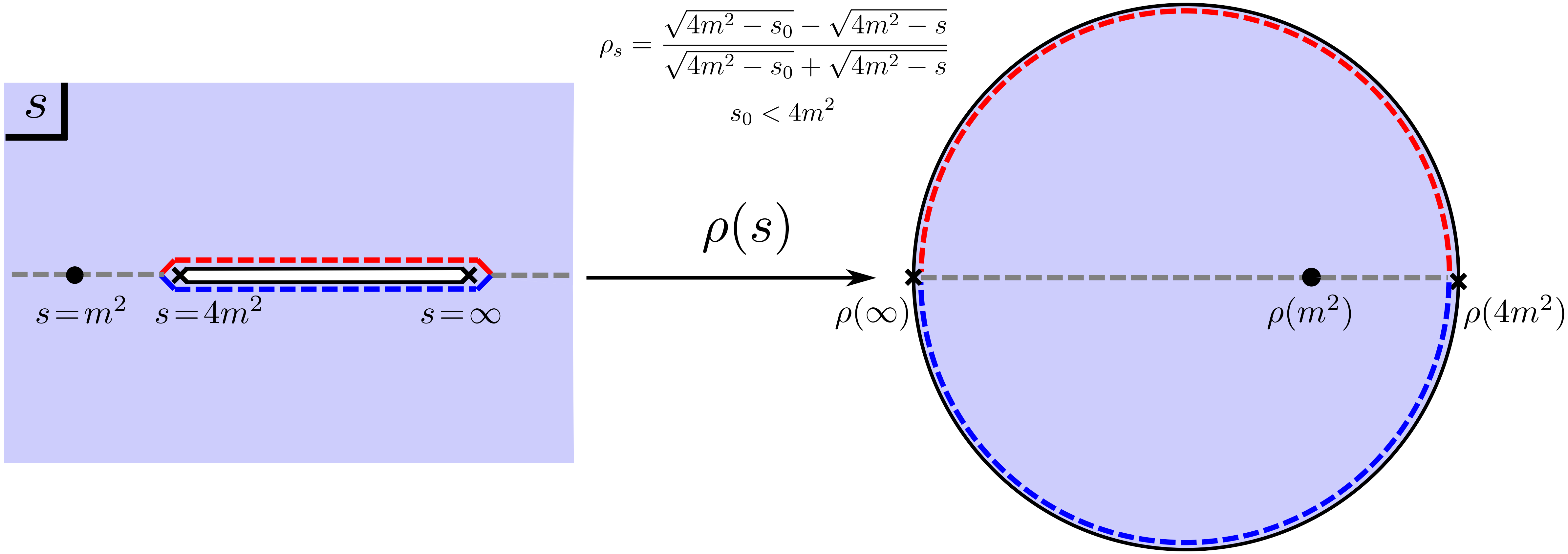


3 stable particles

$$S_{\text{max } m_1 \text{ residue}} = \begin{cases} -[m_1][m_2] & \text{region A} \\ [m_1][m_2][m_3] & \text{region B} \\ -[m_1][m_3] & \text{region C} \\ [m_1][m_3] & \text{region D} \\ -[m_1][m_2][m_3] & \text{region E} \\ [m_1][m_3] & \text{region F} \\ -[m_1] & \text{region G} \end{cases}$$



Numerical approach



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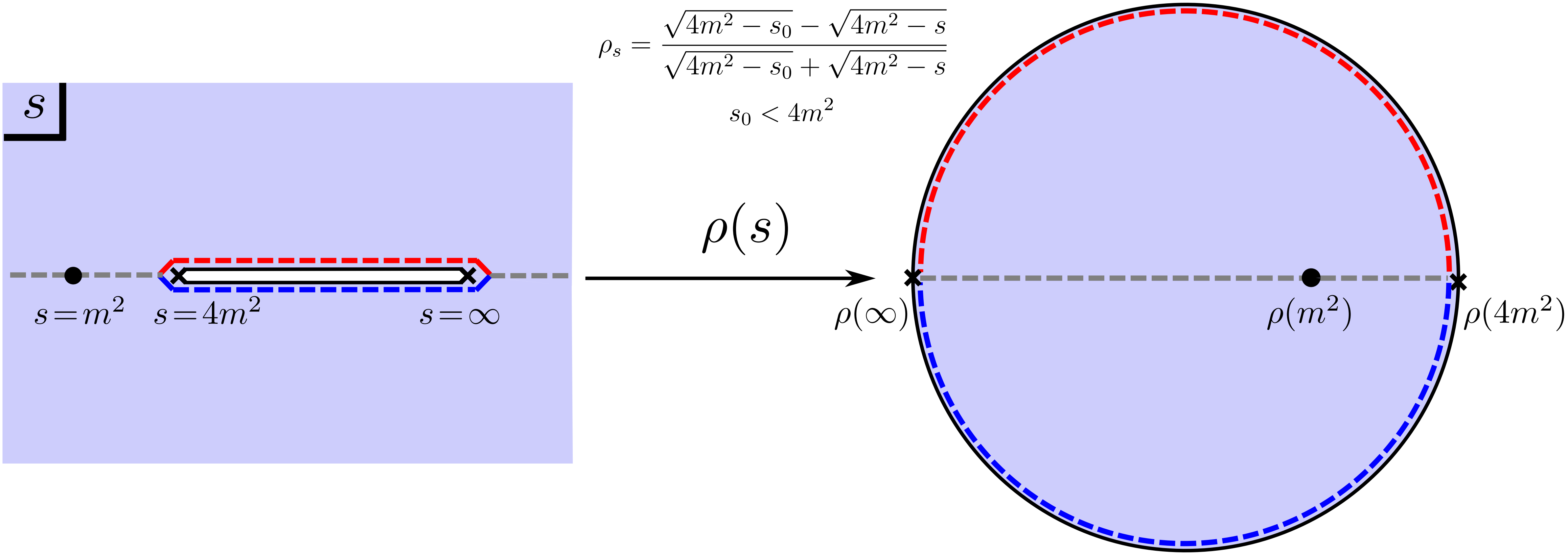

Ansatz:

$$S_{ext}(s,t) = \frac{g_b^2}{s - m_b^2} + \frac{g_b^2}{t - m_b^2} + \sum_{a,b=0} c_{(ab)} \rho_s^a \rho_t^b$$

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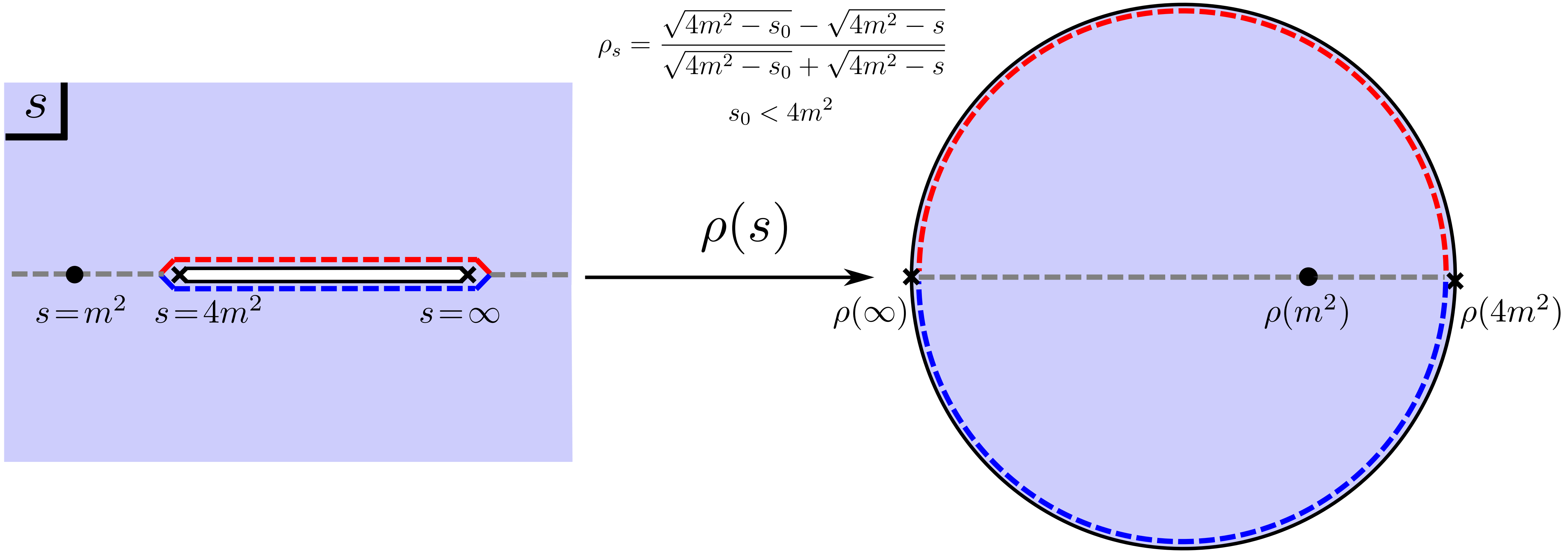

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Crossing symmetry and analyticity are automatic.

Unitarity gives quadratic constraints:

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Truncate to finite number of variables and **quadratic constraints**

$a + b \leq N_{\max}$ ↓ $\{g_b^2, c_{(ab)}\}$

↓ at $s = s_1, s_2, \ldots, s_M$

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[Simmons-Duffin '15]

Use semidefinite programming (SDPB) to maximize g_b^2 subject to these constraints. This reproduces the analytic solution as $N_{\max} \to \infty$

S-matrix Bootstrap
in d+1 QFT

2 to 2 Scattering Amplitude

$$\langle \mathbf{p}_3, \mathbf{p}_4 | S | \mathbf{p}_1, \mathbf{p}_2 \rangle = \mathbb{1} + i(2\pi)^{d+1} \delta^{(d+1)}(p_1 + p_2 - p_3 - p_4)\, T\,(s, t, u)$$

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Partial waves:

Gegenbauer polynomial
$x = \cos\theta$

$$S_\ell(s) = 1 + i \frac{(s - 4m^2)^{\frac{d-2}{2}}}{\sqrt{s}} \int_{-1}^{1} dx (1 - x^2)^{\frac{d-3}{2}} P_\ell^{(d)}(x)\, T(s,t,u)\Big|_{\substack{t \to -\frac{1-x}{2}(s-4m^2) \\ u \to -\frac{1+x}{2}(s-4m^2)}}$$

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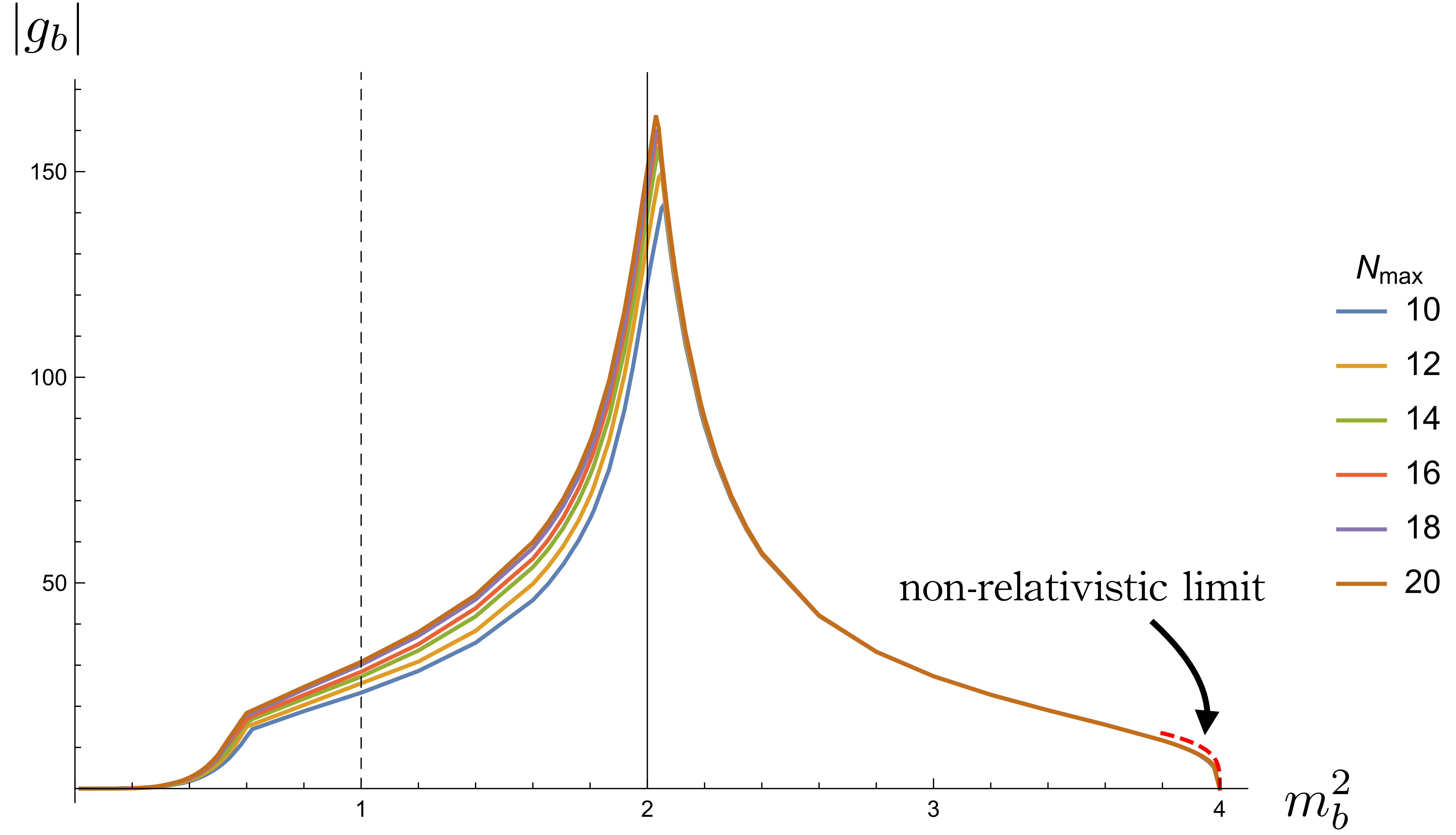
$\Rightarrow$ Quadratic constraints on the variables $\{g_b^2, \alpha_{(abc)}\}$

$a + b + c \leq N_{\max}$

Maximal cubic coupling in 3+1 QFT



Maximal quartic coupling

Ansatz with **no poles**. Maximize $\lambda = \frac{1}{32\pi} T(s = t = u = \frac{4}{3} m^2)$
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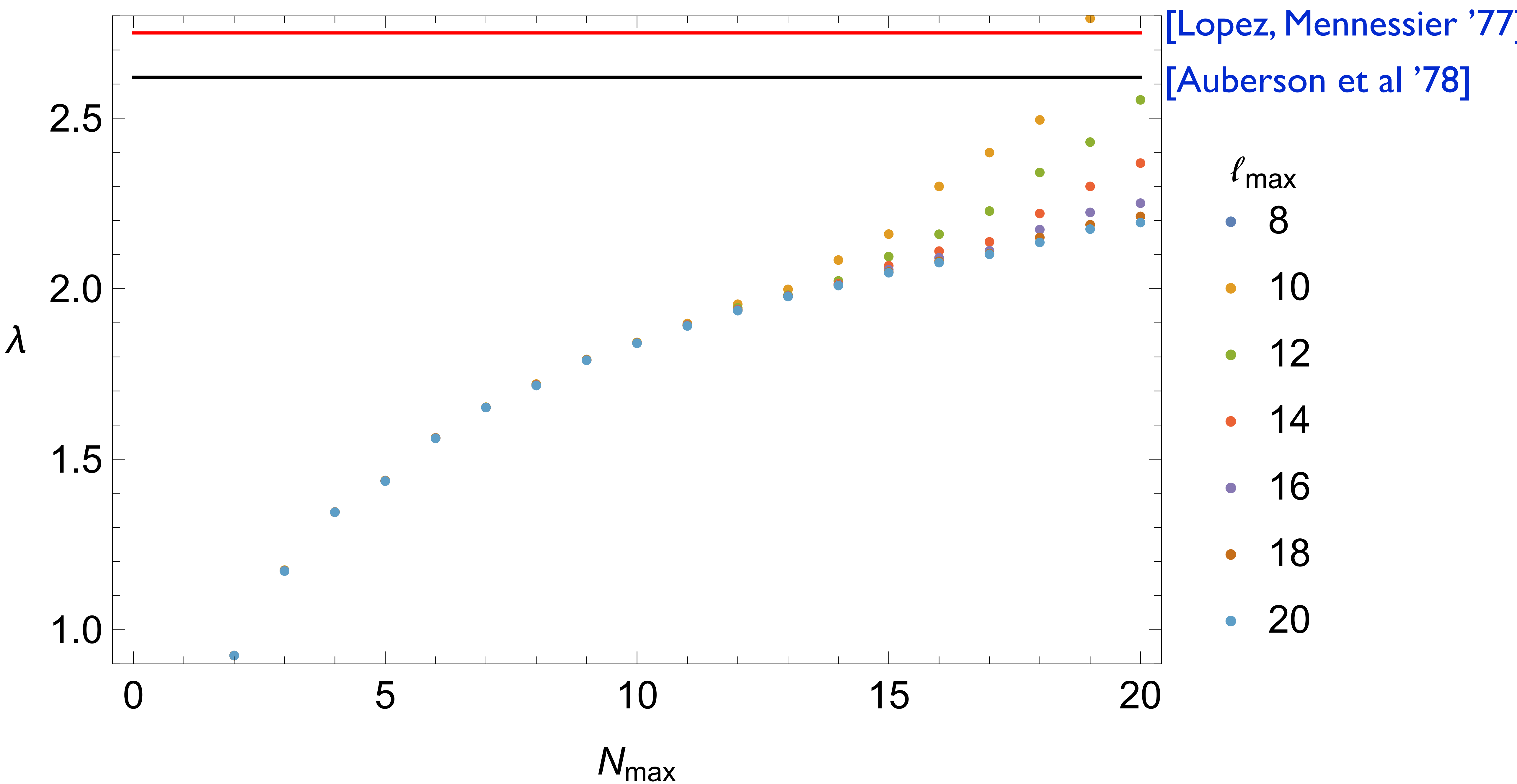

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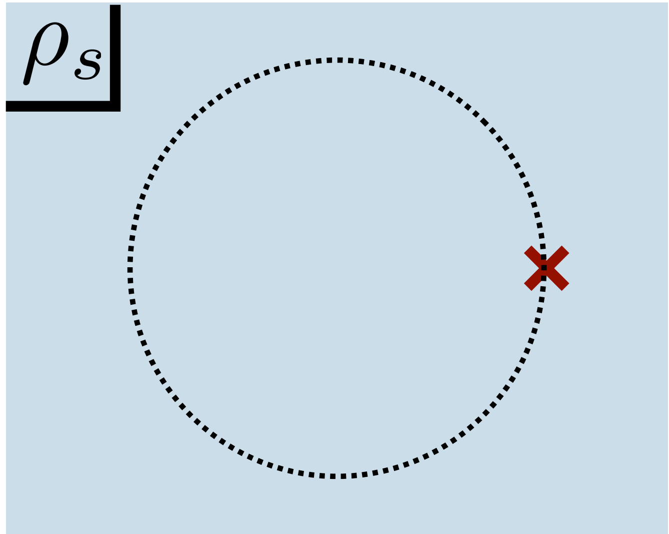

Improved ansatz with threshold bound state:

$$T(s,t,u) = \beta\left(\frac{1}{\rho_s - 1} + \frac{1}{\rho_t - 1} + \frac{1}{\rho_u - 1}\right) + \sum_{\substack{a,b,c=0 \\ a+b+c\leq N_{\max}}} \alpha_{(abc)}\rho_s^a \rho_t^b \rho_u^c$$

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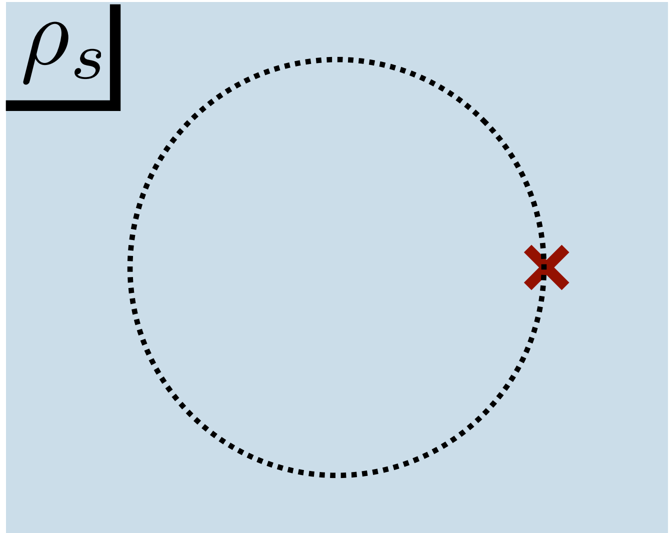

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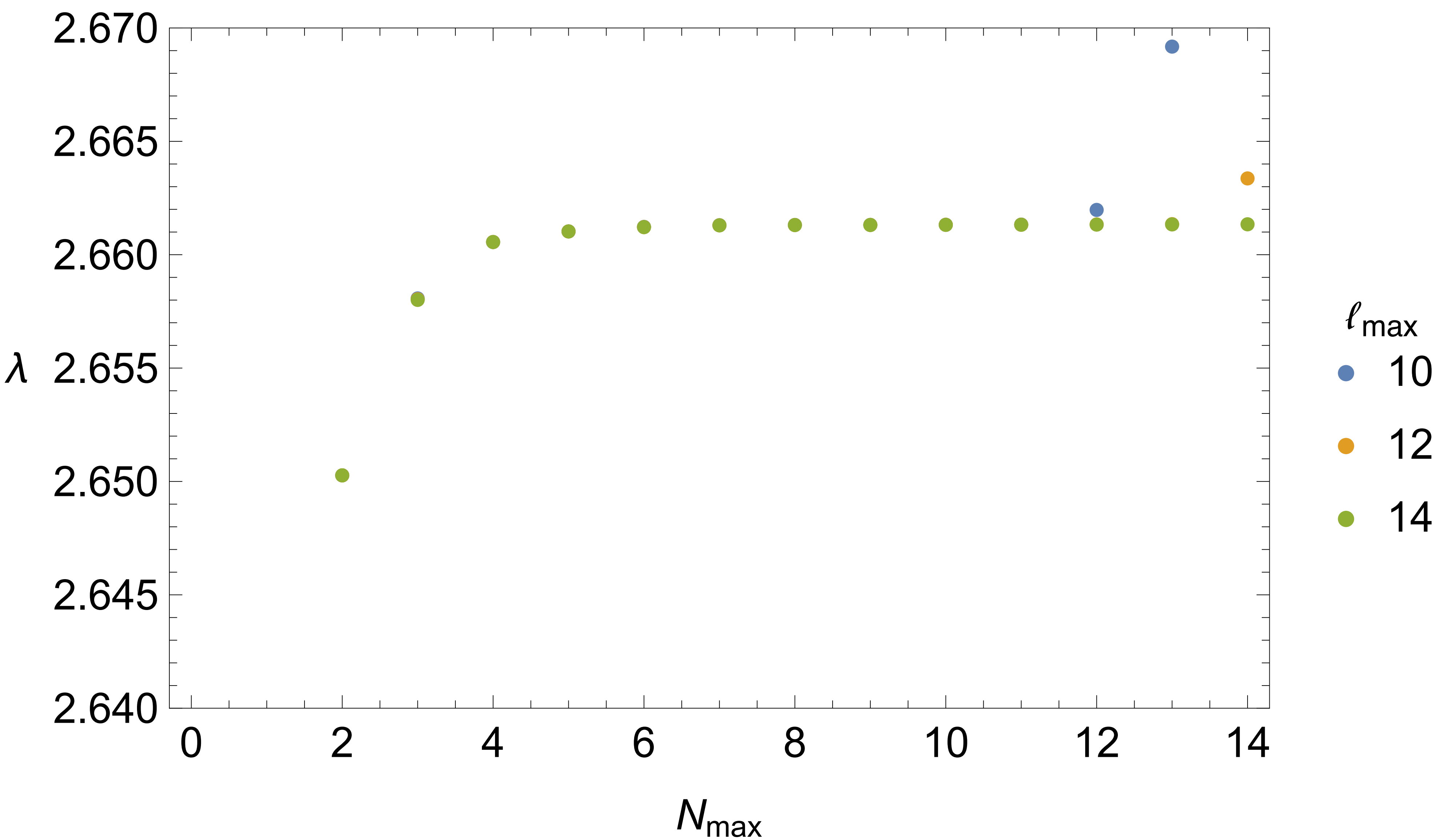
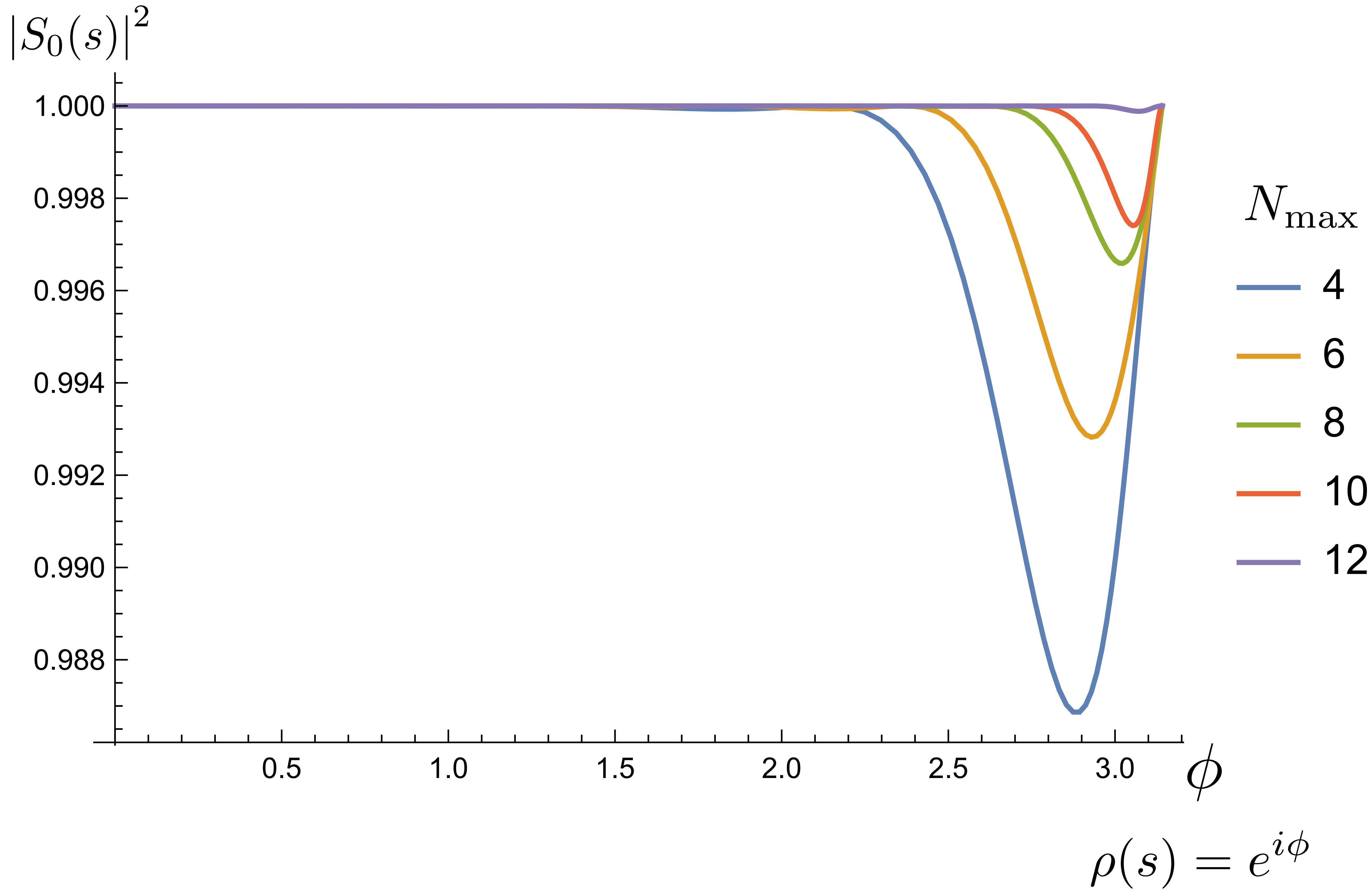
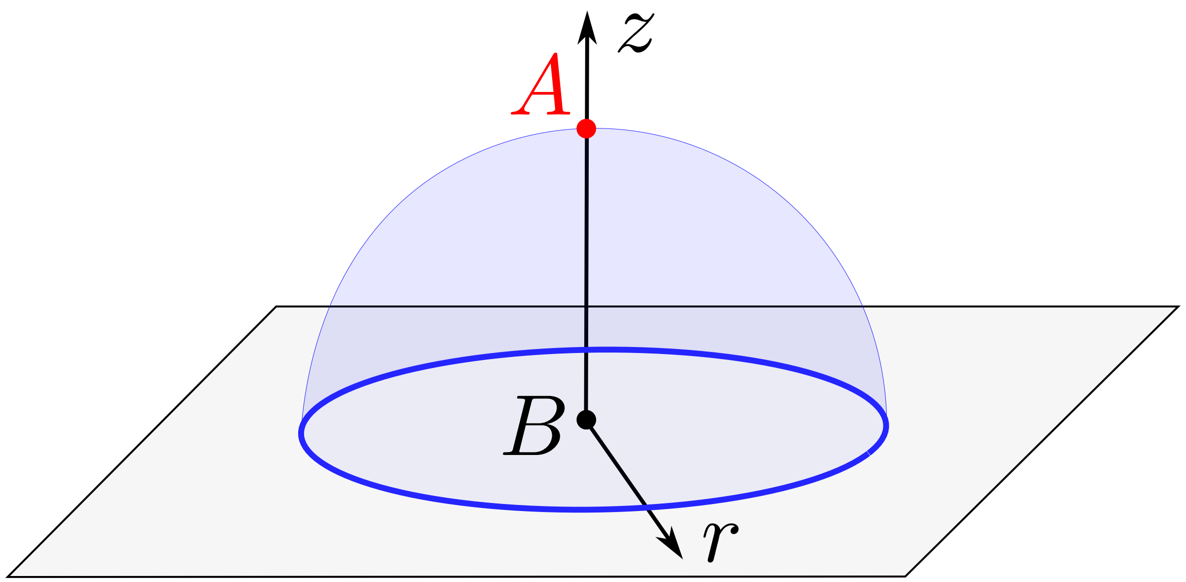


No particle production?

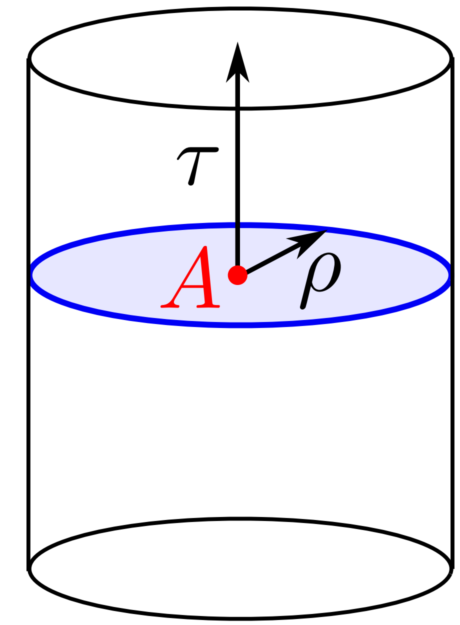


S-matrix from the Conformal Bootstrap

QFT in AdS



$$z = e^{\tau} \cos \rho$$
$$r = e^{\tau} \sin \rho$$

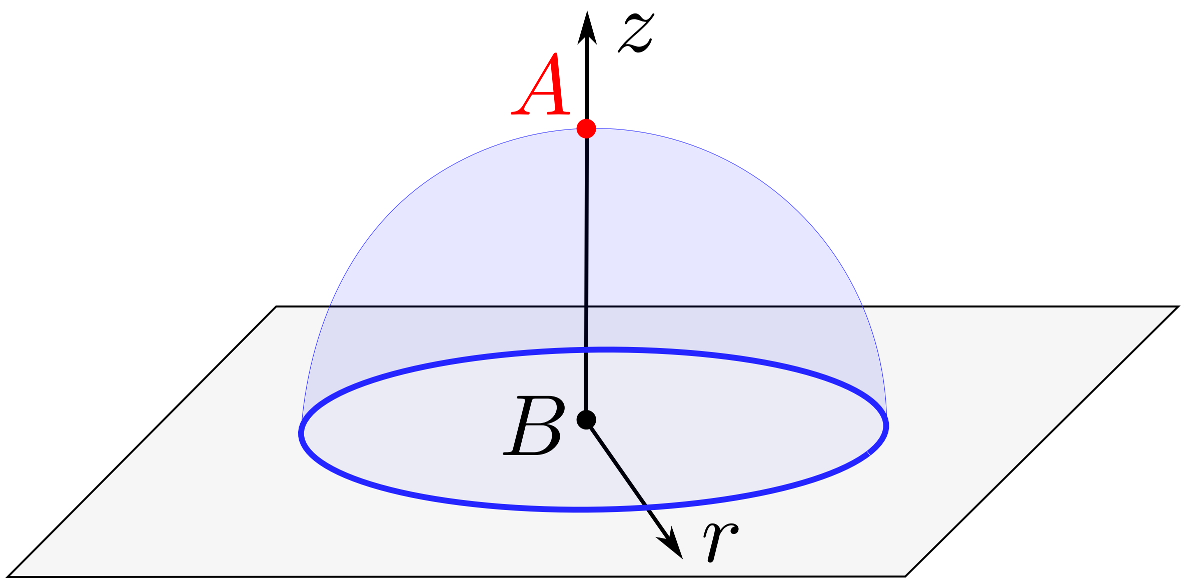


Correlation functions of boundary operators

$$\langle \mathcal{O}(x) \ldots \rangle = \lim_{z \to 0} z^{-\Delta} \ldots \langle \phi(z, x) \ldots \rangle$$

bulk operator

QFT in AdS



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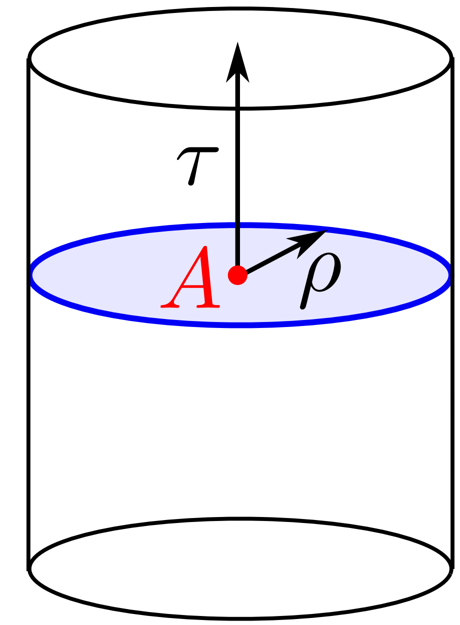

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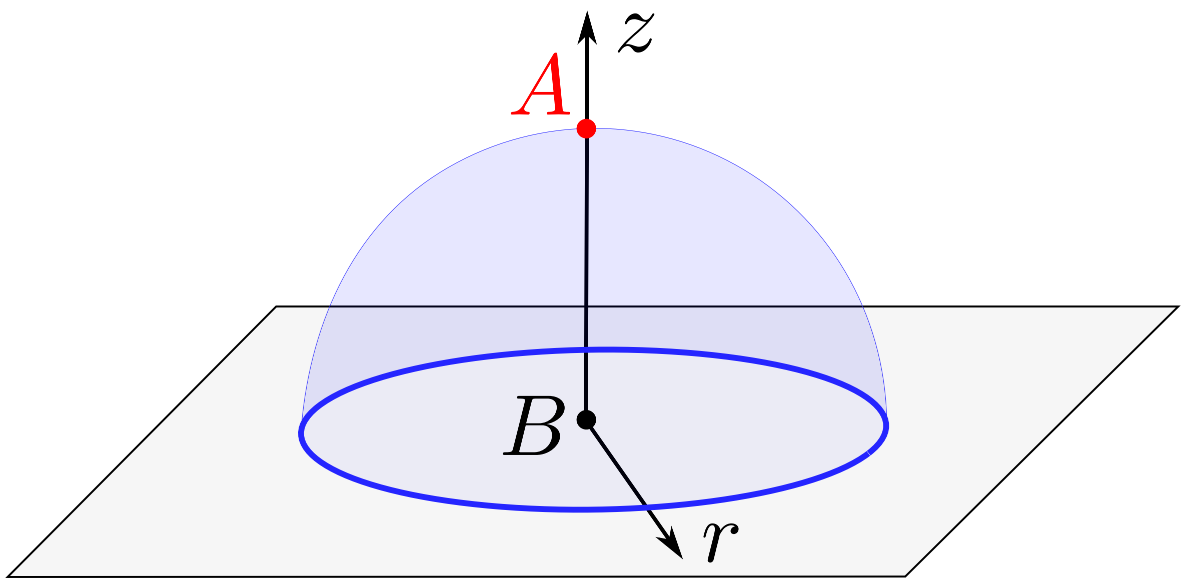
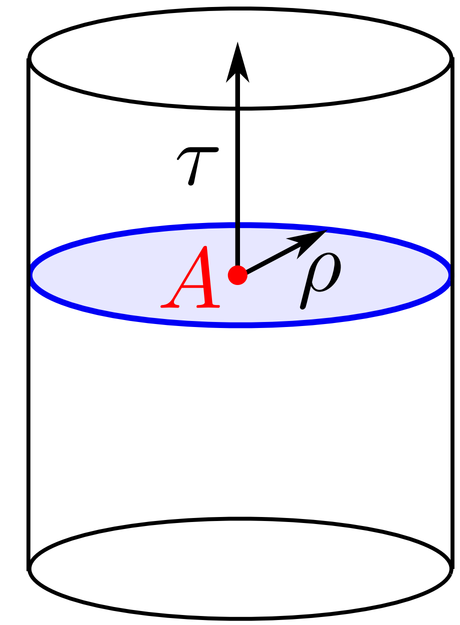
Isometry group of AdS = SO(d+1,1) = Conformal group

QFT in AdS



$$z = e^{\tau} \cos\rho$$
$$r = e^{\tau} \sin\rho$$



Correlation functions of boundary operators

$$\langle \mathcal{O}(x) \ldots \rangle = \lim_{z \to 0} z^{-\Delta} \ldots \langle \phi(z,x) \ldots \rangle$$

bulk operator

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Convergent OPE for boundary operators

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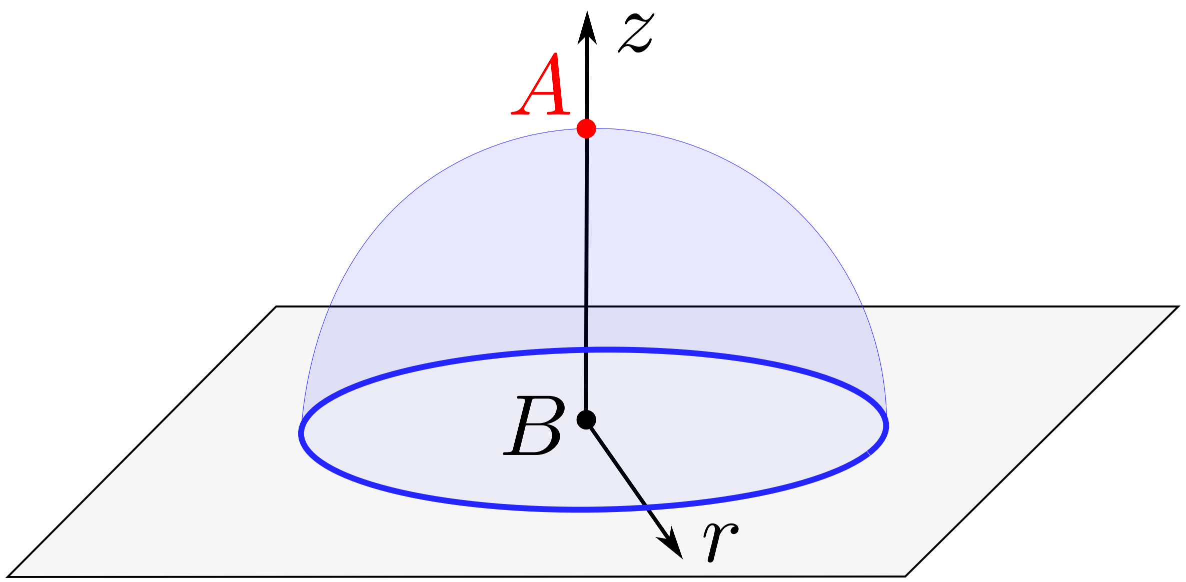

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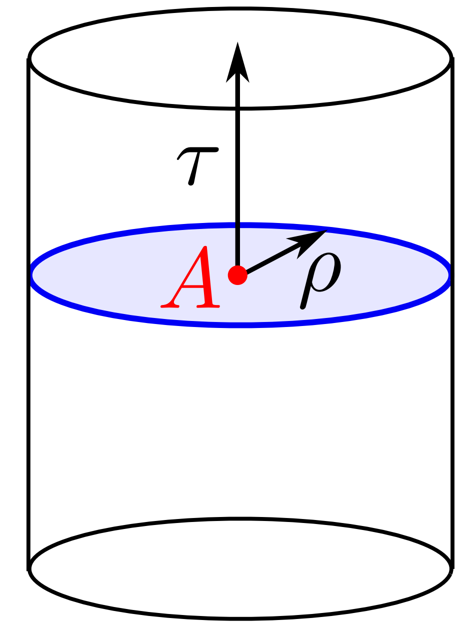

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Convergent OPE for boundary operators

$\Rightarrow$ Use **conformal bootstrap** to study $\langle \mathcal{O}_1(x_1) \ldots \mathcal{O}_n(x_n) \rangle$

Flat space limit of AdS

AdS radius $R \to \infty$

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Mass spectrum:

$$\Delta_i \sim m_i R$$

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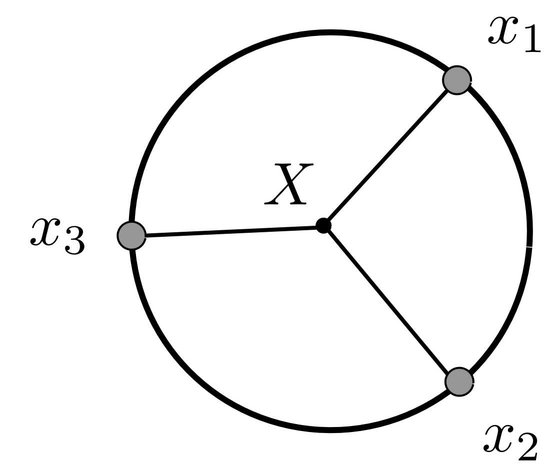

Cubic couplings:

OPE coefficient

$$g_{123} = \lim_{\Delta_i \to \infty} \lambda_{123} \times \frac{2(\Delta_1)^{\frac{d-5}{2}}}{\pi^{\frac{d}{2}} \Gamma(\frac{1}{2}\sum_{i=1}^{3} \Delta_i - \frac{d}{2})} \prod_{i=1}^{3} \frac{\Gamma(\Delta_i)}{\Gamma(\frac{1}{2}\sum_{i=1}^{3} \Delta_i - \Delta_i) \sqrt{\mathcal{C}_{\Delta_i}}} .$$

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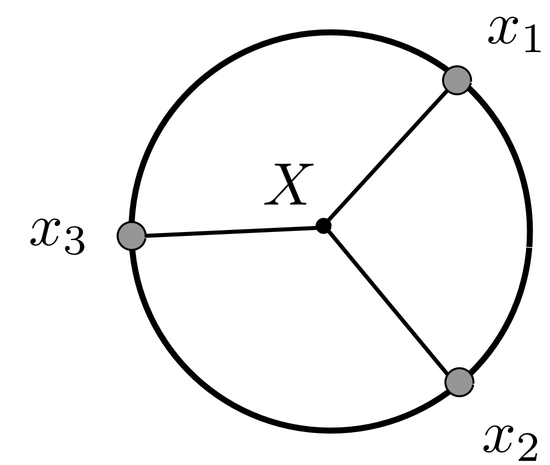

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Scattering amplitudes:

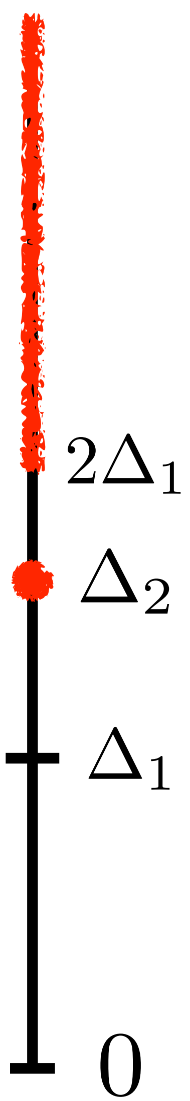
Mellin amplitude

$$(m_1)^a\, T(k_i) = \lim_{\Delta_i \to \infty} \frac{(\Delta_1)^a}{\mathcal{N}} M\left(\gamma_{ij} = \frac{\Delta_i \Delta_j}{\Delta_1 + \cdots + \Delta_n}\left(1 + \frac{k_i \cdot k_j}{m_i m_j}\right)\right)$$

$$a = n(d-1)/2 - d - 1 \qquad \mathcal{N} = \frac{1}{2}\pi^{\frac{d}{2}} \Gamma\left(\frac{\sum \Delta_i - d}{2}\right) \prod_{i=1}^{n} \frac{\sqrt{\mathcal{C}_{\Delta_i}}}{\Gamma(\Delta_i)}, \qquad \mathcal{C}_\Delta \equiv \frac{\Gamma(\Delta)}{2\pi^{\frac{d}{2}}\Gamma\left(\Delta - \frac{d}{2} + 1\right)} \,.$$

Numerical Conformal Bootstrap

$$\mathcal{O}_1 \times \mathcal{O}_1 = 1 + \lambda_{112}\mathcal{O}_2 + \ldots (\text{operators with } \Delta > 2\Delta_1) \ldots$$



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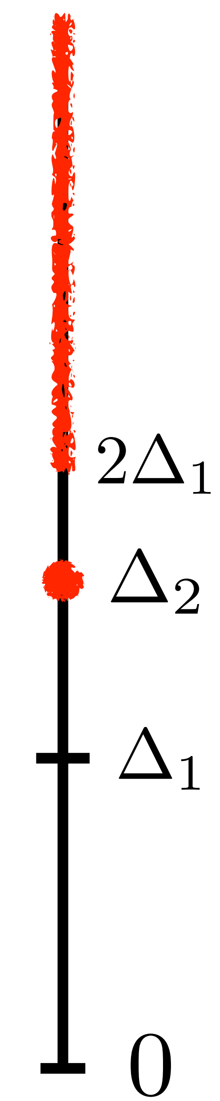

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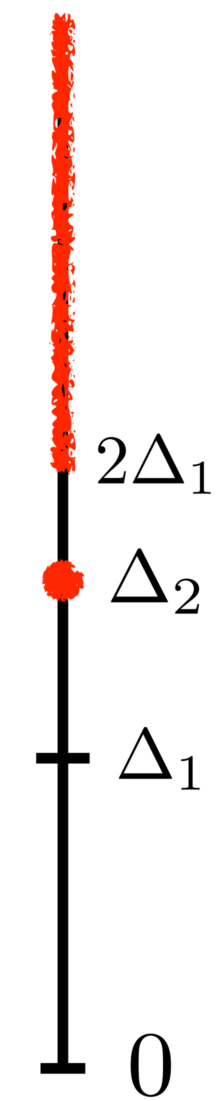

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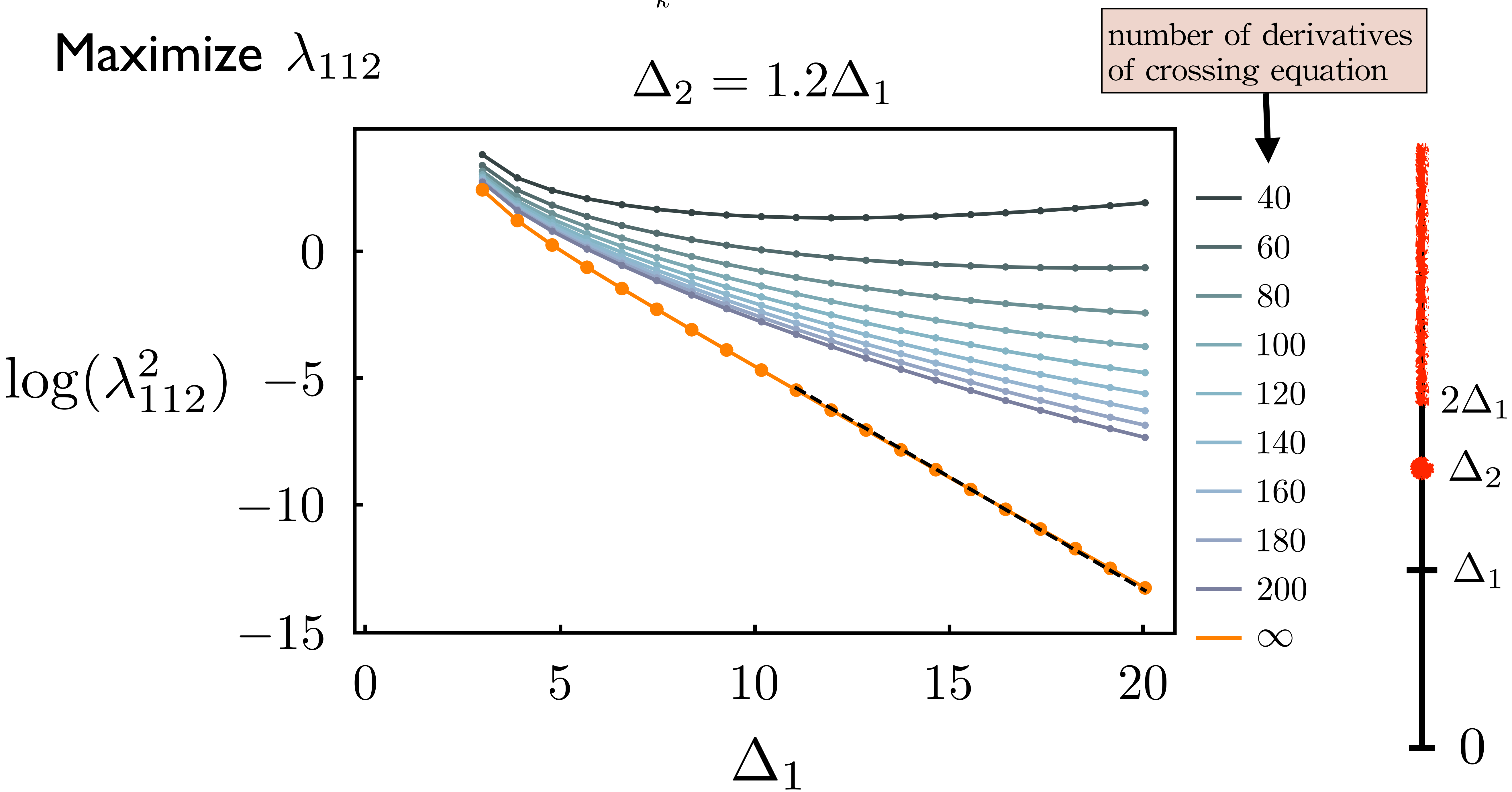
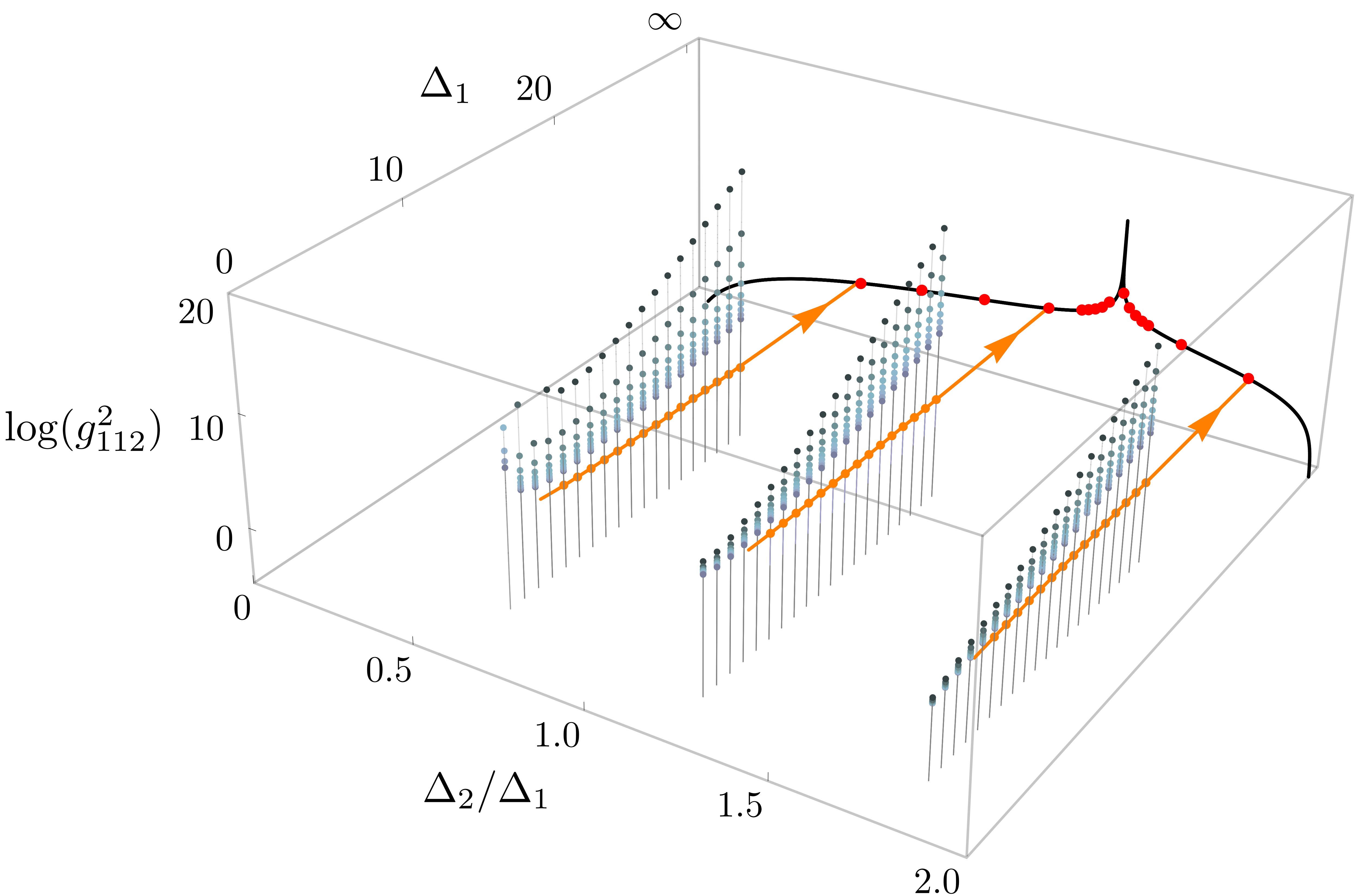
Maximize λ_{112}

$\Delta_2 = 1.2\Delta_1$



Extrapolation^2



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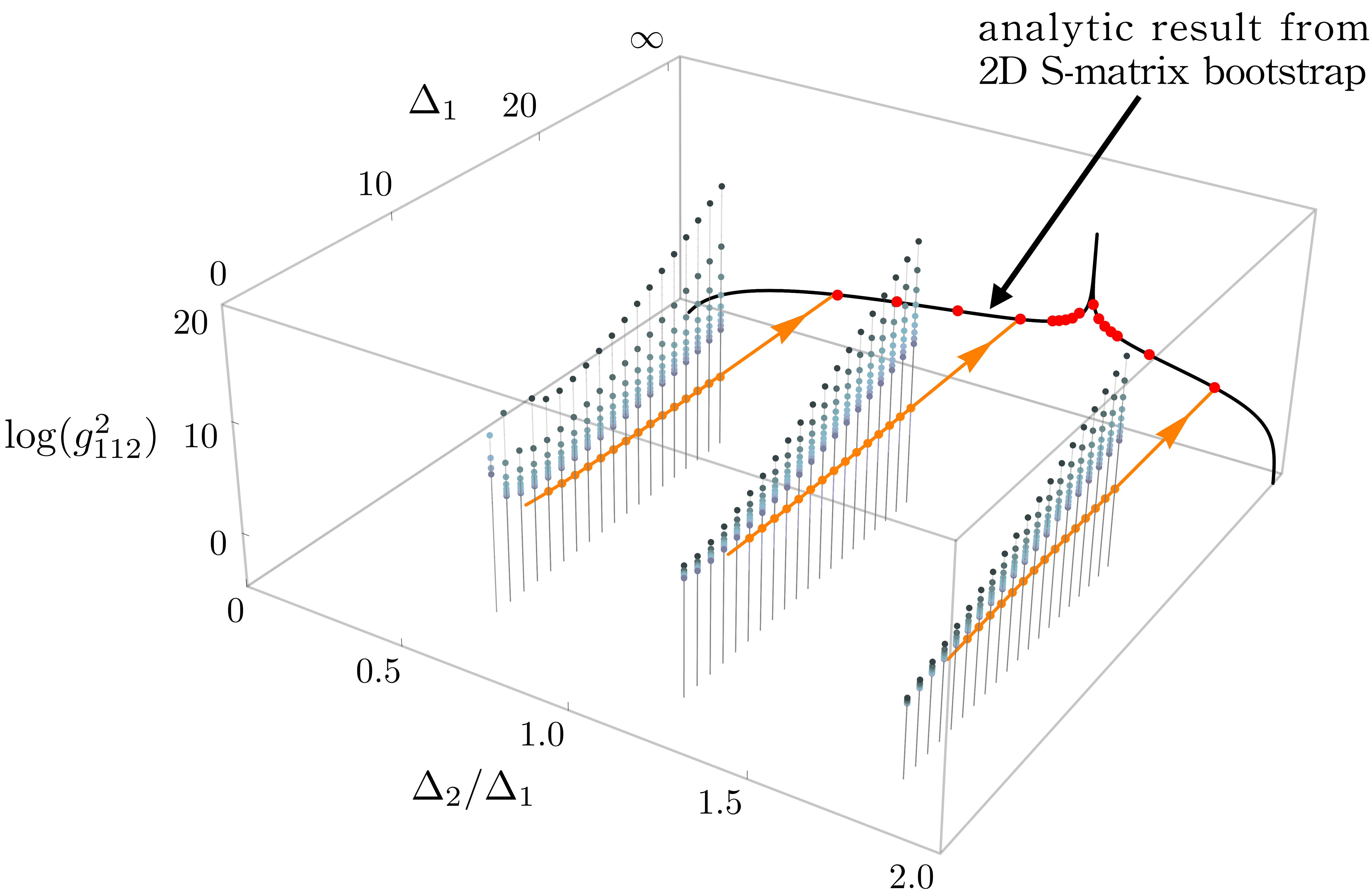
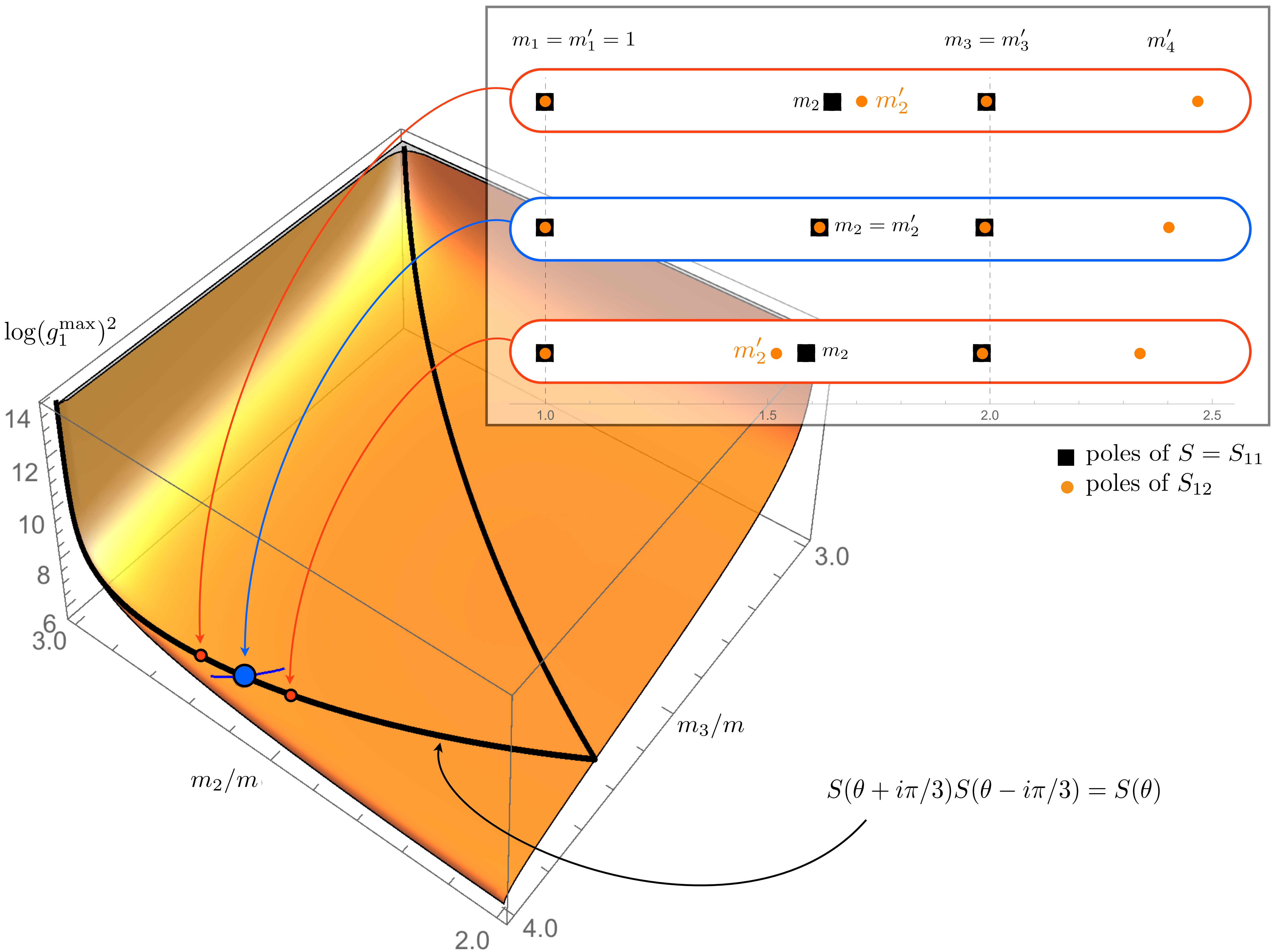


Open questions

Future work

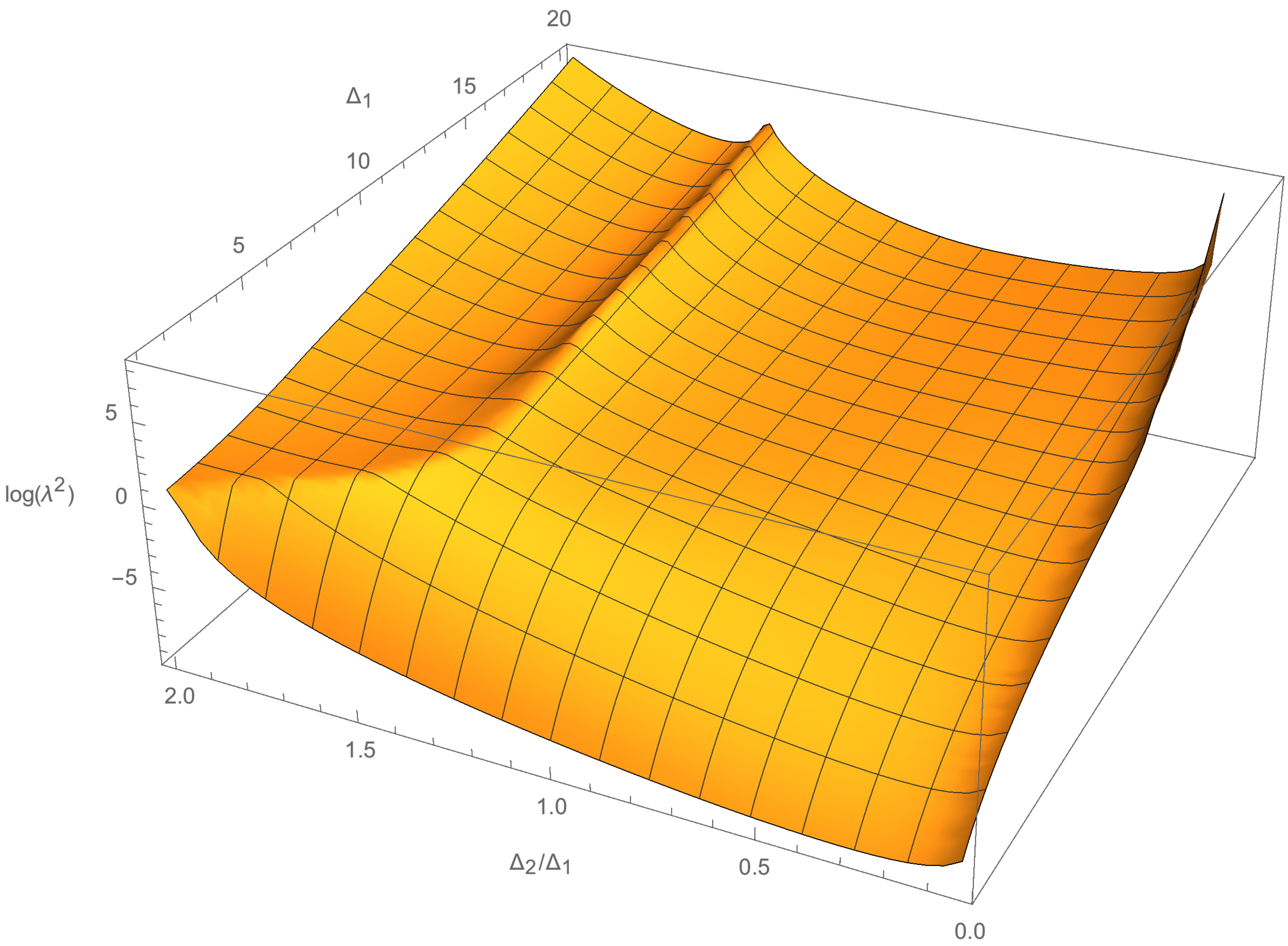
- Bootstrap multiple amplitudes
- Anomalous thresholds (Landau diagrams)
- Particles with spin (internal and external)
- Particles with flavour (global symmetries)
- Use analyticity beyond the physical sheet
- Connect with conformal bootstrap for D>2
- Other interesting questions? Maximize particle production?
- Can we input UV data about the QFT? Hard scattering?

Thank you!



$m_1 = m_1' = 1$
$m_3 = m_3'$
m_4'
m_2
m_2'
$m_2 = m_2'$
m_2'
m_2
1.0
1.5
2.0
2.5
poles of $S = S_{11}$
poles of S_{12}
$\log(g_1^{\max})^2$
14
12
10
8
6
3.0
m_2/m
2.0
m_3/m
3.0
4.0
$S(\theta + i\pi/3)S(\theta - i\pi/3) = S(\theta)$

2D Conformal bootstrap - preliminary



RG flows from QFT in AdS