Amplitudes + Correlators as Differential Forms;

Worldsheets as Positive Geometries

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Unique Gauge Invariants

Amplitudes as Canonical Forms of Positive Geometries

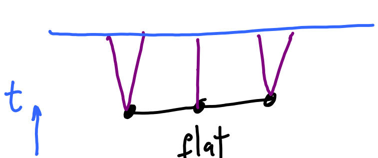
Amplituhedron

Pert. String Amps

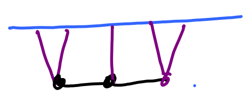
Scattering Eqns

Twistor String

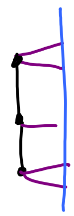
Worldsheets as Positive Geometries



$\Psi^{\text{flat}}_{\text{Vac}}[\varphi]$



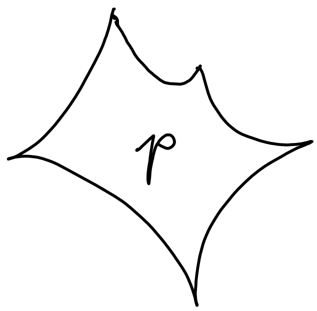
$\Psi^{dS}_{\text{Vac}}[\varphi]$



AdS Bndy Correlators

Canonical Form of "Cosmological Polytopes"

Positive Geometries $\longleftrightarrow$ Canonical Forms



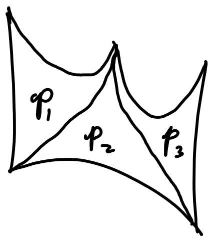
Region $\mathcal{P}$ w/ boundaries of all Codimension

$\Omega_{\mathcal{P}}$: unique form with logarithmic singularities on (+ only on) boundaries of $\mathcal{P}$ $\left\{\text{locally } \Omega \to \frac{dx_1}{x_1} \cdots \frac{dx_D}{x_D}\right\}$

Real Geometry

Complex Form

"Triangulation"

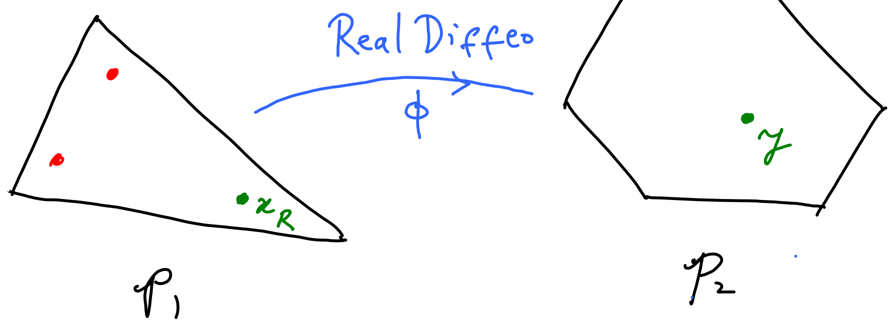


$\mathcal{P}$ tiled by $\mathcal{P}_i$

$$\Downarrow$$

$$\Omega_{\mathcal{P}} = \sum_i \Omega_{\mathcal{P}_i}$$

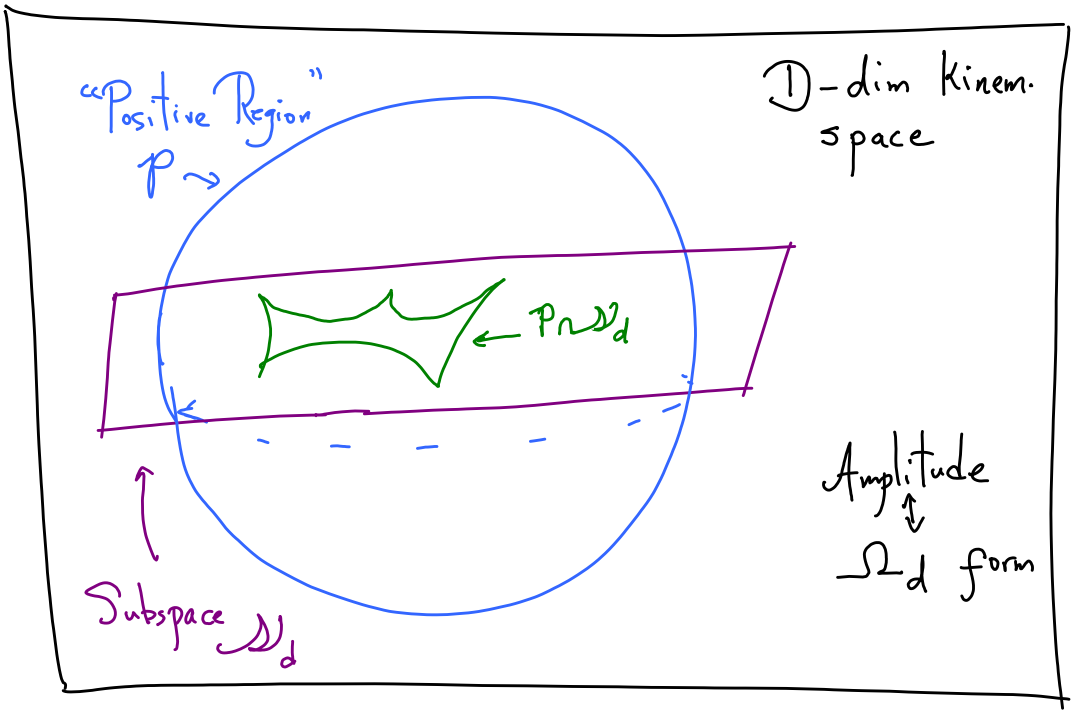
"Push-Forward"



$$\Omega_{\mathcal{P}_2}[y] = \sum_{\phi^{-1}(y)} \Omega_{\mathcal{P}_1}[\phi^{-1}(y)]$$

Magic: single real solution $x_R \in \mathcal{P}_1$ iff $y_R \in \mathcal{P}_2$

General Picture



Ω_d fixed thusly:

$\mathcal{S}_d$ intersects $\mathcal{P}$ in a d-dimensional positive geometry.

Ω_d, pulled back to $\mathcal{S}_d$, is the associated canonical form

Scattering Forms in Planar $\mathcal{N}=4$ SYM

Usually : $\mathcal{M}_{n,K}[Z_a^I, \eta_a^I]$ e.g. $\dfrac{(\langle Z_1 Z_2 Z_3 Z_4\rangle \eta_5 + \cdots)^4}{\langle Z_1 Z_2 Z_3 Z_4\rangle \cdots \langle Z_5 Z_1 Z_2 Z_3\rangle}$

4 : Mom. Twistors

4: Grassmann η's

weight 4K

$\eta_a^I \to dZ_a^I$

Now : Just replace $\eta_a^I \longrightarrow dZ_a^I$

$$\boxed{\mathcal{M}_{n,K} \longrightarrow \Omega_{4K}[Z_a]}$$

$d\log \dfrac{\langle 2345\rangle}{\langle 1234\rangle} \wedge \cdots \wedge d\log \dfrac{\langle 5123\rangle}{\langle 1234\rangle}$

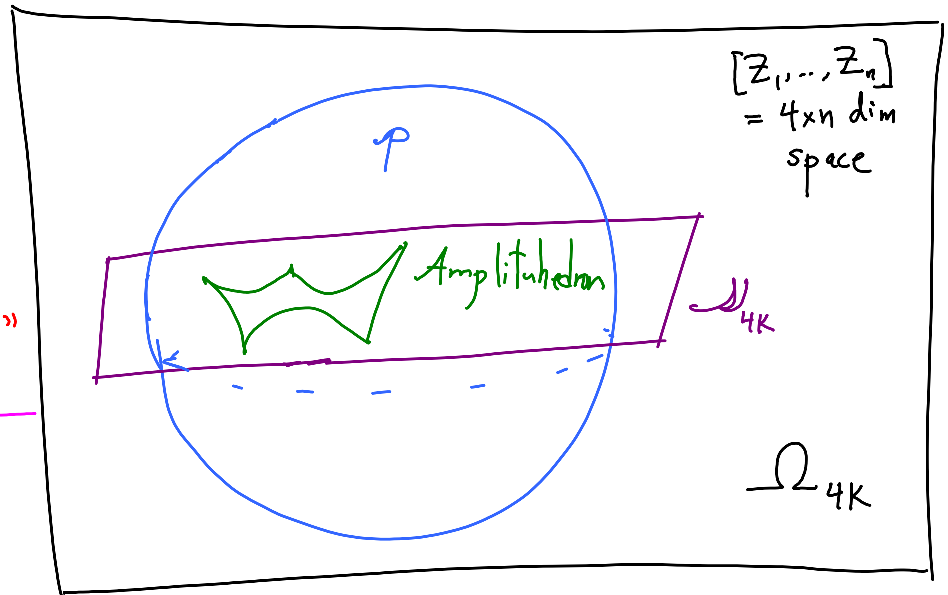
exposes underlying positive geometry!

{ Pushforward from Pos. Grassmannian, SUSY + … follows }

$\mathcal{P}$: Config. of $\{Z_1, \dots, Z_n\}$ has fixed "binary code"

$\Rightarrow$ physical poles > 0

$+$ maximal "winding #"



In full:

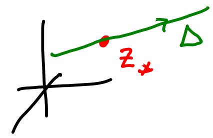
$\langle i\, i{+}1\; j\, j{+}1 \rangle > 0$

$\{\langle 1234 \rangle, \dots, \langle 123n \rangle\}$ has k sgn flips

$\langle AB_\alpha\, i\, i{+}1 \rangle > 0$, $\langle AB_\alpha AB_\beta \rangle > 0$

$\{\langle AB_\alpha 12 \rangle, \dots, \langle AB_\alpha 1n \rangle\}$ has $K+2$ sgn flips

$\mathcal{S}_{4K}$: Affine subspace



$$Z^I_a = Z^I_{*a} + y^I_\alpha \Delta^\alpha_a$$

$$\begin{pmatrix} Z_* \\ \hline \Delta \end{pmatrix} \cap G_+(4+K, n)$$

Scattering Forms in Momentum Space

$\Omega = \mathcal{M}^+ \, d^2\lambda + \mathcal{M}^- \, d^2\tilde{\lambda} + \cdots$ [Can do even for non-SUSY QED, QCD !]

(not just planar) $\mathcal{N}=4$, Parity-symmetric on-shell superspace:

$$\mathcal{M}[\lambda_a, \tilde{\lambda}_a, \eta_a, \tilde{\eta}_a] \xrightarrow[\tilde{\eta}_a^{\dot{\alpha}} \to d\tilde{\lambda}_a^{\dot{\alpha}}]{\eta_a^{\alpha} \to d\lambda_a^{\alpha}} (Q)^4 \; \Omega_{(2n-4)}$$

$Q \to \sum_a \lambda_a d\tilde{\lambda}_a = -\sum_a \tilde{\lambda}_a d\lambda_a$

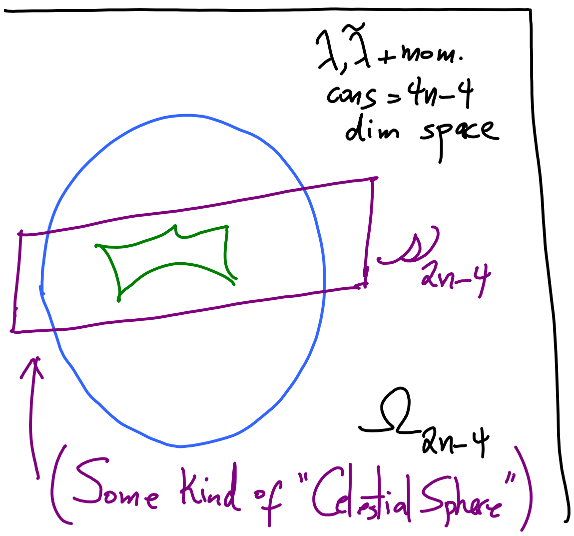
$\Omega_{(2n-4)}$: dlog form, e.g. $\Omega_{n=4} = d\log \frac{\langle 12 \rangle}{\langle 13 \rangle} \cdot\cdot \, d\log \frac{\langle 14 \rangle}{\langle 13 \rangle}$

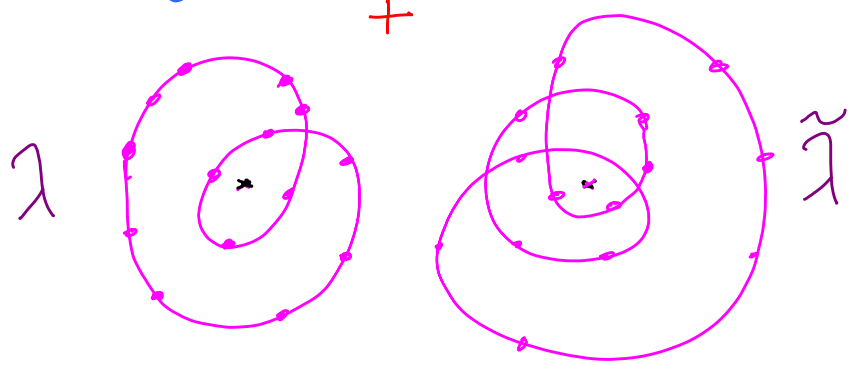
{ Again, ↑ follows from pushforward from Pos. Grassmannian }

The Tree Amplituhedron Avatar in Mom. Space

$\{Z_1, \ldots, Z_n\}$ "correct binary code" $\longrightarrow$ Phys. Poles $(p_i + p_{i+1} + \cdots + p_j)^2 > 0$

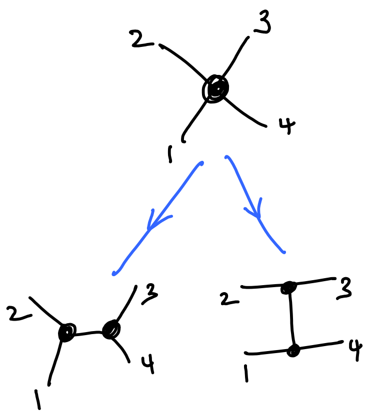
$+$

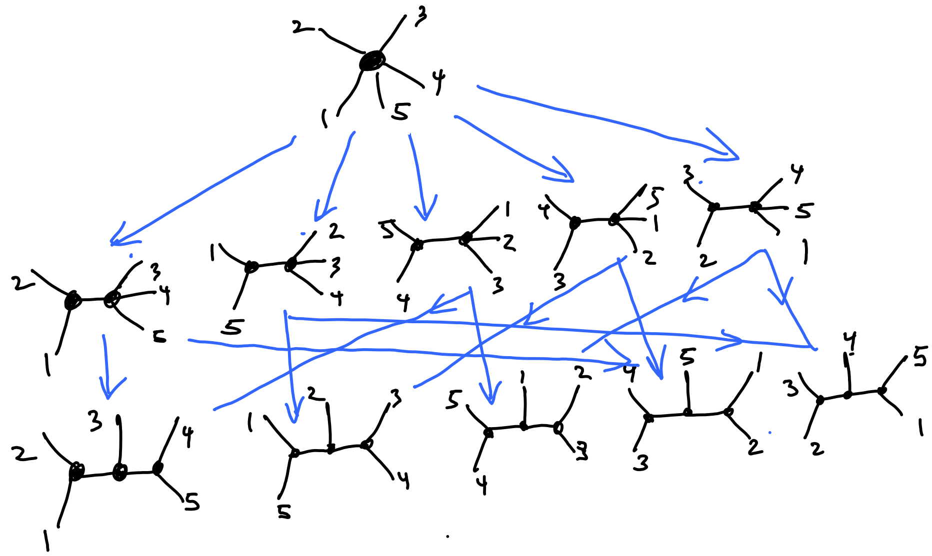




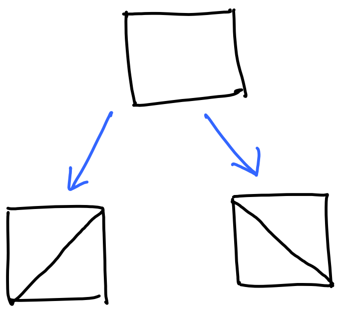
	K		$K+2$
winding #	$n-K-2$	or	$n-K-4$

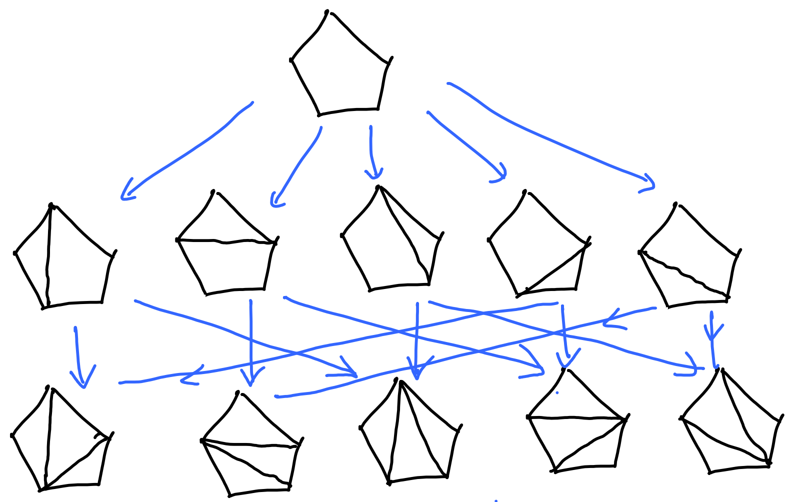
The Associahedron





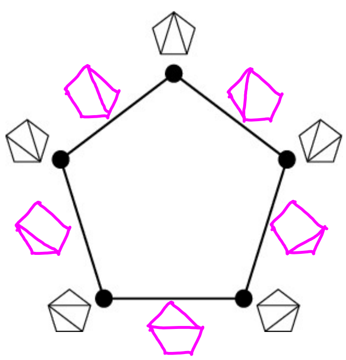
The Associahedron



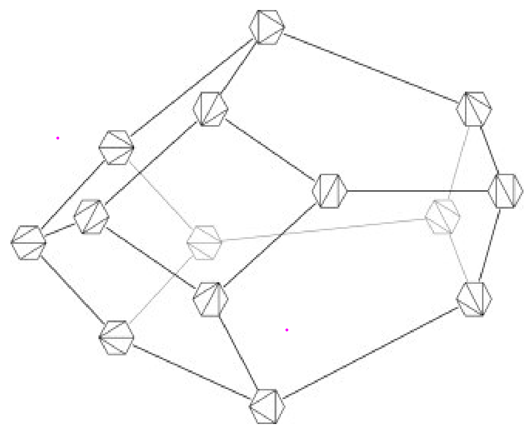


Quite beautifully, this network of inclusion relationships is that of a convex polytope in $(n-3)$ dimensions:

$n = 4$



$n = 5$

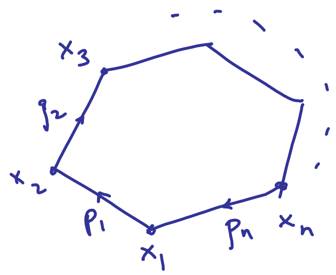


$n = 6$

Mandelstam Kinematics

* n particles ; $s_{ab} = p_a \cdot p_b$; $\sum_{b \neq a} s_{ab} = 0$; $\binom{n}{2} - n = \underbrace{\frac{n(n-3)}{2}}_{\text{dim space.}}$

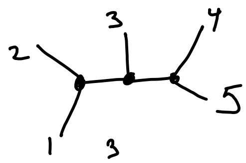
* Order



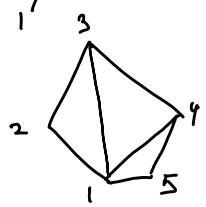
$p_a^\mu = x_{a+1}^\mu - x_a^\mu$

$X_{ij} \equiv (x_i - x_j)^2 = (p_i + \cdots + p_{j-1})^2$

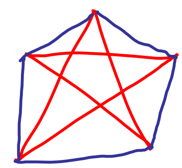
*



$\frac{1}{s_{12} \; s_{123}}$



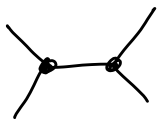
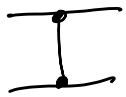
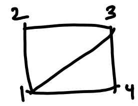
$\frac{1}{X_{13} \; X_{14}}$

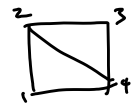


$\#$ diagonals $= \frac{n(n-3)}{2}$

X_{ij} a basis for kinematic space

Scattering Form



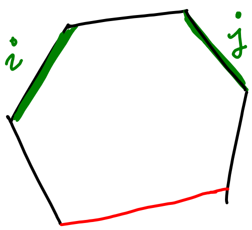
$$+\frac{ds}{s} \qquad -\frac{dt}{t} \quad = \quad +\frac{dX_{13}}{X_{13}} \qquad -\frac{dX_{24}}{X_{24}}$$

$$\Omega_{(n-3)} = \sum_{T} \underset{\substack{\uparrow \\ \text{later}}}{(\pm)} \prod_{(ab)\in T} \frac{dX_{ab}}{X_{ab}}$$

Associahedron In Mandelstam Space!

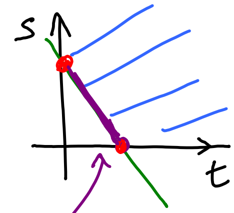
"Positive Region" $X_{ij} \geq 0$ $\{\text{All poles} \geq 0\}$

Linear Subspace:



$-2p_i \cdot p_j = c_{ij}$ fixed, with $c_{ij} > 0$.

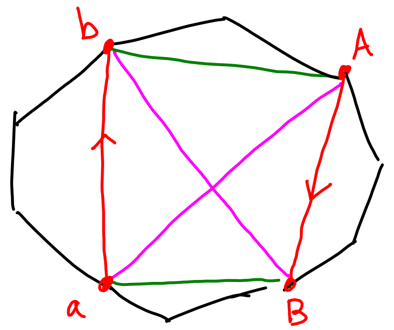
The intersection of this Subspace with Positive Region is $(n-3)$ dimensional — and is an associahedron!



$-u = s + t = c > 0$

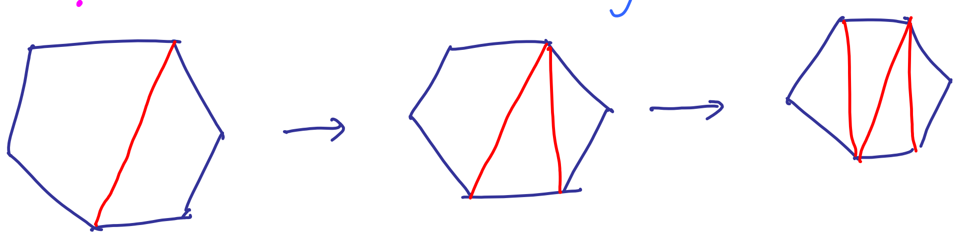
$n = 4$ Assoc.

Why?



$$-2\, p_{a\to b} \cdot p_{A\to B} = \sum -2 p_i \cdot p_j > 0$$
$$\parallel$$
$$X_{bB} + X_{aA}$$
$$- X_{bA} - X_{aB}$$

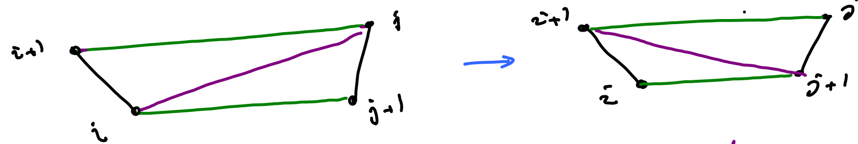
So, we can't set crossing diagonals X_{bB}, X_{aA} both to 0! Thus boundaries correspond to triangulations,



precisely defn of associahedron!

Pull-back of $\Omega_{(n-3)}$ on Subspace

Note



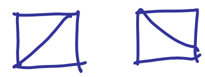
But $dX_{i+1,j} + dX_{i,j+1} - dX_{i,j} - dX_{i+1,j+1} = 0$

So $dX_{i+1,j} \wedge dX_{i,j+1} \wedge dX_{i,j} = - dX_{i+1,j} \wedge dX_{i,j+1} \wedge dX_{i+1,}$

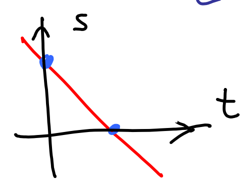
$$\Longrightarrow \Omega_{(n-3)}\Big|_{\text{subspace}} = \left(\prod_{i=3}^{n-1} dX_{1i}\right) \times \Big[\underbrace{\text{Sum of all Cubic Graphs}}_{\text{Amplitude!}}\Big]$$

= Canonical Form of Associahedron "

ex:;



$$\Omega = \frac{ds}{s} - \frac{dt}{t}$$



$s + t = c$

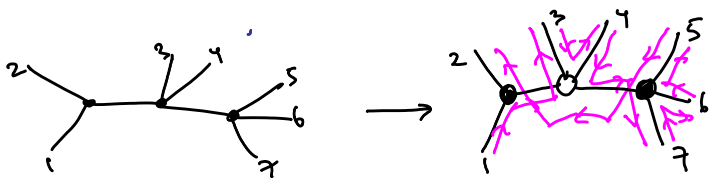
$\Omega|_{\text{subspace}}$

$$= ds\underbrace{\left[\frac{1}{s} + \frac{1}{t}\right]}_{\text{Amp}}$$

What's the theory? Bi Colored $(\phi_{aA})^3$

Double partial amps, $\{1\,2 \cdots n \,|\, \pi_1 \pi_2 \cdots \pi_n\}$ Where is this?

Facets:



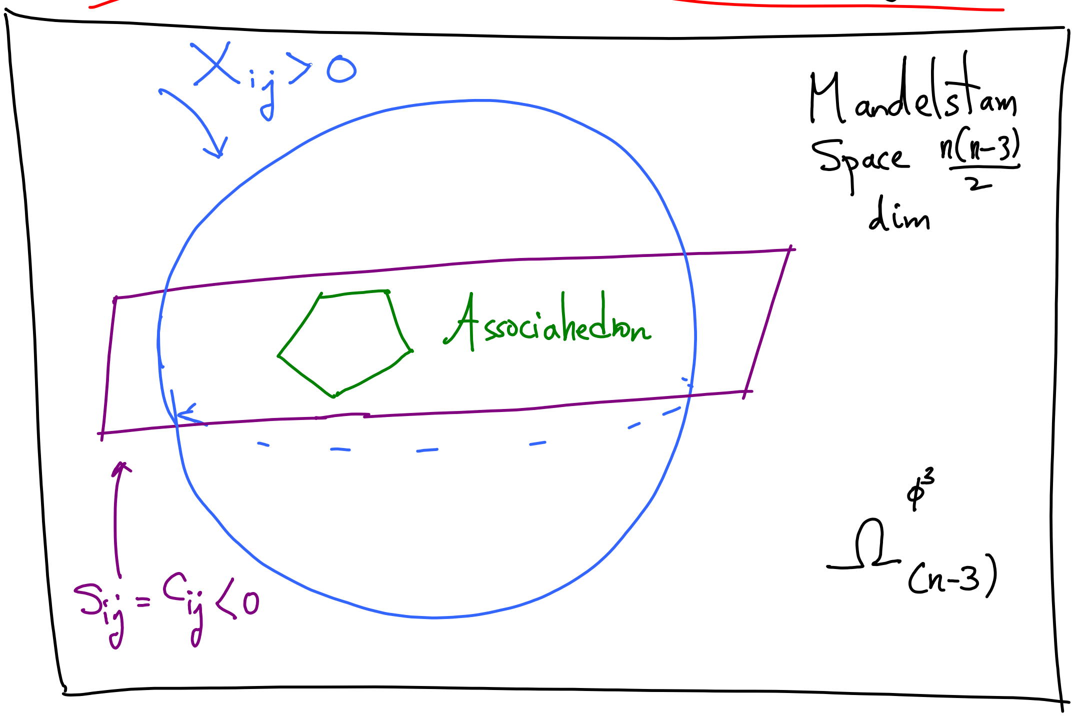
Left-Right Paths Give permutation $(\pi_1 \pi_2 \cdots \pi_n)$!

Subspace: "zoom into" facet, send rest of assoc $\longrightarrow \infty$

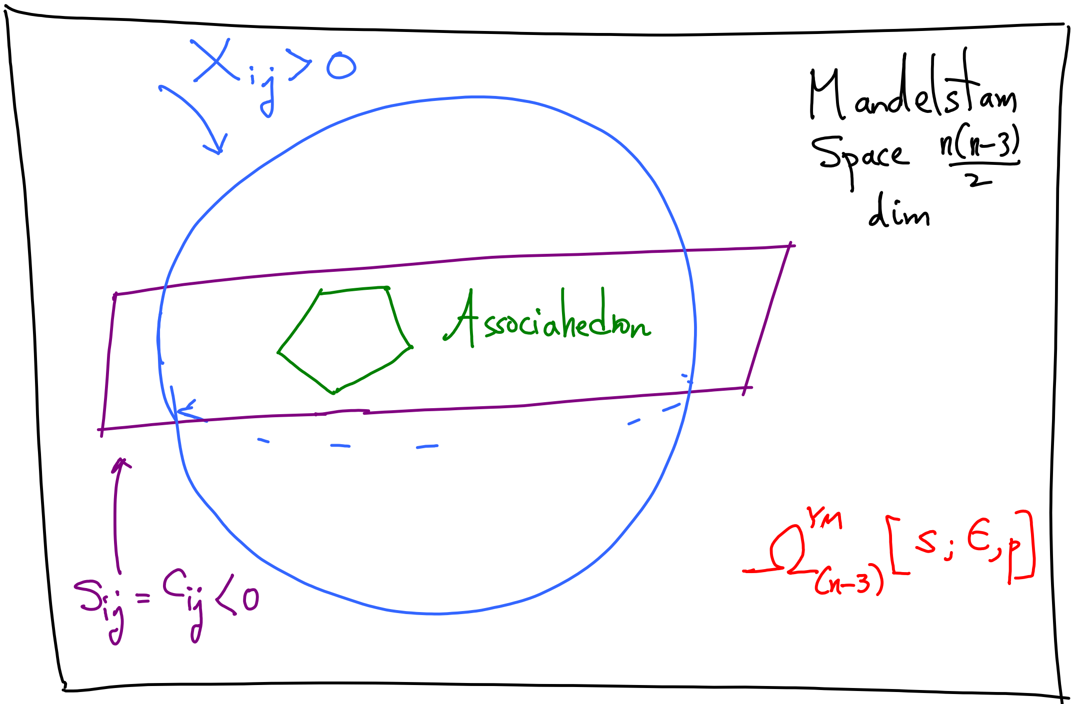
Pull-back of $\underline{\text{same}}$ $\Omega_{(n-3)}\Big| = \left(\prod dX\right) \times \{1 2 \cdots n \,|\, \pi_1 \cdots \pi_n\}$

Double-Partial Amp

Bi-Colored $(\phi_{aA})^3$ theory



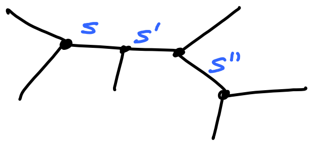
Gluons



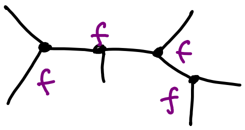
* Perm. inv
* On-shell GI
* Minimal pole @ ∞

Ω^{YM} is Unique!
{non-trivial!}

Color IS Kinematics (Form)

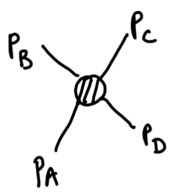


$\rho \equiv ds \wedge ds' \wedge ds''$



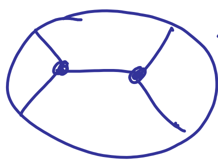
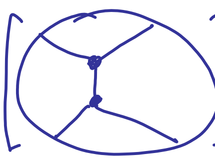
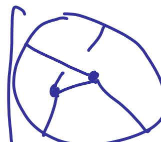
$f\,f\,f\,f$

Exactly the same Algebraic Relations

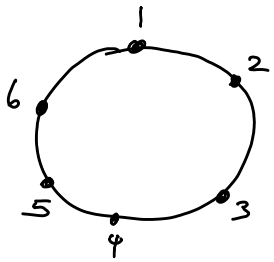
Why? 

$$(p_A+p_B)^2+(p_A+p_C)^2+(p_A+p_D)^2 = p_A^2+p_B^2+p_C^2+p_D^2$$

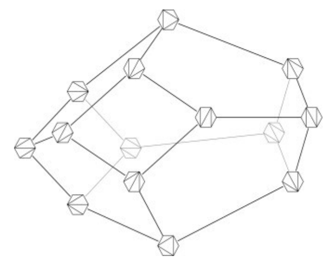
So $\left[d(p_A+p_B)^2+d(p_A+p_C)^2+d(p_A+p_D)^2\right] \wedge dp_A^2\, dp_B^2\, dp_C^2\, dp_D^2 = 0$

Hence ρ [] $+\rho$ [] $+\rho$ [] $= 0$!

Worldsheet as Positive Geometry I : Associahedron



$\longleftrightarrow$



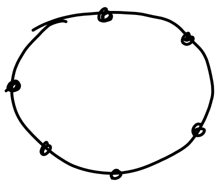
$$2\updownarrow \underbrace{\begin{pmatrix} \sigma_1 & \cdots & \sigma_n \end{pmatrix}}_{\longleftarrow\, n \,\longrightarrow} \Big/ \begin{matrix} GL(1)^n \\ \times SL(2) \end{matrix}$$

$$\langle \sigma_a \sigma_b \rangle > 0 \quad a < b$$

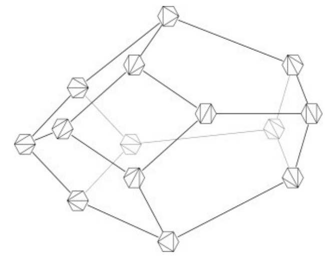
$$\Omega^{WS}_{(n-3)} = \frac{d^2\sigma_1 \cdots d^2\sigma_n}{(12)(23)\cdots(n1)} \Big/ \begin{matrix} GL(1)^n \\ \times SL(2) \end{matrix}$$

Pushing Forward From Worldsheet

WS Assoc.

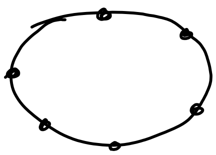


Mandelstam Assoc.



Pushing Forward From Worldsheet

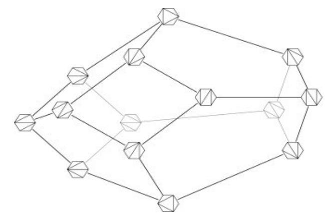
WS



CHY Scattering Eqns!

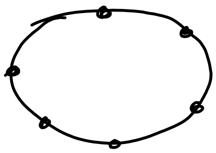
$$\sum_{b \neq a} s_{ab} \frac{\sigma_a^{\alpha} \sigma_b^{\beta}}{(\sigma_a \sigma_b)} = 0$$

Mandelstam



On $(n-3)$ mandelstam subspace $\rightarrow$ solve for s's in terms of σ's $\rightarrow$ Mandelstam associahedron is literally image of WS! Conversely: single real solution for σ's in WS assoc, iff s_{ab} live in some Mandelstam assoc.

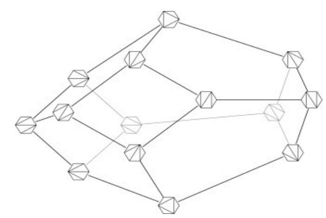
Pushing Forward From Worldsheet



CHY Scattering Eqns!

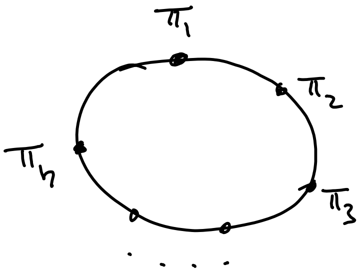
$$\xrightarrow{\hspace{6cm}}$$

$$\sum_{b\neq a} s_{ab} \frac{\sigma_a^{\alpha} \sigma_b^{\beta}}{(\sigma_a \sigma_b)} = 0$$



$$\Omega^{WS}_{(n-3)} \xrightarrow{\text{Push forward}} \Omega^{(\phi_{aA})^3}_{(n-3)}$$

= CHY formula



$$\tilde{\Omega}^{WS}_{(n-3)} = \sum_{\pi} \Omega^{WS}_{(n-3)}[\pi_1, \ldots, \pi_n] \, \mathcal{N}(\epsilon, p)$$

$$\Big\downarrow$$

Push forward on SE

Demand minimal # p + On-shell Gauge Inv.

$\tilde{\Omega}^{WS}_{(n-3)}$ is Unique! + Pushes forward to $\Omega^{YM}_{(n-3)}$

$\tilde{\Omega}^{WS} = \mathrm{P_f}[M]$ big + mysterious part of CHY magic $\longleftrightarrow$ But locked by G.I.

Assoc. is a baby version of simplest positive Grassmannian $G_+(2,n)$:

Assoc

$$\begin{pmatrix} \sigma_1 & \cdots & \sigma_n \end{pmatrix} \Big/ GL(1)^n \times SL(2), \quad \underset{\text{dim}}{(n-3)}$$

$$(\sigma_a\, \sigma_b) > 0 \quad a < b$$

$$\Omega_{(n-3)} = \frac{d^2\sigma_1 \cdots d^2\sigma_n}{(12) \cdots (n1)} \Big/ GL(1)^n \times SL(2)$$

$G_+(2,n)$

$$\begin{pmatrix} \sigma_1 & \cdots & \sigma_n \end{pmatrix} \Big/ GL(2), \quad \underset{\text{dim}}{(2n-4)}$$

$$(\sigma_a\, \sigma_b) > 0 \quad a < b$$

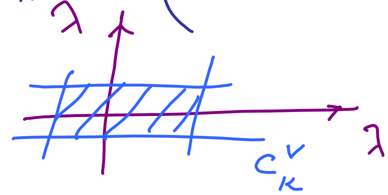
$$\Omega_{(2n-4)} = \frac{d^2\sigma_1 \cdots d^2\sigma_n}{(12) \cdots (n1)} \Big/ GL(2)$$

Worldsheet as Pos. Geometry II : $G_+(2,n)$

In 4d, CHY + Twistor Strings become the same, WS becomes $G_+(2,n)$

$$\sigma = \begin{pmatrix} a \\ b \end{pmatrix} \xrightarrow{\text{Veronese}} \sigma_V = \begin{pmatrix} a^k \\ a^{k-1} b \\ \vdots \\ b^k \end{pmatrix} \; ; \quad C^{V}_{k\,\tilde{\lambda}} = \begin{pmatrix} \sigma_V^{(1)} \cdots & \sigma_V^{(n)} \end{pmatrix}$$

"RSV Eqns"



$$\Omega^{WS}_{(2n-4)} = \frac{d^2\sigma_1 \cdots d^2\sigma_n}{(12)\cdots(n1)} \Big/ GL(2) \xrightarrow{\text{Push Forward}} \Omega^{SYM}_{(2n-4)}[\lambda, \tilde{\lambda}]$$

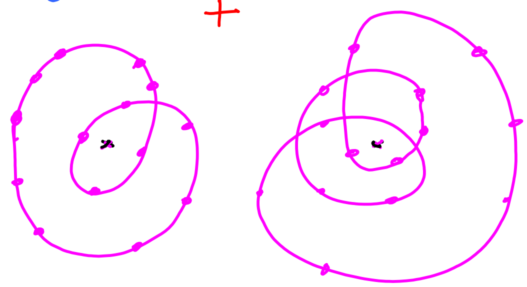
Twistor String WS $\xrightarrow[\text{Pushforward}]{\text{RSV}}$ Mom. Space Amplituhedron!

Strong Evidence:

RSV equations have single, positive soln in WS $G_+(2,n)$ iff

Phys. Poles $(p_i+p_{i+1}+\dots+p_j)^2>0$

+

λ  $\tilde{\lambda}$

winding #	K		$K+2$
	$n-K-2$	or	$n-K-4$

Q: Why pushforward Ω^{WS}? It's a form, why not integrate it inside WS positive geometry?

A: It is logarithmically divergent!

Canonical way of dealing w/ this [c.f. Gelfand et. al. gen. hypergeometric functions]

$$\Omega^{WS}_{\alpha'} = \Omega^{WS} \times \underbrace{\prod_{(ab)} (\sigma_a \sigma_b)^{\alpha' s_{ab}}}_{\text{Koba-Nielsen}}$$

$\left\{ \sum_{b \neq a} s_{ab} = 0 \text{ for } SL(2) \text{ weight} \right\}$

So e.g. take $\tilde{\Omega}^{WS}$, whose "Pfaffian" structure was completely fixed by G.I. on support of Scatt Eqn.

Then

$$(\alpha')^{n-3} \int_{\text{Assoc}(12\cdots n)} \tilde{\Omega}^{WS}_{\alpha'} = \text{Open Superstring Gluon Amplitude!}$$

Units

Unique Gauge Invariants

Amplitudes as Canonical Forms of Positive Geometries

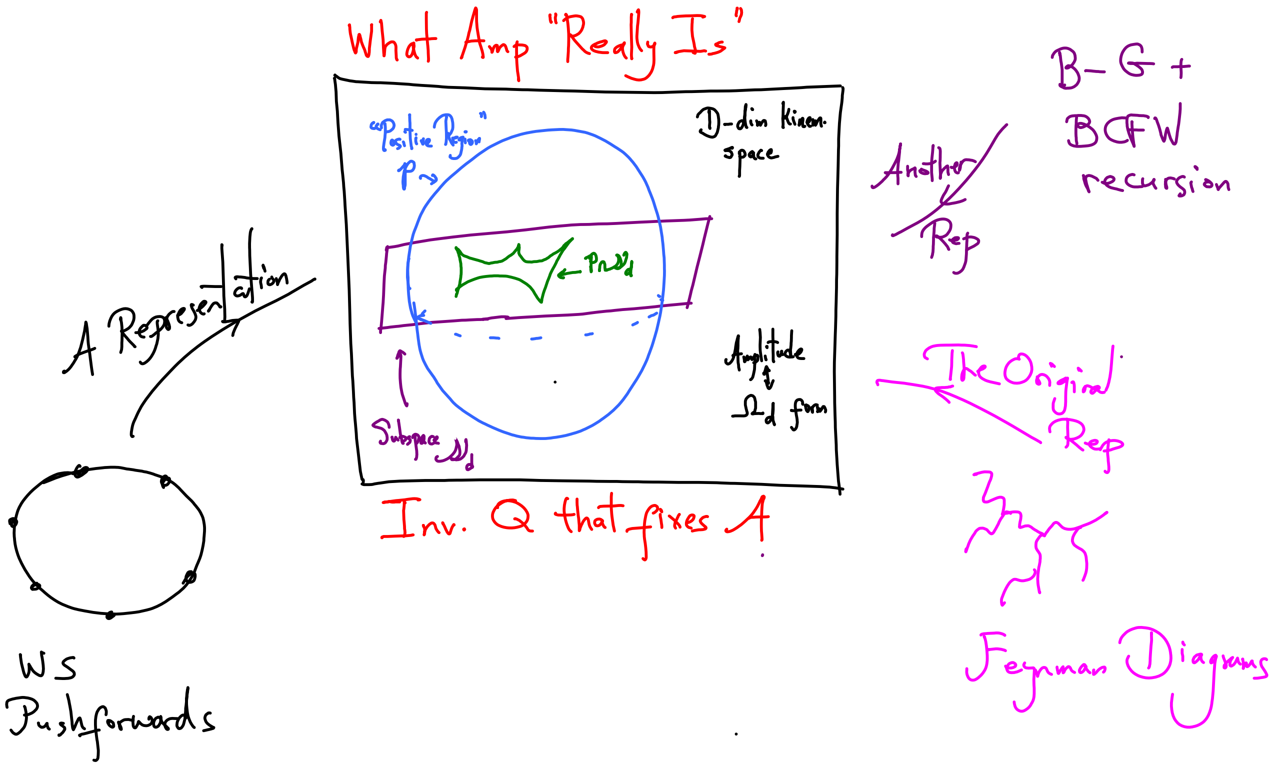
Amplituhedron

Pert. String Amps

Scattering Eqns

Twistor String

Worldsheets as Positive Geometries


What Amp "Really Is"
"Positive Region" P
D-dim Kinem. space
P ∩ S_d
Amplitude
Ω_d form
Subspace S_d
Inv. Q that fixes A!
A Representation
WS Pushforwards
Another Rep
B-G + BCFW recursion
The Original Rep
Feynman Diagrams

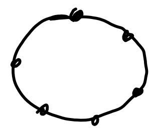
{ Of course the invariant picture makes impossible calculations trivial too — e.g. new picture of Amplituhedron lets you compute in a few lines, infinite classes of integrand cuts, for arbitrarily large # of particles n and loops L }

q + Q

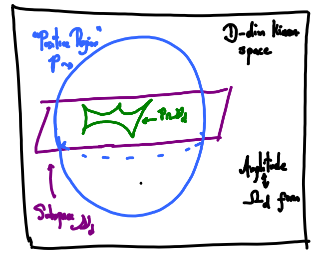
q: What is WS push-forward for original Mom-Twistor Amplituhedron?

[Very likely related to still somewhat mythical "Dual Amplituhedron"

Q:



CHY →



Koba-Nielsen α' deform.

Superstring Amps →

What are String Amps "Really"?