VERTEX ALGEBRAS

AND

SUPERCONFORMAL FIELD THEORIES

IN

FOUR DIMENSIONS

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PREFACE

I will be talking about $\mathbf{CFT}_4$ with $\mathcal{N} \geqslant 2$ supersymmetry.

There has been a huge expansion in the known landscape of such theories:

- Conformal gauge theories; class $\mathcal{S}$ theories
- Argyres-Douglas points
- Type II strings on CY3 singularities

Generically don't admit (conventional) Lagrangian descriptions.

This motivates a program (superconformal bootstrap) to develop robust methods to study these models, independent of microscopic formulation.

This has led to a powerful new organisational principle (and an interesting piece of mathematical physics) through a connection to vertex operator algebras.

I will give an overview and update.

EXTRACTING A VERTEX ALGEBRA

[CB, LEMOS, LIENDO, PEELAERS, RASTELLI, VAN REES (2013)]

Start by characterising SCFTs by their OPE algebra of local operators.

$$\mathcal{O}_1(x_1)\mathcal{O}_2(x_2) = \sum_k \frac{c_{12}{}^{k}\mathcal{O}_k(x_2)}{|x_1 - x_2|^{E_1+E_2-E_k}} \; .$$

Operators organize into irreps. of $\mathfrak{su}(2,2|2)$, and in particular of the usual bosonic subalgebra:

$\mathfrak{so}(1,3)$ $\qquad$ $\mathfrak{u}(1)_r$

$$\{\mathcal{O}_{E,j_1,j_2,R,r}\}$$

scaling $\qquad$ $\mathfrak{su}(2)_R$

Carve out tractable algebraic structure by passing to cohomology of carefully chosen supercharge:

$$\mathcal{Q}_{\mathbb{V}} = \mathcal{Q}^1_- + \widetilde{\mathcal{S}}^{\dot{-}}_1$$

EXTRACTING A VERTEX ALGEBRA

[CB, LEMOS, LIENDO, PEELAERS, RASTELLI, VAN REES (2013)]

In $\mathcal{Q}_{\mathbb{V}}$-cohomology, operators restricted to lie in a complex plane $\mathbb{C}_{[z,\bar{z}]} \subset \mathbb{C}^2$.

OPE acquires meromorphic dependence on z, giving a vertex operator algebra/chiral algebra. Consequently we get a map from SCFTs to VOAs:

$$\mathcal{T} \quad \stackrel{\mathbb{V}}{\longrightarrow} \quad \mathbb{V}[\mathcal{T}]$$

Operators in cohomology in one-to-one correspondence with Schur operators, obeying

$$E + (j_1 + j_2) = 2R \ , \qquad j_1 - j_2 = r$$

Includes:

- Higgs branch chiral ring operators.
- Generic elements are "semi-short" operators in $\hat{\mathcal{C}}$ multiplets.

Notable absence of Coulomb branch chiral ring operators.

PROPERTIES OF THE VERTEX ALGEBRA

Some operators in four dimensions have universal images in the VOA:

$$T_{\mu\nu}(z,\bar{z}) \quad \longrightarrow \quad [(J_R)_z^{++}]_{\mathcal{Q}_{\mathbb{V}}} = T(z)$$

$T(z)$ generates Virasoro algebra with central charge $c = -12c_{4d} < 0$.

$$J_\mu^A(z,\bar{z}) \quad \longrightarrow \quad [\mu^A]_{\mathcal{Q}_{\mathbb{V}}} = \mathcal{J}^A(z) \ , \quad A = 1,\ldots,\dim \mathfrak{g}_F$$

$\mathcal{J}^A(z)$ generates $\hat{\mathfrak{g}}_F$ affine Kac-Moody algebra at level $k = -\frac{1}{2}k_{4d} < 0$.

Additional properties follow from general considerations:

- Higgs chiral ring generators are strong VOA generators.
- Schur superconformal index reproduces vacuum character.
- VOA independent of marginal couplings (non-renormalization theorem).

EXAMPLES

In many cases can determine the VOA associated to a given SCFT. There is now a large catalogue, e.g.,

- Free hyper-multiplet $\longrightarrow$ dimension $\left(\frac{1}{2}, \frac{1}{2}\right)$ (β, γ) (symplectic boson)
- (A_1, A_{2n}) Argyres-Douglas SCFTs $\longrightarrow$ $Vir_{2,2n+3}$
- (A_1, D_{2n+1}) Argyres-Douglas SCFTs $\longrightarrow$ $V_{\frac{-4n}{2n+1}}\left(\mathfrak{sl}(2)\right)$
- Rank one F-theory SCFTs $\longrightarrow$ $V_{-1-\frac{h^\vee}{6}}\left(\mathfrak{g}\right), \quad \mathfrak{g} \in \{\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{d}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8\}$

Despite simplicity of above examples, general cases are non-linear VOAs.

- *E.g.*, $N_f = 2N_c$ SCQCD includes non-linear $\mathscr{W}$-currents with spin $N_c/2$.

We see everything from rational Virasoro to irrational $\mathscr{W}$-algebras.

APPLICATION: UNITARITY BOUNDS

These VOAs are specified by finite amount of data (singular OPEs of strong generators), but encode infinite number of 4d two- and three-point functions.

These VOAs are not manifestly unitary (obviously!), all of this data, appropriately reinterpreted in four-dimensional terms, must be compatible with unitarity.

This is a restriction on the VOAs that can arise -- for example, saw that negativity of Virasoro central charge and Kac-Moody levels required by 4d unitarity.

This is just the tip of the iceberg.

UNITARITY BOUNDS: SIMPLE FLAVOUR SYMMETRY

[CB ET. AL. (2013)]

Let's consider $\mathbb{V}[\mathscr{T}]$ when $\mathscr{T}$ enjoys (simple) $\mathfrak{g}_F$ global symmetry.

$h = R = 1$

$$\mathcal{J}^A \sim [\mu^A]$$

$h = R = 2$

$$(\mathcal{J}^A \mathcal{J}^B) \in \text{span}\left\{[\mu^A \mu^B], T\right\}$$

$h = 2,\ R = 1$

Non-singlet $\mathfrak{g}_F$ representations are pure Higgs branch operators.

Singlet channel is an admixture, have to orthogonalize with $T(z)$.

VOA norms of these states are fixed algebraically, 4d unitarity dictates that they be positive!

UNITARITY BOUNDS: SIMPLE FLAVOUR SYMMETRY

[CB ET. AL. (2013)]

Positivity of singlet norm: $kd_{\mathfrak{g}} \leqslant c(k+h^{\vee}_{\mathfrak{g}}) \longrightarrow c \geqslant \dfrac{kd_{\mathfrak{g}}}{k+h^{\vee}_{\mathfrak{g}}}$ if $k > -h^{\vee}_{\mathfrak{g}}$

Positivity of non-singlets:

Flavor group factor	Level bound
$\mathfrak{su}(2)$	$k \leqslant -\frac{1}{3}$
$\mathfrak{su}(n)$	$k \leqslant -\frac{n}{2}$
$\mathfrak{so}(n)\ ,\quad n = 4,\ldots,8$	$k \leqslant -2$
$\mathfrak{so}(n)\ ,\quad n \geqslant 8$	$k \leqslant 2 - \frac{n}{2}$
$\mathfrak{usp}(n)\ ,\ n \geqslant 3$	$k \leqslant -1 - \frac{n}{2}$
$\mathfrak{g}_2$	$k \leqslant -\frac{5}{3}$
$\mathfrak{f}_4$	$k \leqslant -\frac{5}{2}$
$\mathfrak{e}_6$	$k \leqslant -3$
$\mathfrak{e}_7$	$k \leqslant -4$
$\mathfrak{e}_8$	$k \leqslant -6$

Saturating bounds implies Higgs chiral ring relations.

UNITARITY BOUNDS: REDUCTIVE FLAVOUR SYMMETRY

[CB (2018)]

Now let $\mathfrak{g}_F = \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_n$ with $\mathfrak{g}_i$ simple or $\mathfrak{u}_1$.

Unitarity for $\mathfrak{g}_F$ -singlet sector is now substantially more intricate.

Matrix $\mathbb{M}$ of two-point functions of $\mathbf{Tr}(\mu_i^2)$ operators not diagonal.

$$\mathbb{M} = \mathrm{Diag}(\delta_1, \ldots, \delta_n) - \frac{c}{2} \overrightarrow{\alpha} \otimes \overrightarrow{\alpha}$$

$$\delta_i = 2k_i d_i (k_i + h_i^\vee) \ , \quad \alpha_i = \frac{2k_i d_i}{c}$$

Positive semi-definiteness of $\mathbb{M}$ implies delicate constraints on central charges.
Can summarise as follows...

UNITARITY BOUNDS: REDUCTIVE FLAVOUR SYMMETRY

[CB (2018)]

Call simple factor:

- subcritical if $k_i > -h_i^\vee$
- critical if $k_i = -h_i^\vee$
- supercritical *if* $k_i < -h_i^\vee$

Unitarity then demands:

- For any two critical factors i and j, have $\mathbf{Tr}\mu_i^2 \propto \mathbf{Tr}\mu_j^2$ in Higgs chiral ring.
- If at least one critical factor, can be no subcritical factors.
- If any subcritical factor, $c \geqslant \sum_i \frac{k_i d_i}{k_i + h_i^\vee}$ (Sugawara bound).
- There can be at most one subcritical factor.

UNITARITY BOUNDS BEYOND $h = 2$?

Algebraic rigidity of VOAs plus four-dimensional unitarity leads to constraints when we can unambiguously map operators.

Difficult to push further for lack of dictionary between VOA states and operators with definite 4d quantum numbers.

Problems arise because the space $\mathscr{V}$ of Schur operators is triply graded...

$$\mathscr{V} = \bigoplus_{h,R,r} \mathscr{V}_{h,R,r} , \qquad h = \frac{E + (j_1 + j_2)}{2} = E - R$$

...but only bi-grading by h and r is manifest in VOA, OPE violates R-conservation (but only downward, defines an R-filtration).

Have to ask how much four-dimensional physics we can really see without R-grading. Can we see the Higgs branch?

GEOMETRY OF THE ASSOCIATED VOA

[CB, RASTELLI (2017)]

Apparently yes! One can canonically associate a finite-dimensional Poisson variety to any vertex operator algebra: the associated variety.

$$X_{\mathscr{V}} = \text{mSpec}\left(\mathscr{V}/C_2(\mathscr{V})\right) \qquad C_2(\mathscr{V}) = \text{span}\left\{\left(\partial\mathcal{O}_1\mathcal{O}_2\right)\right\}$$

Conjecture [CB, Rastelli (2017)]: For VOAs arising from four dimensional SCFTs

$$X_{\mathbb{V}[\mathcal{T}]} = \mathscr{M}_H[\mathcal{T}]\,.$$

Higgs branch of vacua

This amounts to a rather technical condition on the R-filtration.

Verified in infinity of examples.

GEOMETRY OF THE ASSOCIATED VOA

[CB, RASTELLI (2017)]

Conjecture implies in particular that the associated variety is symplectic. This is a strong constraint (in generic case it is just Poisson).

e.g.

$$X_{Vir_c} = \begin{cases} \mathbb{C} & c \text{ generic} \\ \{\text{pt.}\} & c = c_{p,q} = 1 - 6\frac{(p-q)^2}{pq} \end{cases}$$

$$X_{V_k(\mathfrak{g})} = \begin{cases} \mathfrak{g}_{\mathbb{C}} & k \text{ generic} \\ \{\text{pt.}\} & k \in \mathbb{N} \\ \overline{\mathbb{O}_q(\mathfrak{g}_{\mathbb{C}})} & k = -h^\vee + \frac{p}{q} \end{cases}$$

Condition generalises rational VOAs, which have trivial associated varieties (C_2 co-finite).

VOAs with symplectic associated variety have been given the moniker quasi-Lisse by Tomoyuki Arakawa and Kazuya Kawasetsu.

MODULAR DIFFERENTIAL EQUATIONS

[CB, RASTELLI (2017); CB, PEELAERS (TO APPEAR)]

Theorem (Arakawa, Kawasetsu 2016): The vacuum character of a quasi-Lisse VOA solves a linear modular differential equation of weight zero.

Implies that Schur limit of the superconformal index is always the solution of such a differential equation, and so is a component of a vector-valued modular form.

(quasi-)

For example, for rank-one F-theory SCFTs:

$$X_{V_{-(h^\vee+6)/6}(\mathfrak{g})} = \mathcal{M}_H[\mathcal{T}_{\mathfrak{g}}] = \overline{\mathbb{O}_{\min}(\mathfrak{g})}$$

Serre derivative

$$\left(D_q^{(2)} - 5(h^\vee+1)(h^\vee-1)\mathbb{E}_4(q)\right)\mathcal{I}_{\rm Sch}(q) = 0$$

Generalises results by N. Drukker et. al. for $\mathcal{N}=4$ SYM and earlier observations by S. Razamat.

VOA FROM GEOMETRY: FREE FIELDS

[CB, MENEGHELLI, RASTELLI (2019)]

VOAs possess nice geometric structure in the associated variety, but this arises upon taking a huge quotient.

$$\mathbb{V}[\mathcal{T}] \xrightarrow{X_{\mathbb{V}}} \mathcal{M}_H$$

A closer connection to the Higgs branch seems to arise through free field realisations.

$$\mathcal{M}_H[\mathcal{T}] \xrightarrow{\text{FFR}} \mathbb{V}[\mathcal{T}]$$

We have found highly economical realisations in terms of the lattice VOA for a lattice $\Pi_{d,d}$ of signature (d,d) with $d = \dim_{\mathbb{H}} \mathcal{M}_H$.

E.g., F-theory SCFTs have free field realisation in terms of $\Pi_{1,1}$ lattice VOA and $h^\vee - 2$ symplectic bosons [compare $\dim \mathfrak{g}$ chiral bosons for Wakimoto].

(For $\mathfrak{g} = \mathfrak{e}_8$, compare 58 chiral bosons to 248 for Wakimoto).

VOA FROM GEOMETRY: FREE FIELDS

[CB, MENEGHELLI, RASTELLI (2019)]

In more detail, the picture is that we specify the Higgs branch physics of the theory:

- Higgs branch as a holomorphic-symplectic variety
- Low energy effective field theory at generic point

From this concoct "minimal" free field realisation of the VOA (currently requires some artistry)

$$(A_1, D_3)\ \text{Argyres-Douglas} \qquad \mathcal{M}_H = \mathbb{C}^2/\mathbb{Z}_2 = \{xy + z^2 = 0\} \subset \mathbb{C}^3$$

Consider principal open subset where $x \neq 0$, which has the structure of $T^*\mathbb{C}^\times$

$$\mathbb{V}[T^*\mathbb{C}^\times] = \mathbb{V}\Pi_{1,1} \qquad (x, z) \longmapsto (e^{\delta+\varphi}, \partial\varphi - \partial\delta)$$

VOA images of Higgs chiral ring generators fixed up to "quantum corrections"

$$X = e^{\delta+\varphi}\ , \quad Z = k\partial\varphi\ , \quad Y = \left(-(\tfrac{k}{2}\partial\delta)^2 + \tfrac{k(k+1)}{2}\partial^2\delta\right) e^{-\delta-\varphi}$$

This reproduces a free field construction of $V_{-4/3}(\mathfrak{sl}(2))$ due to Adamović (2004)

VOA FROM GEOMETRY: FREE FIELDS

[CB, MENEGHELLI, RASTELLI (2019); CB, MENEGHELLI, PEELAERS, RASTELLI (TO APPEAR)]

Many more examples! Each one a beautiful gem.

- Rank-two F-theory SCFTs (built from 2 bosons and 2 x rank-one VOA)
- $\mathcal{N} = 4$ SYM theories [Bonetti, Meneghelli, Rastelli (2018)]
- Linear conformal quivers

In general one has are generalised free field realisations that utilize C_2 co-finite VOAs as irreducible ingredients along with lattice/free field VOAs.

What's more, these realisations come equipped with canonical (geometric) filtration,

$$R[e^{n(\delta+\varphi)}] = n \ , \quad R[\partial^n(\delta-\varphi)] = 1 \ , \quad R[\partial^n(\delta+\varphi)] = 0$$

With this filtration can make precise the notion of "quantum corrections" in construction.

In examples, passes highly nontrivial checks to reproduce physical R-filtration.

PARTING THOUGHTS

Correspondence between VOAs and SCFTs in four dimensions is a deep one, encodes detailed algebraic information about strongly coupled SCFTs and brings new intuition to the study of irrational VOAs.

Structural rigidity of VOAs leads to significant consequences for higher-dimensional SCFTs.

- Intricate unitarity bounds
- Modular behaviour of superconformal index.

Associated VOA closely connected to Higgs branch physics; may be possible to reconstruct VOA from Higgs branch data, which in turn gives back 4d R-structure.

Possible classification scheme? Needs to be made more systematic.

Important question: which C_2 co-finite VOAs allowed? Expect unitarity play a key role.

Many more developments I didn't have time to mention -- please take a look!

Merci beaucoup!

Danke je wel!