How to Succeed at Holographic Correlators without Really Trying

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Computing efficiently in AdS/CFT

Still far from harnessing the full computational power of AdS/CFT.

Dramatic illustration:

In $\mathcal{N} = 4$ SYM, correlators $\langle \mathcal{O}_{p_1} \dots \mathcal{O}_{p_n} \rangle$ of one-half BPS local operators

$$\mathcal{O}_p^{I_1 \dots I_p}(x) = \mathrm{Tr}\, X^{\{I_1} \dots X^{I_p\}}(x) \,, \quad I_k = 1, \dots 6 \,.$$

Trivial for $n = 2, 3$ but wild for $n \geqslant 4$.

Seemingly very hard even for $N \to \infty$, $\lambda \to \infty$, the regime of classical IIB supergravity on $AdS_5 \times S^5$.

Classical sugra is a complicated non-linear theory.

Prior to our work, only very few holographic correlators were known, even for the most susy backgrounds, and despite heroic efforts:

- For $AdS_5 \times S^5$:
  - Three cases with $p_i = p$, namely $p = 2, 3, 4$
    Arutyunov Frolov, Arutyunov Dolan Osborn, Arutyunov Sokatchev
  - The class $p_1 = n + k$, $p_2 = n - k$, $p_3 = p_4 = k + 2$
    ("next-to-next-to-extremal" cases) Uruchurtu

- For $AdS_7 \times S^4$: Only $p_i = 2$ (supergraviton) Arutyunov Sokatchev

- For $AdS_3 \times S^3 \times \mathcal{M}_4$: None
  [But see recent indirect results for $\langle \mathcal{O}_H \mathcal{O}_H \mathcal{O}_L \mathcal{O}_L \rangle$ Galliani Giusto Russo]

- For $AdS_4 \times S^7$: None

A new approach

In this talk, I will describe a new approach to holographic correlators.

- Conceptually, on-shell "bootstrap" approach, inspired by modern methods for perturbative scattering amplitudes.

  The diagrammatic expansion hides the true simplicity of the final on-shell result. Instead, directly fix the full answer by consistency.
- Technically, Mellin representation of CFT correlators.

  Mack, Penedones, ...

Our principal new result is a very simple explicit formula for $\langle \mathcal{O}_{p_1} \mathcal{O}_{p_2} \mathcal{O}_{p_3} \mathcal{O}_{p_4} \rangle$ in $\mathcal{N} = 4$ SYM, in the sugra limit, for arbitrary p_i.

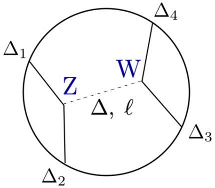
Partial results for other backgrounds.

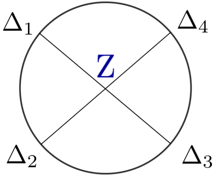
Strong hints of hidden elegant structure.

Review of traditional method

To leading $O(1/N^2)$ order,

$$\mathcal{A}_{\text{sugra}} = \mathcal{A}_{\text{exchange}} + \mathcal{A}_{\text{contact}}$$





- External legs: bulk-to-boundary propagators $K_{\Delta_i}(x_i, Z)$.
- Internal legs: bulk-to-bulk propagators $G_{\Delta,\ell}(Z, W)$.
- Vertices: effective action obtained by KK reduction of IIB sugra on S^5.

Contact diagrams with no derivatives are knowns as D-functions,

$$D_{\Delta_1\Delta_2\Delta_3\Delta_4} = \int [dZ]\; K_{\Delta_1}(x_1, Z) K_{\Delta_2}(x_2, Z) K_{\Delta_3}(x_3, Z) K_{\Delta_4}(x_4, Z)\,.$$

D_{1111} is the scalar box integral in $4d$; the higher D-functions are obtained by taking derivatives in x_{ij}^2.

These are the basic building blocks. Remarkably, all exchange diagrams that occur in $AdS_5 \times S^5$ can be written as finite sums of $D_{\Delta_1\Delta_2\Delta_3\Delta_4}$. D'Hoker Freedman LR, "How to succeed at z-integrals without really trying".

Example: scalar exchange.
For external dimensions $\Delta_i = p$, internal dimension Δ (always even),

$$\mathcal{A}^{\Delta,\ell=0}_{p\,p\,p\,p}(x_1, x_2, x_3, x_4) = \sum_{k=\Delta/2}^{p-1} a_k \, |x_{12}|^{-2p+2k} D_{k\,k\,p\,p}(x_1, x_2, x_3, x_4)$$

field	s_k	$A_{\mu,k}$	$C_{\mu,k}$	ϕ_k	t_k	$\varphi_{\mu\nu,k}$
irrep	$[0,k,0]$	$[1,k-2,1]$	$[1,k-4,1]$	$[2,k-4,2]$	$[0,k-4,0]$	$[0,k-2,0]$
m^2	$k(k-4)$	$k(k-2)$	$k(k+2)$	k^2-4	$k(k+4)$	k^2-4
Δ	k	$k+1$	$k+3$	$k+2$	$k+4$	$k+2$
twist τ	k	k	$k+2$	$k+2$	$k+4$	k

Sugra modes exchanged in the holographic one-half BPS four-point function.
Twist cut-off: $\tau \leqslant \min\{p_1+p_2, p_3+p_4\} - 2$ for an s-channel exchange.

Heroic calculation of quartic couplings by Arutyunov and Frolov:
15 pages just to write them down.

Proliferation of diagrams and tedious combinatorics make this very hard already at $p_i \sim 4$.

With rearrangements, some hopeful hints of simplication:
1, 4, 14 Ds for $p = 2, 3, 4$, but answers completely non-transparent.

Kinematics

Eliminate $SO(6)$ indices by contracting them with a null vector

$$\mathcal{O}_p(x,t) = t_{I_1} \dots t_{I_p} \, \mathcal{O}_p^{I_1 \dots I_p}(x) \,, \quad t \cdot t = 0 \,.$$

For simplicity, I will focus on $p_i = p$ in this talk.

Using bosonic subgroup $SU(2,2) \times SU(4) \subset PSU(2,2|4)$,

$$\langle \mathcal{O}_p(x_1,t_1) \dots \mathcal{O}_p(x_4,t_4) \rangle = \left(\frac{t_{12} t_{34}}{x_{12}^2 x_{34}^2} \right)^p \mathcal{G}(U,V;\sigma,\tau) \,,$$

where $x_{ij} = x_i - x_j$, $t_{ij} = t_i \cdot t_j$ and

$$U = \frac{(x_{12})^2 (x_{34})^2}{(x_{13})^2 (x_{24})^2} \,, \qquad V = \frac{(x_{14})^2 (x_{23})^2}{(x_{13})^2 (x_{24})^2}$$

$$\sigma = \frac{t_{13} t_{24}}{t_{12} t_{34}} \,, \qquad \tau = \frac{t_{14} t_{23}}{t_{12} t_{34}} \,.$$

Note that $\mathcal{G}(U,V;\sigma,\tau)$ is a polynomial of degree p in σ and τ,

$$\mathcal{G}(U,V;\sigma,\tau) = \sum_{0 \leqslant m+n \leqslant p} \sigma^m \tau^n \mathcal{G}^{(m,n)}(U,V) \,.$$

We should finally impose the constraints from the fermionic symmetries.

In position space, the superconformal Ward identity reads
Eden Petkou Schubert Sokatchev, Nirschl Osborn

$$\partial_{\bar{z}}[\mathcal{G}(z\bar{z},(1-z)(1-\bar{z});\alpha\bar{\alpha},(1-\alpha)(1-\bar{\alpha}))\big|_{\bar{\alpha}\to 1/\bar{z}}]=0\,,$$

where we have performed the useful change of variables
$U=z\bar{z}$, $V=(1-z)(1-\bar{z})$, $\sigma=\alpha\bar{\alpha}$, $\tau=(1-\alpha)(1-\bar{\alpha})$.

The solution is

$$\mathcal{G}(U,V;\sigma,\tau)=\mathcal{G}_{\rm free}(U,V;\sigma,\tau)+R\;\mathcal{H}(U,V;\sigma,\tau)$$

$$R=(1-z\alpha)(1-\bar{z}\alpha)(1-z\bar{\alpha})(1-\bar{z}\bar{\alpha})\,.$$

All dynamical information is contained in $\mathcal{H}(U,V;\sigma,\tau)$.

A position space method

Write an ansatz as a *finite* sum of D-functions,

$$\mathcal{A}_{\rm sugra} = \mathcal{A}_{\rm exchange} + \mathcal{A}_{\rm contact} = \sum a_{ijkl}(\sigma,\tau)\, D_{ijkl}(U,V)$$

$$= R_\Phi\, \Phi(U,V) + R_U \log U + R_V \log V + R_1$$

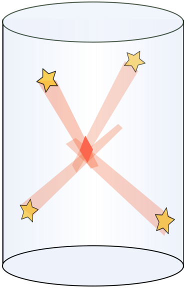
$\Phi(U,V) \equiv$ scalar box function,
$R_{\Phi,U,V,1} \equiv$ *rational* functions of U and V, polynomials in σ and τ, depending on a finite set of undetermined parameters.

Parameters uniquely fixed by imposing the superconformal Ward identity.
No details of sugra effective action needed. Easier than standard method.

Known results reproduced. New result for $p_i = 5$.
But still too hard for larger p_i.

CFT correlators as AdS scattering amplitudes

CFT correlators are the best analog we have in AdS for an S-matrix. They are on-shell objects. Mellin space makes this analogy manifest.



Scattering in AdS

Penedones, Fitzpatrick Kaplan Penedones Raju, Paulos

Mellin representation of CFT correlators

$$G(x_i) = \langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\dots\mathcal{O}_n(x_n)\rangle_{\text{conn}} = \int [d\delta_{ij}]\, M(\delta_{ij})(x_{ij}^2)^{-\delta_{ij}}\,.$$

The integration variables obey the constraints

$$\delta_{ij} = \delta_{ji}\,, \quad \delta_{ii} = -\Delta_i\,, \quad \sum_{j=1}^{n} \delta_{ij} = 0$$

$M(\delta_{ij})$ is the so-called reduced Mellin amplitude [Mack].

The operator product expansion

$$\mathcal{O}_1(x_1)\mathcal{O}_2(x_2) = \sum_i \frac{C_{12i}}{(x_{12}^2)^{\frac{\Delta_1+\Delta_2-\Delta_i}{2}}}\left(\mathcal{O}_i(x_2) + \text{descendants}\right)$$

implies that $M(\delta_{ij})$ has simple poles at

$$\delta_{12} = \frac{\Delta_1 + \Delta_2 - (\Delta_i + 2n)}{2}$$

with residue $\sim C_{12i}\,\langle \mathcal{O}_i\mathcal{O}_3\dots\mathcal{O}_n\rangle$. Poles with $n > 0$ come from descendants.

Factorization properties analogous to tree-level scattering in flat space.

In 4pt case, two independent variables. We can use "Mandelstam" invariants

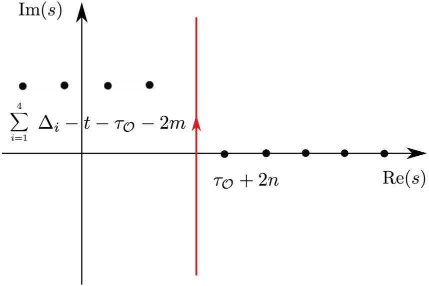
$$s = \Delta_1 + \Delta_2 - 2\delta_{12}\,, \quad t = \Delta_1 + \Delta_4 - 2\delta_{14}, \quad u = \Delta_1 + \Delta_3 - 2\delta_{13},$$

with $s + t + u = \sum_i \Delta_i$.

- $M(s,t)$ has the usual crossing symmetry properties of a $2 \to 2$ S-matrix.
- Primary operator $\mathcal{O}$ in the s-channel OPE $(x_{12} \to 0) \to$ poles at

$$s_0 = \tau_{\mathcal{O}} + 2n$$

$$\tau \equiv \Delta - J \text{ is the twist}$$



Mellin amplitude at large N

Now define the Mellin amplitude $\mathcal{M}$ by [Mack]

$$M(\delta_{ij}) = \mathcal{M}(\delta_{ij}) \prod_{i<j} \Gamma[\delta_{ij}]\,.$$

The factors of Gamma's are such that $\mathcal{M}$ has polynomial residues.

This is a particularly natural definition at large N. The explicit poles in the Gamma's account precisely for the double-trace contributions. [Penedones]

For example, $\Gamma(\delta_{12})$ gives poles corresponding to the double-traces

$$\mathcal{O}_{\Delta_1} \partial^J \Box^n \mathcal{O}_{\Delta_2}\,,$$

of twist $\tau = \Delta_1 + \Delta_2 + 2n + O(1/N^2)$.

In the planar limit, $\mathcal{M}$ is meromorphic, with only single-trace poles.

Mellin amplitudes for Witten diagrams

$D_{\Delta_1\Delta_2\Delta_3\Delta_4}$ has $\mathcal{M} = 1$.

The simplification of sugra calculations in Mellin space descends from this fact.

Contact diagrams with $2k$ spacetime derivatives have $\mathcal{M} = P_k(s,t)$, polynomial of degree k. Penedones

Exchange diagrams (in the s-channel) take the general form

$$\mathcal{M}_{\Delta,J}(s,t) = \sum_{m=0}^{\infty} \frac{Q_{J,m}(t)}{s-(\Delta-J)-2m} + P_{J-1}(s,t)$$

for arbitrary internal dimension Δ and spin J.

When specialized to the quantum numbers of $AdS_5 \times S^5$, sum over m truncates at some $m_{\max}$. This is the translation of the fact that exchange diagrams are finite sums of D-functions.

This is actually necessary for consistency of the OPE interpretation. Single-trace poles must truncate before they overlap with the double-trace poles from $\Gamma(-s/2+p)^2$.

A double pole in s corresponds to a $\log U$ term in $\mathcal{G}(U,V)$, and is needed to account for the $O(1/N^2)$ anomalous dimensions of the double-trace operators. But a triple pole cannot arise at this order.

Conditions on $\mathcal{M}$ for 1/2 BPS correlator

$$\langle \mathcal{O}_p(x_1,t_1)\dots\mathcal{O}_p(x_4,t_4)\rangle = \left(\frac{t_{12}t_{34}}{x_{12}^2 x_{34}^2}\right)^p \mathcal{G}(U,V;\sigma,\tau)$$

$$\mathcal{G}(U,V;\sigma,\tau) \leftrightarrow \mathcal{M}(s,t;\sigma,\tau)$$

In the sugra limit $\mathcal{M}$ is a very constrained function.
Our goal is to characterize it from a set of abstract conditions:

1. Crossing symmetry

$$\sigma^p \mathcal{M}(u,t;1/\sigma,\tau/\sigma) = \mathcal{M}(s,t,;\sigma,\tau)$$
$$\tau^p \mathcal{M}(t,s;\sigma/\tau,1/\tau) = \mathcal{M}(s,t;\sigma,\tau)$$

2. Analytic properties:

$\mathcal{M}$ has a finite number of simple poles in s, t, u at $2,4,\dots 2p-2$.
The residue at each pole is a polynomial in the other variable.

3. Asymptotics for large s, t:

$$\mathcal{M}(\beta s, \beta t; \sigma, \tau) \sim O(\beta) \quad \text{for } \beta \to \infty \,.$$

Necessary for consistency of the flat space limit, which is dominated by graviton exchange. Obvious for exchange diagrams (since $J \leqslant 2$). True if contact diagrams with at most two derivatives contribute. Non-trivial but true, as confirmed by Arutyunov Frolov Klabbers Savin

4. Superconformal Ward identity:
Need to translate into Mellin space the solution of the Ward identity,

$$\mathcal{G}(U,V;\sigma,\tau) = \mathcal{G}_{\rm free}(U,V;\sigma,\tau) + R(U,V;\sigma,\tau)\,\mathcal{H}(U,V;\sigma,\tau)\,,$$

where R is a quadratic polynomial in U, V and in σ, τ.

$\mathcal{G} \to \mathcal{M}$, $\mathcal{G}_{\rm free} \to$ "zero", $\mathcal{H} \to \widetilde{\mathcal{M}}$ (new object). Hence:

$$\mathcal{M}(s,t;\sigma,\tau) = \widehat{R} \circ \widetilde{\mathcal{M}}(s,t;\sigma,\tau)$$

where $\widehat{R}$ is a certain difference operator shifting s and t.

Our solution

These conditions define a very constrained bootstrap problem. Experimentation at low p leads to the conjecture

$$\widetilde{\mathcal{M}}(s,t;\sigma,\tau) = \sum_{\substack{i+j+k=p-2 \\ 0 \leqslant i,j,k \leqslant p-2}} \frac{C_{pppp} \binom{p-2}{i\ j\ k}^2 \sigma^i \tau^j}{(s-s_M+2k)(t-t_M+2j)(\tilde{u}-u_M+2i)},$$

where $s_M = t_M = u_M = 2p-2$, $\tilde{u} \equiv u-2$.

This satisfies all conditions and reproduces the known sugra results for $p = 2, 3, 4, 5$. As simple as it could be.

We believe this is the unique solution but lack a complete proof.

General external weights

$$\widetilde{\mathcal{M}}(s,t;\sigma,\tau) = \sum_{\substack{i+j+k=L-2 \\ 0 \leqslant i,j,k \leqslant L-2}} \frac{a_{ijk}\,\sigma^i\tau^j}{(s-s_M+2k)(t-t_M+2j)(\tilde{u}-u_M+2i)}\,,$$

where

$$\begin{aligned} L &= \max\{p_4,(p_2+p_3+p_4-p_1)/2\} \\ s_M &= \min\{p_1+p_2,p_3+p_4\}-2 \\ t_M &= \min\{p_1+p_4,p_2+p_3\}-2 \\ u_M &= \min\{p_1+p_3,p_2+p_4\}-2 \end{aligned}$$

and

$$\begin{aligned} a_{ijk} &= \left(1+\tfrac{|p_1-p_2+p_3-p_4|}{2}\right)_i^{-1}\left(1+\tfrac{|p_1+p_4-p_2-p_3|}{2}\right)_j^{-1} \\ &\quad\times\left(1+\tfrac{|p_1+p_2-p_3-p_4|}{2}\right)_k^{-1}\binom{L-2}{i\ j\ k}\,C_{p_1p_2p_3p_4} \end{aligned}$$

The next-to-next extremal cases are correctly reproduced.

$AdS_7 \times S^4$

Structurally similar, but more involved.

- Position space method:
  $p_i = 2$ reproduced, new results for $p_i = 3, 4$.

- Mellin space method:
  $\mathcal{M}(s, t; \sigma, \tau)$ satisfies stringent constraints analogous to AdS_5 case, *e.g.*, a difference operator acting on an auxiliary amplitude $\widetilde{\mathcal{M}}$ must yield a certain pole structure. We believe that they fix it uniquely.

  General solution still missing.

  For $p_i = 2$ (supergraviton 4pt function),

$$\widetilde{\mathcal{M}}_2(s, t; \sigma, \tau) = \frac{8}{N^3(s-6)(s-4)(t-6)(t-4)(\tilde{u}-6)(\tilde{u}-4)}$$

  where $\tilde{u} = u - 6$.

For $p_i = 4$,

$$\widetilde{\mathcal{M}}_4(s,t;\sigma,\tau) = \widetilde{\mathcal{M}}_{4,200}(s,t) + \sigma^2\widetilde{\mathcal{M}}_{4,020}(s,t) + \tau^2\widetilde{\mathcal{M}}_{4,002}(s,t)$$
$$+\sigma\widetilde{\mathcal{M}}_{4,110}(s,t) + \sigma\tau\widetilde{\mathcal{M}}_{4,011}(s,t) + \tau\widetilde{\mathcal{M}}_{4,101}(s,t)$$

$$\widetilde{\mathcal{M}}_{4,200}(s,t) = \frac{1}{29700N^3}\prod_{i=2}^{7}\frac{1}{s-2i}\prod_{j=6}^{7}\frac{1}{t-2i}\prod_{k=6}^{7}\frac{1}{\tilde{u}-2i}$$
$$\times\left(165s^4 - 6820s^3 + 102620s^2 - 661648s + 1525632\right) ,$$

$$\widetilde{\mathcal{M}}_{4,101}(s,t) = \frac{1}{7425N^3}\prod_{i=4}^{7}\frac{1}{s-2i}\prod_{j=4}^{7}\frac{1}{t-2i}\prod_{k=6}^{7}\frac{1}{\tilde{u}-2i}$$
$$\times(165s^2t^2 - 4180s^2t + 26180s^2 - 4180st^2 + 105980st$$
$$- 664424s + 26180t^2 - 664424t + 4170432) .$$

$$\widetilde{\mathcal{M}}_{4,200}(s,t) = \widetilde{\mathcal{M}}_{4,020}(\tilde{u},t) = \widetilde{\mathcal{M}}_{4,002}(t,s) ,$$
$$\widetilde{\mathcal{M}}_{4,011}(s,t) = \widetilde{\mathcal{M}}_{4,101}(\tilde{u},t) = \widetilde{\mathcal{M}}_{4,110}(t,s) .$$

Rather unwieldy, but still vastly simpler than position space answer ($\mathcal{G}$ = sum of 137 D-functions).

Some future directions

- Higher n-point functions:
  Is there a more "constructive" way to obtain our formula and its generalization to $n > 4$ using on-shell recursion relations?

- Quantum corrections:
  Very interesting bootstrap-style approach to loops in sugra. $O(1/N^4)$ results enforcing consistency with OPE and crossing.
  Aharony Alday Bissi Perlmutter, Alday Bissi, Aprile Drummond Heslop Paul

- Other backgrounds:
  - Partial results for $AdS_7 \times S^4$, to appear.
  - $AdS_3 \times S^3 \times \mathcal{M}_4$, in progress. LR Roumpedakis Zhou
  - $AdS_4 \times S^7$ significantly harder. Exchanges = *infinite* sums of Ds.

Conclusions

The remarkable simplicity of $\mathcal{M}$ for $AdS_5 \times S^5$ is a welcome surprise.

Like the Parke-Taylor formula for tree-level MHV gluon scattering,
it succinctly encodes the sum of an intimidating number of diagrams.

Holographic correlators are much simpler than previously understood.

We believe that they should be studied following the blueprint of the
modern on-shell approach to perturbative gauge theory amplitudes.