# Analyticity Properties of Superstring Loop Amplitudes

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**Brussels, July 2019**

String theory has extended objects as elementary constituents.

Does this lead to any violation of locality in perturbative amplitudes?

In QFT, locality is related to the analytic properties of the S-matrix.

Does the S-matrix of string theory display the same analytic behaviour as that of a local QFT?

– may provide an indirect test of locality of string theory.

Consider a scattering amplitude of n particles with ingoing momenta $p_1, \cdots, p_n$ satisfying on-shell condition

$$p_a^2 + m_a^2 = 0, \qquad m_a > 0$$

$$p_a^0 \gtrless 0 \leftrightarrow \begin{matrix}\text{ingoing}\\ \text{outgoing}\end{matrix} \text{ particle}, \quad p_a^2 \equiv -(p_a^0)^2 + \vec{p}_a^{\,2}$$

Define

$$s_{ab} = -(p_a + p_b)^2 = m_a^2 + m_b^2 - 2p_a.p_b$$

In what domain in the <u>complex</u> $s_{ab}$ space is the scattering amplitude analytic?

– useful starting point for studying many other properties, e.g. crossing symmetry, dispersion relations etc.

## How is this studied in local QFT?

**Consider the off-shell amputated Green's function $G(p_1, \cdots p_n)$.**

$$S(p_1, \cdots p_n) \propto G(p_1, \cdots p_n)|_{p_a^2 = -m_a^2}$$

**We have dropped the Lorentz indices for notational convenience but they are present implicitly.**

**Define $P_{(\alpha)} = \sum_{a \in A_\alpha} p_a$, $\qquad A_\alpha \subset \{1, 2, \cdots, n\}$**

Based on the locality of the position space Green's function one can show that $G(p_1, \cdots p_n)$ is analytic in the following domain

1. If $Im(P_{(\alpha)}) \neq 0$ then $Im(P_{(\alpha)})$ must be time-like.

2. If $Im(P_{(\alpha)}) = 0$, then $P_{(\alpha)}$ must be below the threshold of production of multi-particle states.

– primitive domain of analyticity D

Simple kinematics shows that D has no intersection with the subspace in which the external states are on-shell

$$p_a^2 + m_a^2 = 0 \quad \text{for a = 1,2,} \cdots \text{,n}$$

Apparent conclusion: Analyticity of G in D is useless for exploring analyticity properties of S-matrix

The way out:

Suppose $f(z_1, \cdots z_n)$ is function of multiple complex variables, known to be analytic in some domain D.

Without knowing anything else about f, we can often prove that the function is analytic in an extended domain W.

W depends only on D and not on f.

This is different for function f(z) of one complex variable.

Given <u>any</u> connected domain R, one can find f(z) such that it is analytic in R and non-analytic on the boundary of R.

## Example

Suppose $f(z_1, z_2)$ is known to be analytic in

$$\text{Disk} \times \text{Annulus} + \text{Annulus} \times \text{Disk}$$

Then $f(z_1, z_2)$ is analytic in

$$\text{Disk} \times \text{Disk}$$

Proof is elementary, using Cauchy's integration formula.

Using this kind of argument we can extend the domain of analyticity of $G(p_1, \cdots p_n)$ beyond the primitive domain D.

Jost, Lehmann; Dyson; Bros, Messiah, Stora; $\cdots$

The extended domain W includes points satisfying $p_a^2 + m_a^2 = 0$

– can be used to prove interesting results for $2 \to 2$ scattering

1. Crossing symmetry: Existence of an analytic continuation relating

Bros, Epstein, Glaser

$$A + B \to C + D \quad \Rightarrow \quad A + \bar{C} \to \bar{B} + D$$

2. Analyticity of the elastic forward scattering amplitude (t=0) in the full complex s-plane

see Itzykson-Zuber

etc.

What about in superstring theory?

String theory seemingly has non-local interactions.

Is there any signature of this in the S-matrix?

Do the amplitudes in string theory enjoy the same analyticity properties as those in quantum field theory?

**To follow the same approach as in QFT, we need off-shell Green's function**

**– use superstring field theory to define $G(p_1, \cdots, p_n)$.**

**Our goal is to prove analyticity of $G(p_1, \cdots p_n)$ in the primitive domain D as in QFT.**

**However the starting point is missing**

**– we do not have the position space Green's functions.**

**Strategy: Try to prove analyticity in D directly in the momentum space using <u>Feynman</u> <u>diagrams</u>.**

Superstring field theory has infinite number of fields $\{\phi_\alpha\}$.

Gauge fixed action:

$$\int \frac{d^D k}{(2\pi)^D} K_{\alpha\beta}(k)\phi^\alpha(k)\phi^\beta(-k)$$
$$+\sum_n \int \frac{d^D k_1}{(2\pi)^D}\cdots\frac{d^D k_1}{(2\pi)^D}\,(2\pi)^D\delta^{(D)}(k_1+\cdots+k_n)$$
$$V^{(n)}_{\alpha_1\cdots\alpha_n}(k_1,\cdots k_n)\,\phi^{\alpha_1}(k_1)\cdots\phi^{\alpha_n}(k_n)$$

$K_{\alpha\beta} \propto (k^2+m_\alpha^2)\,\delta_{\alpha\beta}$

$V^{(n)}$ has exponential suppression factors as

$$k_s^0 \to \pm i\infty \quad \text{or} \quad k_s^i \to \pm\infty.$$

Each Feynman diagram is manifestly UV finite as long as the ends of $k_s^0$ integration contours are at $\pm i\infty$ and the ends of $k_s^i$ integration contours are at $\pm\infty$.

## Infrared issues

The analyticity properties in QFT have been derived asuming the absence of massless states.

Our goal is to examine if string theory and local QFT differ in their analytic properties

$\Rightarrow$ we shall study analyticity of string amplitudes after removing the contribution from massless fields.

Any additional non-analyticity due to massless fields affects string theory in the same way as it affects QFT

Define P: projection operator to massless fields.

Replace each internal propagator in Feynman diagram by (1-P)

Analyticity will be analyzed for this amplitude

Physically these amplitudes describe the interaction vertices of the gauge invariant Wilsonian effective action of massless fields.

The effect of massless internal states will have to be taken care of separately, and will produce the usual IR singularities of QFT

We begin by analyzing $G(p_1, \cdots p_n)$ at $p_a = 0$ for $a = 1, \cdots n$.

$\ell_s$: loop momenta, $k_r$: momenta of propagators

Take the $\ell_s^0$ integration contour along imaginary axis and the $\ell_s^i$ integration contour along real axis.

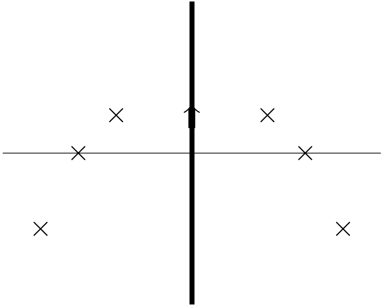
For an internal propagator with momentum $k_r$,

$$k_r^2 + m_r^2 = -(k_r^0)^2 + \vec{k}_r^{\ 2} + m_r^2 > 0$$

since $k_r^0$ is imaginary and $\vec{k}_r$ is real

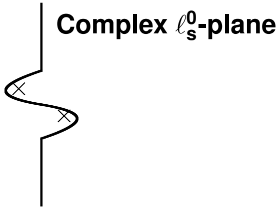
Therefore the integrals are well defined and $G(p_1, \cdots p_n)$ is analytic in the neighborhood of $\{p_a = 0\}$

## Distribution of poles in a complex loop energy ($\ell_s^0$) plane



As we deform the external momenta $\{p_a\}$ away from 0 the pole positions move.

If the poles approach the integration contour, we move the contour away from the poles to avoid singularities, but keep the ends fixed for UV finiteness.




When a contour is 'pinched' by poles from two sides so that we cannot deform the contour, the integral becomes non-analytic.

Our goal will be to show that this does not happen inside the primitive domain D.

We have been able to prove a limited version of the result in which $Im(p_a)$'s lie in a two dimensional Lorentzian plane

Without any loss of generality we can take the two dimensional plane to be the 0-1 plane.

We shall call the corresponding domain $D'$.

Analyticity in $D'$ is sufficient to prove all the known analyticity properties of S-matrix proved for local QFT

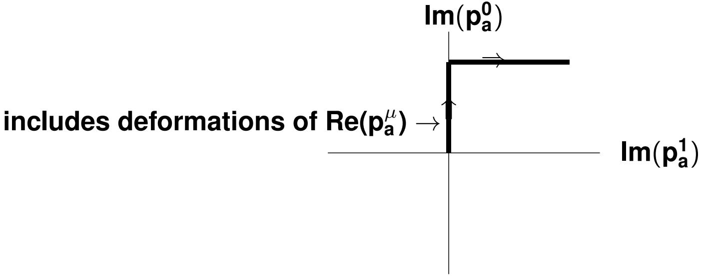
**We carry out the analysis in two steps.**

**1. Keep $\mathrm{Im}(p_a^1) = 0$ and deform all other components of all external momenta from 0 to the desired value remaining inside $D'$.**

**2. Deform $\mathrm{Im}(p_a^1)$ from 0 to the desired value keeping all other components fixed, and remaining inside $D'$.**

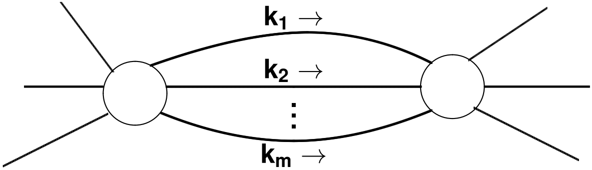
**Our goal is to show that we do not encounter any pinch singularity during either of these deformations.**

**A schematic representation of the deformation in $\mathrm{Im}(p_a)$ plane:**




**Strategy: Assume that the integration contour is pinched and show that there is a contradiction.**

**1. Assuming that the contours are pinched during the first step, we show that at the pinch the momenta in the Feynman diagram must have the following form:**

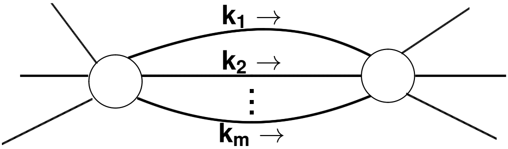



**Each $k_r$ is real, on-shell and has $k_r^0 > 0$.**

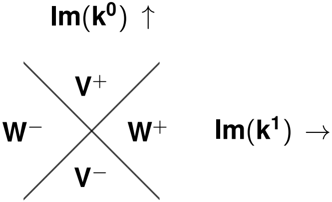
**Since $\sum_r k_r = P_{(\alpha)}$ for some $\alpha$, the corresponding $P_{(\alpha)}$ is real, and above the threshold of production of multi-particles states with momenta $k_1, \cdots, k_m$.**

**– not possible inside D.**

**2. Assuming that the contours are pinched during the 2nd step, we show that at the pinch the momenta in the Feynman diagram must have the following form:**




**Each $k_r$ is on-shell, with $Im(k_r) \in W^+$.**




**$\Rightarrow Im(P_{(\alpha)}) = \sum_r Im(k_r) \in W^+ \Rightarrow Im(P_{(\alpha)})$ is space-like $\Leftarrow$ not possible in D**

**This proves the analyticity of the off-shell Green's function of string field theory in $D'$**

**This is sufficient to prove that all the <u>known</u> analyticity properties of QFT S-matrix are shared by the perturbative S-matrix of superstring theory.**

**Therefore, so far there is no evidence of non-locality of string theory in the perturbative S-matrix.**

## Future directions

1. Extend the analysis to the case where the imaginary parts of external momenta are not restricted to lie in a two dimensional plane.

2. Explore whether the momentum space results may be inverted to give information on position space Green's functions

– could provide a direct handle on the (absence of) non-locality in perturbative string theory.